International Symposium in Approximation Theory
Vanderbilt University
Nashville, May 2011

CHROMATIC DERIVATIVES AND EXPANSIONS

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Towards local signal representation

- Let $f \in \mathbf{BL}(\pi)$, i.e., $f \in L^2$ with $\widehat{f(\omega)}$ supported on $[-\pi, \pi]$

Shannon's Expansion:

(Whittaker–Kotelnikov–Nyquist–Shannon)

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \, \frac{\sin \pi(t-n)}{\pi(t-n)}$$

- **global in nature** – requires samples $f(n)$ for all n;
- **fundamental** to signal processing;
- **poor local signal representation**

Taylor's Expansion:

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \, \frac{t^n}{n!}$$

- **local in nature** – requires $f^{(n)}(t)$ at a single instant $t = 0$.
- **very little use** in signal processing

Why do we need local approximations?

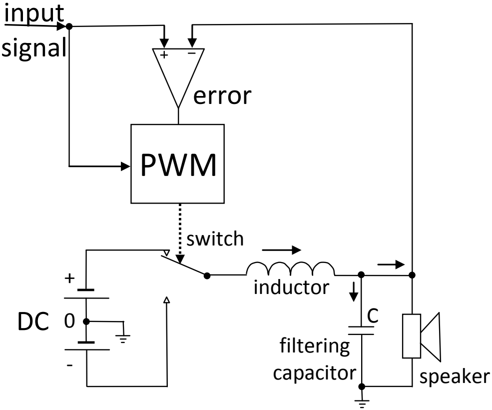
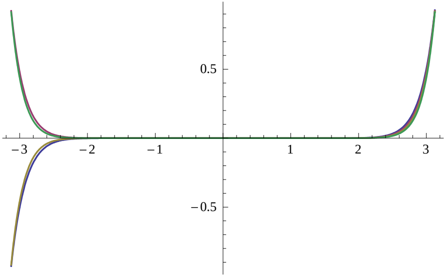


Figure: A switching amplifier

Problems with the Taylor formula:

- derivatives of high order obliterate the spectrum:
  if $f \in \mathbf{BL}(\pi)$ then

$$\frac{f^{(n)}(t)}{\pi^n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}^n \left(\frac{\omega}{\pi}\right)^n \widehat{f(\omega)} e^{\mathrm{i}\,\omega t} d\omega.$$



plots of $\left(\frac{\omega}{\pi}\right)^n$

$(n = 15 - 18)$

- monomials $\dfrac{t^n}{n!}$ do not belong to $\mathbf{BL}(\pi)$.

An orthogonal base for the space of linear differential operators

- Start with normalized and re-scaled Legendre polynomials:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega)P_m^L(\omega)\mathrm{d}\omega = \delta(m-n).$$

- Obtain operator polynomials by replacing ω^k with $\mathrm{i}^k\, d^k/dt^k$:

$$\mathcal{K}_t^n = (-\mathrm{i})^n P_n^L\left(\mathrm{i}\ \frac{d}{dt}\right)$$

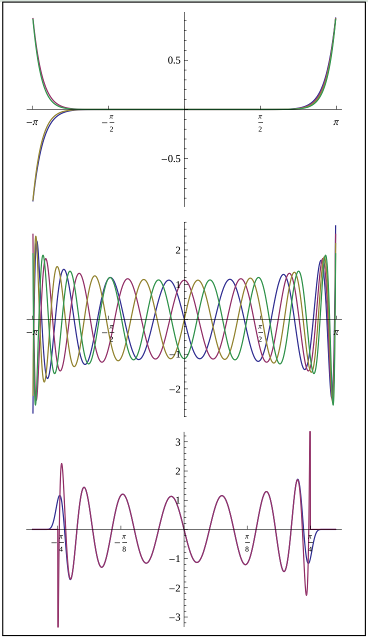
- Definition of $\mathcal{K}^n$ chosen so that

$$\mathcal{K}_t^n[e^{\mathrm{i}\,\omega t}] = \mathrm{i}^n P_n^L(\omega)\, e^{\mathrm{i}\,\omega t}.$$

- Thus, for $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}^n P_n^L(\omega)\widehat{f(\omega)}\, e^{\mathrm{i}\,\omega t} d\omega.$$

Why are chromatic derivatives a better base?



- Compare the graphs of the transfer functions of $1/\pi^n d^n/dt^n$, i.e., $(\omega/\pi)^n$ (first graph) and of $\mathcal{K}^n$, i.e., $P^L_n(\omega)$ (second graph).

- Transfer functions of $\mathcal{K}^n$ form a sequence of **well separated comb filters** which **preserve spectral features** of the signal, thus we call them the **chromatic derivatives**.

- Third graph: transfer function of the ideal filter $\mathcal{K}^{15}$ (red) vs. transfer function of a transversal filter (blue), (128 taps, 2× oversampling.)

Local representation of the scalar product

Proposition: Assume that $f, g \in \mathbf{BL}(\pi)$; then the sums on the left hand side of the following equations do not depend on the choice of the instant t, and

$$\sum_{n=0}^{\infty} K^n[f](t)^2 = \int_{-\infty}^{\infty} f(x)^2 dx = \|f\|^2$$

$$\sum_{n=0}^{\infty} K^n[f](t)\overline{K^n[g](t)} = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx = \langle f, g\rangle$$

$$\sum_{n=0}^{\infty} K^n[f](t) K^n_t[g(u-t)] = \int_{-\infty}^{\infty} f(x) g(u-x) dx = (f * g)(u)$$

- These are the **local equivalents** of the usual, “globally defined” norm, scalar product and convolution!

- **Aim:** **“maximally localized”** signal processing, suitable for real-time applications and transient analysis.

Chromatic expansions

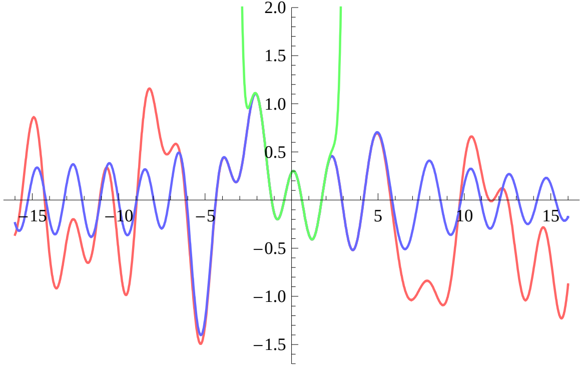
Proposition: Let $\mathrm{sinc}\,(t) = \dfrac{\sin(\pi t)}{\pi t}$ and let $f(t)$ be **any entire function**. Then,

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\ \mathcal{K}^n[\mathrm{sinc}\,(t)] \\ &= \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\ \sqrt{2n+1}\, j_n(\pi t) \end{aligned}$$

$j_n(t)$ — the spherical Bessel functions of the first kind

- **The truncations of the series belong to BL**(π).
- **If** $f \in \mathbf{BL}(\pi)$ **the series converges to** $f(t)$ **both uniformly and in** L^2.

Chromatic versus Taylor's approximation



- **red:** the signal; **blue:** the chromatic approximation of order 15; **green:** the Taylor approximation of order 15.
- Great for extremely robust regularized least square fit applications - efficiently removes out of band noise.

Applications:

- signal interpolation/extrapolation;
- in-painting (with a 2D version)???
- frequency estimation;
- transient classification according to what type of differential equation they satisfy.

CONJECTURE:

One can obtain a useful classification of transients via the minimal degree linear differential equation (with slow time varying coefficients) which the transient satisfies over its relatively short duration. Such differential equation is a local replacement for the spectrum of the "steady state" signals of longer duration which optimally avoids effects of windowing.

General families of chromatic derivatives

- Given a family of orthonormal polynomials $P_n(\omega)$ we can always define differential operators

$$\mathcal{K}_t^n = (-\mathrm{i})^n \; P_n^{\scriptscriptstyle L}\left(\mathrm{i} \; \frac{d}{dt}\right)$$

Question:

What are the families of orthogonal polynomials such that for the corresponding differential operators K^n and some associated function $\boldsymbol{m}(t)$ we have

$$f(t) = \sum_{n=0}^{\infty} (-1)^n K^n[f](u) K^n[\boldsymbol{m}](t-u)$$

for important classes of functions, and when is the convergence uniform?

Legendre Polynomials/Spherical Bessel functions

- For the (normalized) **Legendre polynomials**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) \mathrm{d}\omega = \delta(m-n)$$

and for

$$\boldsymbol{m}(t) = \frac{\sin(\pi t)}{\pi t}$$

we have

$$\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{2n+1}\, j_n(\pi t)$$

and

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t)$$

holds for all entire functions;

- The convergence is uniform for functions in $\mathbf{BL}(\pi)$

Chebyshev polynomials /Bessel functions

- For the (normalized) **Chebyshev polynomials** of the first kind:

$$\int_{-\pi}^{\pi} \frac{P^T_n(\omega)P^T_m(\omega)}{\pi^2\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}}\mathrm{d}\omega = \delta(n-m).$$

for $\boldsymbol{m}(t) = J_0(\pi t)$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2}\,J_n(\pi t)$ and

$$f(t) = f(u)J_0(\pi t) + \sqrt{2}\,\sum_{n=1}^{\infty}\mathcal{K}^n[f](0)J_n(\pi t)$$

- The Neumann series - converges for all entire functions;

- Convergence uniform for band limited functions which satisfy

$$\int_{-\pi}^{\pi}|\widehat{f}(\omega)|^2\sqrt{1-(\omega/\pi)^2}\,\mathrm{d}\omega < \infty$$

Hermite polynomials/Gaussian monomials

- For the (normalized) **Hermite polynomials**

$$\int_{-\infty}^{\infty} P_n^H(\omega) P_m^H(\omega)\, \frac{\mathrm{e}^{-\omega^2}}{\sqrt{\pi}}\, \mathrm{d}\omega = \delta(n-m)$$

and $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \frac{t^n}{\sqrt{2^n\, n!}}\, \mathrm{e}^{-t^2/4}$

- chromatic expansion converges locally for entire functions which satisfy

$$\limsup_{n\to\infty} \frac{|f^{(n)}(z)|^{1/n}}{\sqrt{n}} < \infty$$

- converges uniformly for all entire functions s.t.

$$\int_{-\infty}^{\infty} |\widehat{f(\omega)}|^2\, \mathrm{e}^{\omega^2} \mathrm{d}\omega < \infty$$

The hyperbolic family

Let $L_n(\omega)$ satisfy

$$\frac{1}{2}\int_{-\infty}^{\infty} L_n(\omega)\ L_m(\omega)\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\mathrm{d}\omega = \delta(m-n)$$

and $\boldsymbol{m}(z) = \mathrm{sech}(z)$; then $\mathcal{K}^n[\boldsymbol{m}](z) = (-1)^n\,\mathrm{sech}(z)\mathrm{tanh}^n(z)$ and

$$f(z) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\,\mathrm{sech}(z)\mathrm{tanh}^n(z)$$

converges uniformly inside the strip $|\mathcal{I}m(z)| < \pi/2$ if

$$\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\cosh(\omega)\mathrm{d}\omega < \infty$$

General families of chromatic derivatives

Definition: A family of polynomials $P_n(\omega)$ which is orthonormal with respect to a non-decreasing bounded moment distribution function $a(\omega)$:

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) \mathrm{d}a(\omega)$$

is **chromatic** if the moments μ_n of $a(\omega)$,

$$\mu_n = \int_{-\infty}^{\infty} \omega^n \mathrm{d}a(\omega)$$

satisfy

$$\rho = \limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} < \infty$$

Lemma: $P_n(\omega)$ are chromatic if and only if for every $0 \leq \alpha < \rho$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|} \mathrm{d}a(\omega) < \infty$$

General families of chromatic derivatives

If $P_n(\omega)$ is a chromatic family of polynomials orthonormal with respect to $a(\omega)$, then:

- The Fourier transform of $\mathrm{d}a(\omega)$,

  $$\boldsymbol{m}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega)$$

  is analytic on the strip $S_\rho = \{z : |\mathrm{Im(z)}| < \rho\}$.

- $P_n(\omega)$ are a complete base of the space

  $$L^2_{a(\omega)} = \{\phi(\omega) \ : \ \int_{-\infty}^{\infty} |\phi(\omega)|^2 \mathrm{d}a(\omega) < \infty\}.$$

Definition: Λ^2 is the space of functions $f(t)$ analytic on $S_{\rho/2}$ such that

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty.$$

Theorem: A function $f(z)$ is in Λ^2 if and only if there exists a function $\phi_f(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi_f(\omega) \mathrm{e}^{\mathrm{i}\,\omega z} \mathrm{d}a(\omega)$$

in which case

$$\phi_f(\omega) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) P_n(\omega)$$

Corollary: Thus, for all $t \in \mathbb{R}$,

$$\|f\|_{\Lambda}^2 = \|\phi_f(\omega)\|^2_{a(\omega)} = \sum_{n=0}^{\infty} |\mathcal{K}^n[\boldsymbol{m}](0)|^2 = \sum_{n=0}^{\infty} |\mathcal{K}^n[\boldsymbol{m}](t)|^2$$

General families of chromatic derivatives

Theorem: If $f(z) \in \Lambda^2$, then

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0) \mathcal{K}^n[\boldsymbol{m}](t)$$

with the series converging uniformly on strips $S_{\rho/2-\epsilon}$.

But how about the **local (non-uniform) convergence** of the chromatic series??

For example, in the case of the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions of the first kind $J_n(\omega)$, we know that the chromatic series is just the Newmann series, and that the above equality holds **for every entire function** $f(z)$.

Weakly bounded families

A family of polynomials is orthonormal with respect to a moment distribution function $a(\omega)$ with all odd moments $\mu_{2n+1} = 0$ if and only if there exist $\gamma_n > 0$ such that

$$P_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}(\omega).$$

Definition: Such family of polynomials $P_n(\omega)$ is:

- **bounded** if for some M and all n we have $\frac{1}{M} \leq \gamma_n \leq M$.
- **weakly bounded** if for some $0 \leq p < 1$ we have

$$\frac{1}{M} < \gamma_n < M\,n^p \quad \text{and} \quad \frac{\gamma_n}{\gamma_{n+1}} < M$$

- Bounded families are also weakly bounded with $p = 0$.

Examples:

- **Bounded families** $(p = 0)$:
  - **Legendre** polynomials: $\gamma_n = \frac{\pi(n+1)}{\sqrt{4(n+1)^2-1}} \to \frac{\pi}{2}$
  - **Chebyshev** polynomials: $\gamma_0 = \frac{\pi}{\sqrt{2}}$ and $\gamma_{n+1} = \frac{\pi}{2}$
- **Weakly bounded family** $(p = 1/2)$:
  - **Hermite** polynomials: $\gamma_n = \sqrt{(n+1)/2}$;
- **Non - weakly bounded family** $(p = 1)$:
  - **Hyperbolic** family: $\gamma_n = n + 1$;
- This shows that if we want $\boldsymbol{m}(z)$ to be entire, then the bound $p < 1$ is sharp.

Lemma: Every weakly bounded family of orthonormal polynomials is also chromatic.

Theorem: Let $\{P_n(\omega)\}_{n\in\mathbb{N}}$ be a weakly bounded family and let $f(z)$ be an entire function. If

$$\lim_{n\to\infty}\left|\frac{f^{(n)}(0)}{n!^{1-p}}\right|^{1/n}=0$$

then for every $z\in\mathbb{C}$

$$f(z)=\sum_{j=0}^{\infty}(-1)^j\mathcal{K}^j[f](0)\,\mathcal{K}^j[\boldsymbol{m}](z).$$

The convergence is uniform on every disc of finite radius.

Corollary: If $\mathcal{M}$ is bounded then the chromatic expansion of every entire function $f(z)$ point-wise converges to $f(z)$ for all z.

- It turns out that many of the classical formulas such as

$$
\begin{gathered}
\mathrm{e}^{\mathrm{i}\,\omega t} = \textstyle\sum_{n=0}^{\infty} \mathrm{i}^n\, T_n(\omega) J_n(t) \\
J_0(t+u) = J_0(u) J_0(t) + 2 \textstyle\sum_{n=1}^{\infty} (-1)^n J_n(u) J_n(t) \\
J_0(t)^2 + 2 \textstyle\sum_{k=1}^{\infty} J_n(t)^2 = 1 \\
J_0(z) + 2 \textstyle\sum_{n=1}^{\infty} J_{2n}(z) = 1
\end{gathered}
$$

are special cases of chromatic expansions valid **for all weakly bounded families** of polynomials and their associated $\boldsymbol{m}(z)$:

$$
\begin{gathered}
\mathrm{e}^{\mathrm{i}\,\omega t} = \textstyle\sum_{n=0}^{\infty} \mathrm{i}^n\, P_n(\omega) \mathcal{K}^n[\boldsymbol{m}](t) \\
\boldsymbol{m}(t+u) = \textstyle\sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[\boldsymbol{m}](u) \mathcal{K}^n[\boldsymbol{m}](t) \\
\textstyle\sum_{k=1}^{\infty} \mathcal{K}^n[\boldsymbol{m}](t)^2 = 1 \\
\boldsymbol{m}(z) + \textstyle\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}} \right) \mathcal{K}^{2n}[\boldsymbol{m}](z) = 1
\end{gathered}
$$

Theorem: Assume $\mathcal{M}$ is weakly bounded and let k be an integer such that $k \geq 1/(1-p)$; then:

(a) there exists $K > 0$ such that

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{|Kz|^n}{n!^{1-p}}\,\mathrm{e}^{|Kz|^k};$$

(b) for every $f(t) \in \Lambda_2$ there exists $C, L > 0$ such that

$$|f(z)| \leq C\mathrm{e}^{L|z|^k}.$$

A generalization of the Paley - Wienner Theorem?? (Conjecture)

Assume that $f(z)$ is an entire function for which there exist a symmetric moment distribution function $a(\omega)$ and a function $\varphi(\omega) \in L^2_{a(\omega)}$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi(\omega) \mathrm{e}^{\mathrm{i}\, z\omega} \mathrm{d}a(\omega),$$

and let $k \geq 1$ be an integer. The following are equivalent:

(c) there exist $C, L > 0$ such that

$$|f(z)| < C\mathrm{e}^{L|z|^k}, \quad (z \in \mathbb{C});$$

(d) $a(\omega)$ can be chosen such that the corresponding γ_n satisfy $\gamma_n < Mn^p$ for some $0 \leq p \leq 1 - 1/k$.

Some more open questions:

Qestion: Is it possible to characterize weakly bounded families purely in terms of the properties of the corresponding $a(\omega)$?

Qestion: If not, is it possible to characterize functionals $\mathcal{M}$ for which

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|} \mathrm{d}a(\omega) < \infty$$

purely in terms of the asymptotic behavior of the recursion coefficients γ_n of the corresponding family of orthonormal polynomials?

Periodic functions

- Trigonometric functions do not belong to the spaces Λ_2:

$$\|\mathrm{e}^{\mathrm{i}\omega t}\|_{\Lambda}^2 = \sum_{n=0}^{\infty} |\mathcal{K}^n[\mathrm{e}^{\mathrm{i}\omega t}]|^2 = \sum_{n=0}^{\infty} P_n(\omega)^2 \to \infty$$

Definition: Assume that $\liminf_{n\to\infty} \gamma_n > 0$ and that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\gamma_n} = \infty$

We denote by $\mathcal{C}$ the vector space of entire functions such that the sequence

$$\nu_n^f(t) = \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_n}}$$

converges uniformly on every finite interval.

Definition: Let $\mathcal{C}_0 = \{f \in \mathcal{C} \ : \ \lim_{n\to\infty} \nu_n^f(t) = 0\}$ and $\mathcal{C}_2 = \mathcal{C}/\mathcal{C}_0$.

Theorem: Let $f, g \in \mathcal{C}$ and

$$\sigma_n^{fg}(t) = \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)}{\sum_{k=0}^{n} \frac{1}{\gamma_n}};$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function.

Definition: For $f, g \in \mathcal{C}$ we define

$$\langle f, g\rangle = \frac{\sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)}{\sum_{k=0}^{n} \frac{1}{\gamma_n}}$$

- Do the trigonometric functions belong to $\mathcal{C}_2$?

$$\frac{\sum_{k=0}^{n} |\mathcal{K}^k[\mathrm{e}^{\mathrm{i}\omega t}]|^2}{\sum_{k=0}^{n} \frac{1}{\gamma_n}} = \frac{\sum_{k=0}^{n} P_n(\omega)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_n}}$$

- **Chebyshev polynomials**: $(p = 0)$

for all $-\pi < \omega < \pi$,

$$\|\mathrm{e}^{\mathrm{i}\omega t}\| = \frac{\sum_{k=0}^{n} P_n(\omega)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_n}} \sim \frac{1}{n+1} \sum_{k=0}^{n} P_n^T(\omega)^2 \to 1$$

- **Hermite polynomials**: $(p = 1/2)$

for all ω,

$$\|\mathrm{e}^{\mathrm{i}\omega t}\| = \frac{\sum_{k=0}^{n} P_n(\omega)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_n}} \sim \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^H(\omega)^2 \to \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\omega^2}$$

Conjecture: (in my 2009 EJA paper) Let

$$P_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}(\omega),$$

and assume that for some $0 \leq p < 1$ the recursion coefficients γ_n satisfy

$$0 < \lim_{n\to\infty} \frac{\gamma_n}{n^p} < \infty.$$

Then for the corresponding family of orthogonal polynomials we have

$$0 < \lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_k(\omega)^2 < \infty$$

for all ω in the support $sp(a)$ of $a(\omega)$.

Revised Conjecture

Revised Conjecture: Assume that γ_n satisfy:

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_k} \quad \text{diverges;} \quad \liminf_{n\to\infty} \gamma_n > 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\gamma_n}{\gamma_{n+1}} = 1.$$

Let $s_n = \dfrac{\gamma_n}{\gamma_{n+1}} - 1$ and $d_n = s_n - s_{n+1}$.

If $s_n \in l_2$ and $d_n \in l_1$, then for all ω in the support of $a(\omega)$,

$$0 < \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P_k(\omega)^2}{\sum_{k=0}^{n} \frac{1}{\gamma_k}} < \infty$$

(I believe that) I have just proved this for cases when γ_n grow faster than $n^{1/4}$ …

Thank you!