Abstract
Recently, Dogan, Dogan and Yildiz (2016) presented a new efficiency notion for the random assignment setting called SW (social welfare)-efficiency and characterized it. In this note, we generalize the characterization for the more general domain of randomized social choice.

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1. Introduction
The random assignment setting captures the scenario in which $n$ agents express preferences over $n$ objects and the outcome is a probabilistic assignment. For the setting, two interesting efficiency notions are ex post efficiency and SD (stochastic dominance)-efficiency [1, 3, 5, 6, 8, 10]. The assignment setting can be considered as a special case of voting where each deterministic assignment can be viewed as a voting alternative [2, 4, 7].

Recently, Dogan et al. [9] presented a new notion of efficiency called SW (social welfare)-efficiency for the random assignment setting. They characterize SW-efficiency. In this note, we generalize the characterization to the more general voting setting.

2. Preliminaries
Consider the social choice setting in which there is a set of agents $N = \{1, \ldots, n\}$, a set of alternatives $A = \{a_1, \ldots, a_m\}$ and a preference profile $\succsim = (\succsim_1, \ldots, \succsim_n)$ such that each $\succsim_i$ is a complete and transitive relation over $A$. We write $a \succeq_i b$ to denote that agent $i$ values alternative $a$ at least as much as alternative $b$ and use $> i$ for the strict part of $\succsim_i$, i.e., $a >_i b$ iff $a \succsim_i b$ but not $b \succsim_i a$. Finally, $\sim_i$ denotes $i$'s indifference relation, i.e., $a \sim_i b$ if both $a \succeq_i b$ and $b \succeq_i a$. 
a. The alternatives in $A$ could be any discrete structures: voting outcomes, house allocation, many-to-many two-sided matching, or coalition structures. A utility profile $u = (u_1, \ldots, u_n)$ specified for each agent $i \in N$ his utility for $u_i(a)$ for each alternative $a \in A$. A utility profile is consistent with the preference profile $\succ_i$ if for each $i \in N$ and $a, b \in A$, $u_i(a) \geq u_i(b)$ if $a \succ_i b$. Two alternatives $a, b \in A$ are Pareto indifferent if $a \sim_i b$ for all $i \in N$. For any alternative $a \in A$, we will denote by $D(a)$ the set \{b \in A: \exists i \in N, a \succ_i b\}. An alternative $a \in A$ is Pareto optimal if $a \succ_i b$ for all $i \in N$ and $b \succ_i a$ for some $i \in N$. An alternative is Pareto optimal if it is not Pareto dominated by any alternative.

We will also consider randomized outcomes that are lotteries over $A$. A lottery is a probability distribution over $A$. We denote the set of lotteries by $\Delta(A)$. For a lottery $p \in \Delta(A)$, we denote by $p(a)$ the probability of alternative $a \in A$ in lottery $p$. We denote by support supp$(p)$ the set \{a \in A: p(a) > 0\}. A lottery $p$ is interesting if there exist $a, b \in$ supp$(p)$ such that there exist $i, j \in N$ such that $a \succ_i b$ and $b \succ_j a$. A lottery is degenerate if it puts probability one on a single alternative.

Under stochastic dominance (SD), an agent prefers a lottery that, for each alternative $x \in A$, has a higher probability of selecting an alternative that is at least as good as $x$. Formally, $p \succ^SD_i q$ iff $\forall y \in A$: $\sum_{x \in A: x \succ_i y} p(x) \geq \sum_{x \in A: x \succ_i y} q(x)$. It is well-known that $p \succ^SD_i q$ iff $p$ yields at least as much expected utility as $q$ for any von-Neumann-Morgenstern utility function consistent with the ordinal preferences [4, 8]. A lottery is SD-efficient if it is Pareto optimal with respect to the SD relation. A lottery is ex post efficient if each alternative in the support is Pareto optimal.

3. SW-efficiency

We now consider SW-efficiency as introduced by Dogan et al. [9]. Although Dogan et al. [9] defined SW-efficiency in the context of random assignment, the definition extends in a straightforward manner to the case of voting.

**Definition 1** (SW-efficiency). A lottery $p$ is SW-efficient if there exists no other lottery $q$ that SW dominates it. Lottery $q$ SW dominates $p$ if for any utility profile for which $p$ maximizes welfare, $q$ maximizes welfare, and there exists at least one utility profile for which $q$ maximises welfare but $p$ does not.

We prove a series of lemmas which will help us obtain a characterization of SW-efficiency.

**Lemma 1.** For a preference profile $\succ_i$, consider a Pareto optimal alternative $a \in A$ and a non-empty set $D(a) = \{b \in A: \exists i \in N, a \succ_i b\}$. Then, there exists a utility profile $u$ consistent with $\succ_i$ such that $\sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(b)$ for all $b \in D(a)$.

**Proof.** We can construct the required utility function profile $u$ consistent with $\succ_i$ as follows. Whenever $a \succ_i b$, make the difference $u_i(a) - u_i(b)$ huge. Whenever $b \succ_j a$, make the difference $u_j(b) - u_j(a)$ arbitrarily small. Hence the value
\[ u_i(a) - u_i(b) \text{ is large enough that it makes up for all } j \text{ for which } u_j(b) - u_j(a) > 0. \]

Hence \( \sum_{i \in N} (u_i(a) - u_i(b)) > 0. \)

\textbf{Lemma 2.} \textit{SW-efficiency implies SD-efficiency, which implies ex post efficiency.}

\textit{Proof.} It is well-known that SD-efficiency implies ex post efficiency [4].

Consider a lottery \( p \) that is not SD-efficient. Then there exists another lottery \( q \) that SD-dominates it. Hence \( p \) does not maximize welfare for any consistent utility profile because \( q \) yields more utility for each utility profile.

\textbf{Lemma 3.} \textit{An interesting lottery is not SW-efficient.}

\textit{Proof.} If an interesting lottery \( p \) is not SD-efficient, we are already done because by Lemma 2, \( p \) is not SW-efficient. So let us assume \( p \) is SD-efficient and hence ex post efficient. Since \( p \) is interesting, there exists at least one \( a \in \text{supp}(p) \) such that \( a \succ_i b \) for some \( b \in \text{supp}(p) \) and \( i \in N \). Note that \( a \) is Pareto optimal. By Lemma 1, there exists a utility profile \( u \) such that \( \sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(b) \) for all \( b \in D(a) \) where \( D(a) \cap \text{supp}(p) \neq \emptyset \). Hence, there exists a utility profile \( u \) such that \( \sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(b) \) for all \( b \in \text{supp}(p) \cap D(a) \). This means that \( \sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(p) \). Hence for the lottery \( q \) that puts probability 1 on alternative \( a \), \( \sum_{i \in N} u_i(q) > \sum_{i \in N} u_i(p) \). Also note that for any utility profile for which \( p \) maximizes welfare, \( q \) maximizes welfare as well since \( a \in \text{supp}(p) \). Thus \( q \) SW dominates \( p \).

\textbf{Lemma 4.} \textit{An uninteresting lottery over Pareto optimal alternatives is SW-efficient.}

\textit{Proof.} An uninteresting lottery \( p \) over Pareto optimal alternatives is SD-efficient. Assume that there is another lottery \( q \) that SW-dominates \( p \). Then \( \text{supp}(q) \) contains one alternative \( b \) that is not Pareto indifferent to alternatives \( \text{supp}(p) \). This means that there exists a utility profile \( u \) such that welfare is maximized by \( p \) but not by \( b \). Hence \( q \) does not SW-dominate \( p \).

Based on the lemmas proved above, we prove the main result.

\textbf{Theorem 1.} \textit{A lottery is SW-efficient iff it is ex post efficient and uninteresting.}

\textit{Proof.} By Lemma 4, an ex post efficiency and uninteresting lottery is SW-efficient.

We now prove that if a lottery is not ex post efficient or uninteresting, it is not SW-efficient. Due to Lemma 2, if a lottery is not ex post efficient, it is not SW-efficient. Similarly, by Lemma 3, if a lottery is interesting, it is not SW-efficient.

Next we prove that if \( A \) contains no Pareto indifferent alternatives, then a lottery is SW-efficient iff it is ex post efficient and degenerate.

\textbf{Lemma 5.} \textit{If \( A \) contains no Pareto indifferent alternatives, then if a lottery is uninteresting and not degenerate, then it is not ex post efficient.}
Proof. Assume that a lottery $p$ is uninteresting and not degenerate. Since $p$ is not degenerate, $|\text{supp}(p)| \geq 2$. Since $p$ is uninteresting, there do not exist $a, b \in \text{supp}(p)$ such that there exist $i, j \in N$ such that $a \succ_i b$ and $b \succ_j a$. Thus either $a$ Pareto dominates $b$, or $b$ or Pareto dominates $a$ or $a$ and $b$ are Pareto indifferent. The third case is not possible because we assumed that $A$ does not contain Pareto indifferent alternatives. Since $a$ Pareto dominates $b$ or $b$ or Pareto dominates $a$, $\text{supp}(p)$ contains a Pareto dominated alternative. Hence $p$ is not ex post efficient.

Theorem 2. If $A$ contains no Pareto indifferent alternatives, then a lottery is SW-efficient iff it is ex post efficient and degenerate.

Proof. Assume that $A$ contains no Pareto indifferent alternatives. If a lottery $p$ is SW-efficient, then by Theorem 1, it is ex post efficient and uninteresting. By Lemma 5, since $p$ is ex post efficient, it is degenerate.

Now assume that a lottery $p$ is ex post efficient and degenerate. Since $p$ is degenerate, it is uninteresting by definition. Since it is both ex post efficient and uninteresting, then by Theorem 1, it is SW-efficient.

Theorem 2 gives us more insight into the results of Dogan et al. [9],

Lemma 6. An assignment problem with strict preferences does not admit Pareto indifferent deterministic assignments.

Proof. Consider two deterministic assignments $M$ and $M'$ such that all agents are indifferent among them. Then this means that each agent gets the same item in both $M'$ and $M$. But this implies that $M' = M$.

Corollary 1 (Dogan et al. [9]). If preferences are strict, the only undominated probabilistic assignments are the Pareto efficient deterministic assignments.

Proof. By Lemma 6, no two deterministic assignments are completely indifferent for all agents. Hence, by Theorem 2, if a random assignment that is SW-efficient, then it is a deterministic Pareto optimal assignment.

References


