Soc Choice Welf
DOI 10.1007/s00355-015-0877-4

A note on the undercut procedure

Haris Aziz

Received: 31 March 2014 / Accepted: 2 February 2015

Abstract The undercut procedure was presented by Brams et al. (Soc Choice Welf 39:615–631, 2012) as a procedure for identifying an envy-free allocation when agents have preferences over sets of objects. We point out some shortcomings of the undercut procedure. We then simplify the undercut procedure of Brams et al. and show that it works under more general conditions where agents may express indifference between objects and they may not necessarily have responsive preferences over sets of objects. Finally, we show that the procedure works even if agents have unequal claims.

JEL Classification C70 · D61 · D71

1 Introduction

Allocation of indivisible resources is one of the most fundamental problems in *fair division* and *multiagent resource allocation* (Brams and Taylor 1996; Chevaleyre et al. 2006; Aziz et al. 2014). Many of the fair division settings feature two agents since disputes often concern two parties. Recently Brams et al. (2012) presented the *undercut procedure* which is an elegant procedure to divide a set of contested indivisible objects fairly among two agents. A crucial assumption in the paper was that agents have a strict ranking over the objects and the preferences over *sets* of objects are *responsive*. Preferences over sets of objects are *responsive* if for any set S in which an object is removed or is replaced by a less preferred object, the new set S' is less preferred. In that case, Brams et al. (2012) term S' as *ordinally less* than S. We first show that the assumption of responsive preferences can be somewhat restrictive. We also identify three shortcomings of the first few steps of the undercut procedure. Finally, we rectify

H. Aziz (✉)
NICTA and UNSW, Sydney 2033, Australia
e-mail: haris.aziz@nicta.com.au

Published online: 10 February 2015

the shortcomings by simplifying the undercut procedure of Brams et al. (2012) and showing that it returns an envy-free allocation (if it exists) under a more general preference restriction called *separability*.

The setting we consider concerns two agents 1 and 2 and a set of objects O. Both agents have *complete* and *transitive* preferences $\succsim_1$ and $\succsim_2$ over the subsets of objects in O. We write $S \succsim_i T$ to denote that agent i values allocation S at least as much as allocation T and use $\succ_i$ for the strict part of $\succsim_i$, i.e., $S \succ_i T$ iff $S \succsim_i T$ but not $T \succsim_i S$. Finally, $\sim_i$ denotes i's indifference relation, i.e., $S \sim_i T$ iff both $S \succsim_i T$ and $T \succsim_i S$. For any set $S \subset O$, we will refer to $O \setminus S$ by $-S$.

An allocation is a complete list of consumptions of both agents. The goal is to identify an envy-free allocation $(S, -S)$ where S is the allocation of agent 1 and $-S = O \setminus S$ is the allocation of agent 2. Sometimes, when we will refer to the agent i, then agent $-i$ is the other agent and $(S, -S)$ is the allocation that gives S to i and $-S$ to agent $-i$.

2 The undercut procedure

The undercut procedure is a discrete generalization of the divide and choose cake cutting protocol (Brams and Taylor 1996, Chap. 1). The elegance of the undercut procedure lies in the fact that although agents have preferences over sets of objects, it is sufficient to only consider or query about the *minimal bundles* of the agents over the contested pile. A subset $S \subseteq O'$ is a *minimal bundle* with respect to set of objects $O' \subseteq O$ for agent i if $S \succsim_i O' \setminus S$ and for any $T \subset S$, $O' \setminus T \succ_i T$. When the context of O' is clear, we will refer to $O' \setminus S$ and $O' \setminus T$ simply as $-S$ and $-T$ respectively. The set of minimal bundles of agent i with respect to O' is denoted by $MB_i(O')$. Any envy-free allocation $(S, -S)$ of O is *trivial* if $S \sim_1 -S$ and $-S \sim_2 S$. The main idea underlying the undercut procedure is that there exists a non-trivial envy-free allocation of the contested pile if the set of minimal bundles of both agents is not the same. The undercut procedure goes through the minimal bundles with respect to the contested pile of the agents to identify an envy-free allocation if it exists (Brams et al. 2012). We refer the reader to Algorithm 1 for an adapted specification of the undercut procedure.

2.1 Limitation of responsive preferences

The undercut procedure was shown to find an envy-free allocation if the preferences of agents are *responsive*. *Responsiveness* is a well-established preference restriction on preferences over sets of objects which assumes that the agents have preferences over the individual objects. Preferences over sets of objects are *responsive* if for any set in which an object is removed or is replaced by a less preferred object, the new set is less preferred (Barberà et al. 2004). We first highlight that responsive preferences can be restrictive.

Example 1 In a divorce dispute, husband h may prefer each of the two family dogs d_1 and d_2 over the car c: $\{d_1\} \succ_h \{d_2\} \succ_h \{c\} \succ_h \emptyset$. If the husband's preferences are responsive, then his preference over the set of issues is as follows: $\{d_1, d_2, c\} \succ_h \{d_1, d_2\} \succ_h \{d_1, c\} \succ_h \{d_2, c\}$.

Algorithm 1 Undercut procedure of Brams et al. (2012)

Output: An envy-free allocation of O if an envy-free allocation of the contested pile exists.

1 **Generation Phase:** Agent 1 and 2's most preferred objects are given to them if they do not coincide. If both most prefer the same object, then it is placed in the contested pile $I_C \subset O$. The process continues until all objects have been named by at least one agent. If the contested pile is empty, the procedure ends. Otherwise, each agent i identifies his set of minimal bundles $MB_i(I_C)$ with respect to I_C.

2 If $MB_1(I_C) \neq MB_2(I_C)$, each agent i reports to the mechanism his ranking of his minimal bundles with respect to I_C. An agent i is chosen at random, and one of i's top-ranked minimal bundle S is considered. If $S \notin MB_{-i}(I_C)$, then it becomes the proposal, and i is the proposer. If $S \in MB_{-i}(I_C)$, then one of $-i$'s top-ranked minimal bundle S' is considered. If $S' \notin MB_i(I_C)$, then it becomes the proposal, and $-i$ the proposer. If $S' \in MB_i(I_C)$, then the process continues until a minimal bundle of one agent with respect to I_C is found that is not a minimal bundle of the other with respect to I_C. Then proceed to Step 4.

3 If $MB_1(I_C) = MB_2(I_C)$, and there exists an S such that $S \in MB_i(I_C)$ and $-S = I_C \setminus S \in MB_i(I_C)$ (and, therefore $S, -S \in MB_{-i}(I_C)$ also), then S becomes the proposal. If there is no minimal bundle S with respect to I_C to such that $-S$ is also a minimal bundle with respect to I_C, then a minimal bundle with respect to I_C is chosen randomly and becomes the proposal.

4 Assume that S is the proposal and the proposer is i. Then $-i$ may respond by (a) accepting $-S$ of I_C or (b) undercutting i's proposal, i.e., taking his most-preferred subset T and giving $-T = I_C \setminus T$ to $-i$. The procedure ends. An agent's subset of O consists of all objects received in Step 1, plus the agent's share of the contested pile determined in Step 4.

However it may be the case that the husband prefers the set of a car and a dog to the set of two dogs: $\{d_1, c\} \succ_h \{d_1, d_2\}$. This way, he will have both a companion and a ride: $\{d_1, d_2, c\} \succ_h \{d_1, c\} \succ_h \{d_2, c\} \succ_h \{d_1, d_2\}$.

A preference relation $\succsim$ is *separable* if for all $S \subset O$ such that $x \notin S$, the following holds: $\{x\} \succ \emptyset$ if and only if $S \cup \{x\} \succ S$ (Barberà et al. 2004). Informally, separability means that if an agent prefers having the object than having nothing, he would also prefer the inclusion of the object in any other set that does not include the object. Whereas responsive preferences are separable, separable preferences are more general than responsive preferences. Just as in (Brams et al. 2012), we will assume that all the objects are desirable. However, we will not use the restriction in (Brams et al. 2012) that preferences over objects do not admit ties.

Issues with the generation phase of the undercut procedure

In the generation phase of the undercut procedure (Algorithm 1), each agent sequentially picks up his maximal object if it is uncontested. Otherwise, each contested object goes into the *"contested pile"*. We argue that the generation phase of the undercut procedure (also referred to as the *generation phase* in Vetschera and Kilgour 2013a) has some drawbacks. *Firstly*, undercut may fail to identify an envy-free allocation because of the generation phase even if agents have responsive preferences. Let us consider the following preferences of agents 1 and 2: $a \succ_1 b \succ_1 c \succ_1 d$ and $b \succ_2 c \succ_2 d \succ_2 a$. If $\{a, d\} \sim_1 \{b, c\}$, we know that the assignment which allocated $\{a, d\}$ to agent 1 and $\{b, c\}$ to agent 2 is envy-free. However undercut fails to compute this assignment. The reason is that in the generation phase, agent 1 takes a and agent 2 takes b. After this the contested pile is $\{c, d\}$. The undercut procedure ends up in a deadlock in this contested pile because $MB_1(\{c, d\}) = MB_2(\{c, d\})$. *Secondly*, the undercut procedure may work for certain responsive preferences, but the generation phase prevents it from working for every separable preference. *Thirdly*, the generation phase uses sequential allocation. It is well-understood that sequential allocation is highly susceptible to manipulation if at least one agent has sufficient information about the other agent's preferences (Kohler and Chandrasekaran 1971; Vetschera and Kilgour 2013a, b). One point which goes in favour of the generation phase of the undercut procedure is that it decreases the size of the contested pile which was presumably the motivation behind the phase.

Simplified undercut procedure

Next, we show that the simplified undercut procedure works for transitive and separable preferences. We define a *simplified undercut* procedure as follows.

> *Simplified Undercut:* Treat the set of all objects O as the contested pile and run the original undercut procedure while ignoring the generation phase of the original undercut procedure.

Proposition 1 *For transitive and separable preferences, there is a non-trivial envy-free allocation of O if and only if the set of minimal bundles of both agents with respect to O is not the same. Furthermore the simplified undercut procedure finds such an allocation.*

Proof The argument is similar to the one for the proof of (Brams et al. 2012, Theorem 1). If a trivial envy-free allocation exists, the simplified undercut procedure will find it since it considers the minimal bundles of the two agents. If a trivial envy-free allocation does not exist but a non-trivial one does, we show that the simplified undercut procedure will find it. Throughout the proof, for any set $S \subset O$, we will refer to $O \setminus S$ by $-S$. We first prove that if a non-trivial envy-free allocation of the set of objects O exists, it implies that the set of minimal bundles with respect to O of both agents is not the same. Let us assume that a non-trivial envy-free allocation $(S, -S)$ exists. Then there must be an agent $i \in \{1, 2\}$ such that $S \succ_i -S$. By the definition of minimal bundle, we know that $-S \notin MB_i(O)$. Without loss of generality, we can assume that $S \in MB_i(O)$. If it were not then we argue that there exists an S' such that $S' \in MB_i(O)$, $S' \subset S$, such that $S' \succsim_i -S'$. For an $S' \subset S$, by separability, we know that $S \succ_i S'$. Similarly, by separability, we know that $-S' \succ_i -S$ because $-S'$ can be obtained from $-S$ by adding to $-S$ the elements that are removed from S to obtain S'. Since $\succsim_i$ is transitive and complete, there exists some S' such that $S' \succsim_i -S'$ for which there exists no subset $S'' \subset S'$ such that $S'' \succsim_i -S''$.

Now if $S' \sim_i -S'$, then we know that $S' \in MB_i(O)$. By separability, we also know that $-S' \succ_{-i} -S \succsim_{-i} S \succ_{-i} S'$. Hence $S' \notin MB_{-i}(O)$ which means that set of minimal bundles of both agents is not the same. If $S \notin MB_{-i}(O)$, then we have already proved that the set of minimal bundles of the two agents are different. Now let us assume that $S \in MB_{-i}(O)$. Then we know that $S \succsim_{-i} -S$. Since $(S, -S)$ is envy-free, then it follows that $-S \succsim_{-i} S$. Hence $S \sim_{-i} -S$. If $-S \in MB_{-i}(O)$ then we are already done. We show that $-S$ is indeed in $MB_{-i}(O)$. Consider any $T \subset -S$ which implies by separability that $-S \succ_{-i} T$. Since $S = O \setminus -S$ and since $T \subset -S$, we know that $O \setminus T = -T \supset S$. This implies by separability that $-T \succ_{-i} S$. Since $-T \succ_{-i} S$, $-S \succ_{-i} T$ and $S \sim_{-i} -S$, we get by transitivity that $-T \succ_{-i} T$. Hence we have shown that $-S \in MB_{-i}(O)$. Since we know that $-S \notin MB_i(O)$, the set of minimal bundles of both agents is not the same.

We now prove that if the two agents do not have the same set of minimal bundles then there exists a non-trivial envy-free allocation. Let us assume that the two agents do not have the same set of minimal bundles i.e., there exists an $S \subset O$ such that $S \in MB_i(O)$ and $S \notin MB_{-i}(O)$. If $-S \succsim_{-i} S$, then $(S, -S)$ is an envy-free allocation. When agent i will propose $(S, -S)$, agent $-i$ will accept it. Therefore, let us look at the other case when $S \succ_{-i} -S$. If $S \succ_{-i} -S$ and $S \notin MB_{-i}(O)$, then by the definition of a minimal bundle we know that there exists a $T \subset S$ such that $T \succsim_{-i} -T$. Agent $-i$ will undercut the proposal $(S, -S)$ of agent i and will be ready to take T. For agent i, we know that $S \succ_i -T \succ_i T \succ_i -S$. Thus $(-T, T)$ is a non-trivial envy-free allocation. □

3 Discussion

We have shown that the undercut procedure of Brams et al. (2012) works under a weaker set of preference restrictions if we do away with the generation phase of the procedure. We note that the simplified undercut procedure need not ask the agents to submit their complete list minimal bundles initially but may elicit them gradually and stop whenever an envy-free allocation has been found.

A drawback of the simplified undercut procedure is that the number of minimal bundles with respect to O can be considerably more than the number of minimal bundles with respect to I_C. Brams et al. (2012) note that if all rankings over objects are equiprobable, the expected size of the contested pile increases without limit as the number of objects to be divided increases, but the increase is slow. However, it is implausible that agents do not have some degree of correlation over which objects they find more valuable. Correlation of preferences is an issue in various two-sided markets (Mongell and Roth 1991; Abdulkadiroğlu et al. 2005). If the preferences over objects have high degree of correlation, one cannot expect in general to obtain small contested piles (Vetschera and Kilgour 2013b). An extreme manifestation of this is when both agents have identical preferences over individual objects in which case the generation phase fails to decrease the size of the contested pile. Finally, even if the contested pile is much smaller after the generation phase, it can be possible that

$|MB_i(I_c)| > |MB_i(O)|$. Since the running time of the undercut and the simplified undercut procedures crucially depends on number of minimal bundles with respect to O and I_C, reducing the contested pile itself may not necessarily gain computational benefit. An interesting direction of future work is an experimental analysis of how much impact the generation phase can have on the number of minimal bundles of the contested pile.

Finally, we point out the that the simplified undercut procedure works if agents have unequal claims. First, we denote by $u_i(T)$ the utility that agent i has for allocation T. If agents have unequal claims say claim c_i for agent i, then the definition of envy-freeness can be easily extended as follows: $u_i(S) \geq \frac{c_i}{c_{-i}} u_i(-S)$ for an allocation where i gets S (Brams and Taylor 1996, Sect. 7.6). If agents have unequal claims, the undercut procedure still works as follows. We simply redefine a minimal bundle S for agent i as a set of objects such that $u_i(S) \geq \frac{c_i}{c_{-i}} u_i(-S)$ and for any $T \subset S$, $u_i(T) < \frac{c_i}{c_{-i}} u_i(-T)$.

Acknowledgments A preliminary version of this paper was accepted as an extended abstract at *AAMAS 2014 (13th International Conference on Autonomous Agents and Multiagent Systems)*. The author thanks Steven Brams, Christian Klamler, and the anonymous reviewers of *AAMAS 2014* as well as *Social Choice and Welfare* for their helpful comments. He also thanks Toby Walsh for pointing out the article by Brams et al. (2012). This material is based upon work supported by the Australian Government's Department of Broadband, Communications and the Digital Economy, the Australian Research Council, the Asian Office of Aerospace Research and Development through grant AOARD-124056.

References

Abdulkadiroğlu A, Pathak P, Roth AE (2005) The new york city high school match. Am Econ Rev 95(2):364–367

Aziz H, Gaspers S, Mackenzie S, Walsh T (2014) Fair assignment of indivisible objects under ordinal preferences. In Proceedings of the 13th international conference on autonomous agents and multi-agent systems (AAMAS), 1305–1312

Barberà S, Bossert W, Pattanaik PK (2004) Ranking sets of objects. In: Barberà S, Hammond PJ, Seidl C (eds) Handbook of utility theory, vol 17. Kluwer Academic Publishers, Boston, pp 893–977

Brams SJ, Taylor AD (1996) Fair division: from cake-cutting to dispute resolution. Cambridge University Press, Cambridge

Brams SJ, Kilgour DM, Klamler C (2012) The undercut procedure: an algorithm for the envy-free division of indivisible items. Soc Choice Welf 39:615–631

Chevaleyre Y, Dunne PE, Endriss U, Lang J, Lemaître M, Maudet N, Padget J, Phelps S, Rodríguez-Aguilar JA, Sousa P (2006) Issues in multiagent resource allocation. Informatica 30:3–31

Kohler DA, Chandrasekaran R (1971) A class of sequential games. Oper Res 19(2):270–277

Mongell S, Roth AE (1991) Sorority rush as a two-sided matching mechanism. Am Econ Rev 81:441–464

Vetschera R, Kilgour DM (2013a) Strategic behavior in contested-pile methods for fair division of indivisible items. Group Decis Negotiat 22:299–319

Vetschera R, Kilgour DM (2013b) Fair division of indivisible items between two players: design parameters for contested pile methods. Theory Decis 76:547–572