

Chromatic Derivatives, Chromatic Expansions and Associated Spaces
(Extended Abstract)

Aleksandar Ignjatović

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Let $\mathbf{BL}(\pi)$ be the space of continuous L^2 functions with the Fourier transform supported within $[-\pi, \pi]$ (i.e., the space of π band limited signals of finite energy), and let $P_n^L(\omega)$ be obtained by normalizing and scaling the Legendre polynomials, so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) d\omega = \delta(m - n).$$

We consider linear differential operators $\mathcal{K}^n = (-i)^n P_n^L \left(i \frac{d}{dt} \right)$; for such operators and every $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} i^n P_n^L(\omega) \widehat{f}(\omega) e^{i\omega t} d\omega.$$

We show that for $f \in \mathbf{BL}(\pi)$ the values of $\mathcal{K}^n[f](t)$ can be obtained in a numerically accurate and noise robust way from samples of $f(t)$, even for differential operators \mathcal{K}^n of high order.

Operators \mathcal{K}^n have the following remarkable properties, relevant for applications in digital signal processing.

Proposition 0.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a restriction of any entire function; then the following are equivalent:*

- (a) $\sum_{n=0}^{\infty} \mathcal{K}^n[f](0)^2 < \infty$;
- (b) for all $t \in \mathbb{R}$ the sum $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ converges, and its values are independent of $t \in \mathbb{R}$;
- (c) $f \in \mathbf{BL}(\pi)$.

Moreover, the following Proposition provides **local representation** of the usual norm, the scalar product and the convolution in $\mathbf{BL}(\pi)$.

Proposition 0.2. *For all $f, g \in \mathbf{BL}(\pi)$ the following sums do not depend on $t \in \mathbb{R}$, and*

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2 &= \int_{-\infty}^{\infty} f(x)^2 dx; \\ \sum_{n=0}^{\infty} \mathcal{K}^n[f](t) \mathcal{K}^n[g](t) &= \int_{-\infty}^{\infty} f(x)g(x) dx; \\ \sum_{n=0}^{\infty} \mathcal{K}^n[f](t) \mathcal{K}_t^n[g(u-t)] &= \int_{-\infty}^{\infty} f(x)g(u-x) dx. \end{aligned}$$

The following proposition provides a form of Taylor's theorem, with the differential operators \mathcal{K}^n replacing the derivatives and the spherical Bessel functions replacing the monomials.

Proposition 0.3. *Let j_n be the spherical Bessel functions of the first kind; then:*

- (1) for every entire function f and for all $z \in \mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0) \mathcal{K}^n[j_0(\pi z)] = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) \sqrt{2n+1} j_n(\pi z);$$

- (2) if $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on \mathbb{R} and in L^2 .

We give analogues of the above theorems for very general families of orthogonal polynomials. We also introduce some nonseparable inner product spaces. In one of them, related to the Hermite polynomials, functions $f_{\omega}(t) = \sin \omega t$ for all $\omega > 0$ have finite positive norms and for **every** two distinct values $\omega_1 \neq \omega_2$ the corresponding functions $f_{\omega_1}(t) = \sin \omega_1 t$ and $f_{\omega_2}(t) = \sin \omega_2 t$ are mutually orthogonal. Related to the properties of such spaces, we also make the following conjecture for families of orthonormal polynomials.

Conjecture 0.4. *Let $P_n(\omega)$ be a family of symmetric positive definite orthonormal polynomials corresponding to a moment distribution function $a(\omega)$,*

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) da(\omega) = \delta(m - n),$$

and let $\gamma_n > 0$ be the recursion coefficients in the corresponding three term recurrence relation for such **orthonormal** polynomials, i.e., such that

$$P_{n+1}(\omega) = \frac{\omega}{\gamma_n} P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n} P_{n-1}(\omega).$$

If γ_n satisfy $0 < \lim_{n \rightarrow \infty} \frac{\gamma_n}{n^p} < \infty$ for some $0 \leq p < 1$, then $0 < \lim_{n \rightarrow \infty} \frac{1}{n^{1-p}} \sum_{k=0}^{n-1} P_k(\omega)^2 < \infty$ for all ω in the support $sp(a)$ of $a(\omega)$.

Numerical tests with $\gamma_n = n^p$ for many $p \in [0, 1)$ indicate that the conjecture is true.

CHROMATIC DERIVATIVES, CHROMATIC EXPANSIONS AND ASSOCIATED SPACES

ALEKSANDAR IGNJATOVIĆ

ABSTRACT. This paper presents the basic properties of chromatic derivatives and chromatic expansions and provides an appropriate motivation for introducing these notions. Chromatic derivatives are special, numerically robust linear differential operators which correspond to certain families of orthogonal polynomials. Chromatic expansions are series of the corresponding special functions, which possess the best features of both the Taylor and the Shannon expansions. This makes chromatic derivatives and chromatic expansions applicable in fields involving empirically sampled data, such as digital signal and image processing.

1. MOTIVATION

Signal processing mostly deals with the signals which can be represented by continuous L^2 functions whose Fourier transform is supported within $[-\pi, \pi]$; these functions form *the space* $\mathbf{BL}(\pi)$ *of* π *band limited signals of finite energy*. Foundations of classical digital signal processing rest on the Whittaker–Kotel’nikov–Nyquist–Shannon Sampling Theorem (for brevity the Shannon Theorem): every signal $f \in \mathbf{BL}(\pi)$ can be represented using its samples at integers and *the cardinal sine function* $\text{sinc } t = \sin \pi t / \pi t$, as

$$(1) \quad f(t) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(t - n).$$

Such signal representation is of *global nature*, because it involves samples of the signal at integers of arbitrarily large absolute value. In fact, since for a fixed t the values of $\text{sinc}(t - n)$ decrease slowly as $|n|$ grows, the truncations of the above series do not provide satisfactory local signal approximations.

On the other hand, since every signal $f \in \mathbf{BL}(\pi)$ is a restriction to \mathbb{R} of an entire function, it can also be represented by the Taylor series,

$$(2) \quad f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{t^n}{n!}.$$

Such a series converges uniformly on every finite interval, and its truncations provide good local signal approximations. Since the values of the derivatives $f^{(n)}(0)$ are determined by the values of the signal in an arbitrarily small neighborhood of zero,

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the Taylor expansion is of *local nature*. In this sense, the Shannon and the Taylor expansions are complementary.

However, unlike the Shannon expansion, the Taylor expansion has found very limited use in signal processing, due to several problems associated with its application to empirically sampled signals.

- (I) Numerical evaluation of higher order derivatives of a function given by its samples is very noise sensitive. In general, one is cautioned against numerical differentiation:

“... numerical differentiation should be avoided whenever possible, particularly when the data are empirical and subject to appreciable errors of observation” [10].

- (II) The Taylor expansion of a signal $f \in \mathbf{BL}(\pi)$ converges non-uniformly on \mathbb{R} ; its truncations have rapid error accumulation when moving away from the center of expansion and are unbounded.
- (III) Since the Shannon expansion of a signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$, the action of a continuous linear shift invariant operator (in signal processing terminology, a *filter*) A can be expressed using samples of f and the *impulse response* $A[\text{sinc}]$ of A :

$$(3) \quad A[f](t) = \sum_{n=-\infty}^{\infty} f(n) A[\text{sinc}](t - n).$$

In contrast, the polynomials obtained by truncating the Taylor series do not belong to $\mathbf{BL}(\pi)$ and nothing similar to (3) is true of the Taylor expansion.

Chromatic derivatives were introduced in [11] to overcome problem (I) above; the chromatic approximations were introduced in [14] to obtain local approximations of band-limited signals which do not suffer from problems (II) and (III).

1.1. Numerical differentiation of band limited signals. To understand the problem of numerical differentiation of band-limited signals, we consider an arbitrary $f \in \mathbf{BL}(\pi)$ and its Fourier transform $\widehat{f}(\omega)$; then

$$f^{(n)}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (i\omega)^n \widehat{f}(\omega) e^{i\omega t} d\omega.$$

Figure 1 (left) shows, for $n = 15$ to $n = 18$, the plots of $(\omega/\pi)^n$, which are, save a factor of i^n , the symbols, or, in signal processing terminology, the *transfer functions* of the normalized derivatives $1/\pi^n d^n/dt^n$. These plots reveal why there can be no practical method for any reasonable approximation of derivatives of higher orders. Multiplication of the Fourier transform of a signal by the transfer function of a normalized derivative of higher order obliterates the Fourier transform of the signal, leaving only its edges, which in practice contain mostly noise. Moreover, the graphs of the transfer functions of the normalized derivatives of high orders and of the same parity cluster so tightly together that they are essentially indistinguishable; see Figure 1 (left).¹

However, contrary to a common belief, these facts *do not* preclude numerical evaluation of all differential operators of higher orders, but only indicate that, from a numerical perspective, the set of the derivatives $\{f, f', f'', \dots\}$ is a very poor base

¹If the derivatives are not normalized, their values can be very large and are again determined essentially by the noise present at the edge of the bandwidth of the signal.

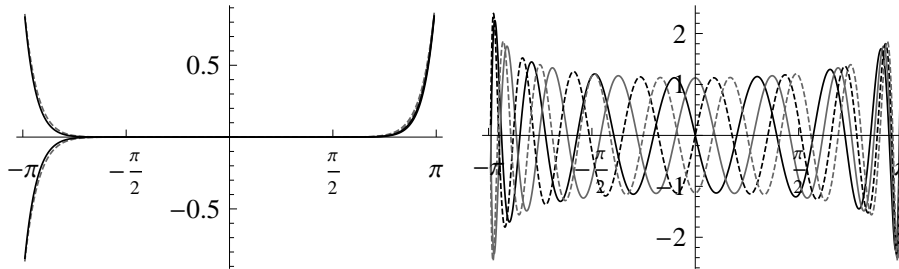


FIGURE 1. Graphs of $(\frac{\omega}{\pi})^n$ (left) and of $P_n^L(\omega)$ (right) for $n = 15 - 18$.

of the vector space of linear differential operators with real coefficients. We now show how to obtain a base for this space consisting of numerically robust linear differential operators.

1.2. Chromatic derivatives. Let polynomials $P_n^L(\omega)$ be obtained by normalizing and scaling the Legendre polynomials, so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) d\omega = \delta(m - n).$$

We define operator polynomials²

$$\mathcal{K}_t^n = (-i)^n P_n^L \left(i \frac{d}{dt} \right).$$

Since polynomials $P_n^L(\omega)$ contain only powers of the same parity as n , operators \mathcal{K}^n have real coefficients, and it is easy to verify that

$$\mathcal{K}_t^n [e^{i\omega t}] = i^n P_n^L(\omega) e^{i\omega t}.$$

Consequently, for $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^n [f](t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} i^n P_n^L(\omega) \widehat{f}(\omega) e^{i\omega t} d\omega.$$

In particular, one can show that

$$(4) \quad \mathcal{K}^n [\text{sinc}](t) = (-1)^n \sqrt{2n+1} j_n(\pi t),$$

where $j_n(x)$ is the spherical Bessel function of the first kind of order n . Figure 1 (right) shows the plots of $P_n^L(\omega)$, for $n = 15$ to $n = 18$, which are the transfer functions (again save a factor of i^n) of the corresponding operators \mathcal{K}^n . Unlike the transfer functions of the (normalized) derivatives $1/\pi^n d^n/dt^n$, the transfer functions of the chromatic derivatives \mathcal{K}^n form a family of well separated, interleaved and increasingly refined comb filters. Instead of obliterating, such operators encode the features of the Fourier transform of the signal (in signal processing terminology, the *spectral features* of the signal). For this reason, we call operators \mathcal{K}^n the *chromatic derivatives* associated with the Legendre polynomials.

Chromatic derivatives can be accurately and robustly evaluated from samples of the signal taken at a small multiple of the usual Nyquist rate. Figure 2 (left)

²Thus, obtaining \mathcal{K}_t^n involves replacing ω^k in $P_n^L(\omega)$ with $i^k d^k/dt^k$ for all $k \leq n$. If \mathcal{K}_t^n is applied to a function of a single variable, we drop index t in \mathcal{K}_t^n .

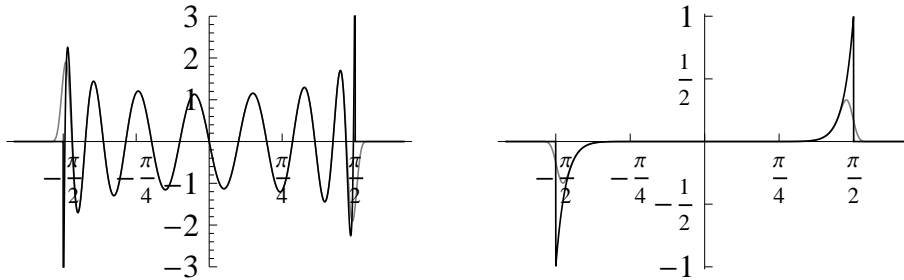


FIGURE 2. Transfer functions of \mathcal{K}^{15} (left, black) and d^{15}/dt^{15} (right, black) and of their transversal filter approximations (gray).

shows the plots of the transfer function of a *transversal filter* given by $\mathcal{T}_{15}[f](t) = \sum_{k=-64}^{64} c_k f(t+k/2)$ (gray), used to approximate the chromatic derivative $\mathcal{K}^{15}[f](t)$, and the transfer function of \mathcal{K}^{15} (black). The coefficients c_k of the filter were obtained using the Remez exchange method [16], and satisfy $|c_k| < 0.2$, ($-64 \leq k \leq 64$). The filter has 129 taps, spaced two taps per Nyquist rate interval, i.e., at a distance of $1/2$. Thus, the transfer function of the corresponding ideal filter \mathcal{K}^{15} is $P_{15}^L(2\omega)$ for $|\omega| \leq \pi/2$, and zero outside this interval. The pass-band of the actual transversal filter is 90% of the bandwidth $[-\pi/2, \pi/2]$. Outside the transition regions $[-11\pi/20, -9\pi/20]$ and $[9\pi/20, 11\pi/20]$ the error of approximation is less than 1.3×10^{-4} .

Implementations of filters for operators \mathcal{K}^n of orders $0 \leq n \leq 24$ have been tested in practice and proved to be both accurate and noise robust, as expected from the above considerations.

For comparison, Figure 2 (right) shows the transfer function of a transversal filter obtained by the same procedure and with the same bandwidth constraints, which approximates the (normalized) “standard” derivative $(2/\pi)^{15} d^{15}/dt^{15}$ (gray) and the transfer function of the ideal filter (black). The figure clearly indicates that such a transversal filter is of no practical use.

Note that (1) and (4) imply that

$$(5) \quad \mathcal{K}^k[f](t) = \sum_{n=-\infty}^{\infty} f(n) \mathcal{K}^k[\text{sinc}](t-n) = \sum_{n=-\infty}^{\infty} f(n) (-1)^k \sqrt{2k+1} j_k(\pi(t-n)).$$

However, in practice, the values of $\mathcal{K}^k[f](t)$, especially for larger values of k , *cannot* be obtained from the Nyquist rate samples using truncations of (5). This is due to the fact that functions $\mathcal{K}^k[\text{sinc}](t-n)$ decay very slowly as $|n|$ grows; see Figure 3 (left). Thus, to achieve any accuracy, such a truncation would need to contain an extremely large number of terms. On the other hand, this also means that signal information present in the values of the chromatic derivatives of a signal obtained by sampling an appropriate filterbank at an instant t is *not redundant* with information present in the Nyquist rate samples of the signal in any reasonably sized window around t , which is a fact suggesting that chromatic derivatives could enhance standard signal processing methods operating on Nyquist rate samples.

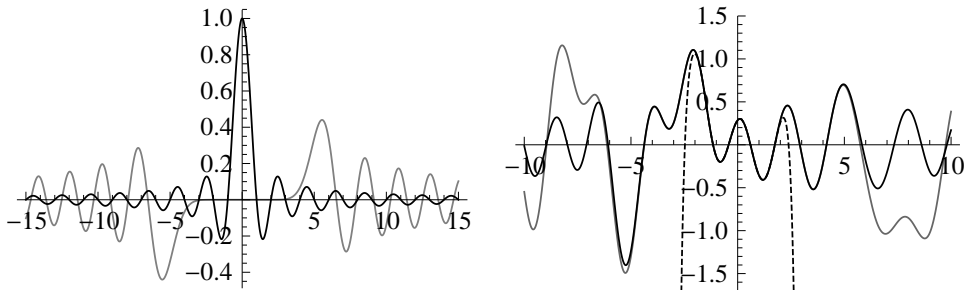


FIGURE 3. LEFT: Oscillatory behavior of $\text{sinc}(t)$ (black), and $\mathcal{K}^{15}[\text{sinc}](t)$ (gray); RIGHT: A signal $f \in \mathbf{BL}(\pi)$ (gray) and its chromatic and Taylor approximations (black, dashed)

1.3. Chromatic expansions. The above shows that numerical evaluation of the chromatic derivatives associated with the Legendre polynomials does not suffer problems which precludes numerical evaluation of the “standard” derivatives of higher orders. On the other hand, the chromatic expansions, defined in Proposition 1.1 below, were conceived as a solution to problems associated with the use of the Taylor expansion.³

Proposition 1.1. *Let \mathcal{K}^n be the chromatic derivatives associated with the Legendre polynomials, let j_n be the spherical Bessel function of the first kind of order n , and let f be an arbitrary entire function; then for all $z, u \in \mathbb{C}$,*

$$(6) \quad f(z) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u) \mathcal{K}_u^n[\text{sinc}(z-u)]$$

$$(7) \quad = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](u) \mathcal{K}^n[\text{sinc}](z-u)$$

$$(8) \quad = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u) \sqrt{2n+1} j_n(\pi(z-u))$$

If $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on \mathbb{R} and in the space $\mathbf{BL}(\pi)$.

The series in (6) is called *the chromatic expansion of f associated with the Legendre polynomials*; a truncation of this series is called a *chromatic approximation* of f . As the Taylor approximation, a chromatic approximation is also a local approximation; its coefficients are the values of differential operators $\mathcal{K}^m[f](u)$ at a single instant u , and for all $k \leq n$,

$$f^{(k)}(u) = \frac{d^k}{dt^k} \left[\sum_{m=0}^n \mathcal{K}^m[f](u) \mathcal{K}_u^m[\text{sinc}(t-u)] \right]_{t=u}.$$

Figure 3 (right) compares the behavior of the chromatic approximation (black) of a signal $f \in \mathbf{BL}(\pi)$ (gray) with the behavior of its Taylor approximation (dashed). Both approximations are of order 16. The signal $f(t)$ is defined using the Shannon

³Propositions stated in this Introduction are special cases of general propositions proved in subsequent sections.

expansion, with samples $\{f(n) : |f(n)| < 1, -32 \leq n \leq 32\}$ which were randomly generated. The plot reveals that, when approximating a signal $f \in \mathbf{BL}(\pi)$, a chromatic approximation has a much gentler error accumulation when moving away from the point of expansion than the Taylor approximation of the same order.

Unlike the monomials which appear in the Taylor formula, functions $\mathcal{K}^n[\text{sinc}](t) = (-1)^n \sqrt{2n+1} j_n(\pi t)$ belong to $\mathbf{BL}(\pi)$ and satisfy $|\mathcal{K}^n[\text{sinc}](t)| \leq 1$ for all $t \in \mathbb{R}$. Consequently, the chromatic approximations also belong to $\mathbf{BL}(\pi)$ and are bounded on \mathbb{R} .

Since by Proposition 1.1 the chromatic approximation of a signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$, if A is a filter, then A commutes with the differential operators \mathcal{K}^n and thus for every $f \in \mathbf{BL}(\pi)$,

$$(9) \quad A[f](t) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](u) \mathcal{K}^n[A[\text{sinc}]](t-u).$$

A comparison of (9) with (3) provides further evidence that, while local just like the Taylor expansion, the chromatic expansion associated with the Legendre polynomials possesses the features that make the Shannon expansion so useful in signal processing. This, together with numerical robustness of chromatic derivatives, makes chromatic approximations applicable in fields involving empirically sampled data, such as digital signal and image processing.

1.4. A local definition of the scalar product in $\mathbf{BL}(\pi)$. Proposition 1.2 below demonstrates another remarkable property of the chromatic derivatives associated with the Legendre polynomials.

Proposition 1.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a restriction of an arbitrary entire function; then the following are equivalent:*

- (a) $\sum_{n=0}^{\infty} \mathcal{K}^n[f](0)^2 < \infty$;
- (b) for all $t \in \mathbb{R}$ the sum $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ converges, and its values are independent of $t \in \mathbb{R}$;
- (c) $f \in \mathbf{BL}(\pi)$.

The next proposition is relevant for signal processing because it provides *local representations* of the usual norm, the scalar product and the convolution in $\mathbf{BL}(\pi)$, respectively, which are defined globally, as improper integrals.

Proposition 1.3. *Let \mathcal{K}^n be the chromatic derivatives associated with the (rescaled and normalized) Legendre polynomials, and $f, g \in \mathbf{BL}(\pi)$. Then the following sums do not depend on $t \in \mathbb{R}$ and satisfy*

$$(10) \quad \sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2 = \int_{-\infty}^{\infty} f(x)^2 dx;$$

$$(11) \quad \sum_{n=0}^{\infty} \mathcal{K}^n[f](t) \mathcal{K}^n[g](t) = \int_{-\infty}^{\infty} f(x)g(x)dx;$$

$$(12) \quad \sum_{n=0}^{\infty} \mathcal{K}^n[f](t) \mathcal{K}_t^n[g(u-t)] = \int_{-\infty}^{\infty} f(x)g(u-x)dx.$$

1.5. We finish this introduction by pointing to a close relationship between the Shannon expansion and the chromatic expansion associated with the Legendre polynomials. Firstly, by (7),

$$(13) \quad f(n) = \sum_{k=0}^{\infty} \mathcal{K}^k[f](0) (-1)^k \mathcal{K}^k[\text{sinc}](n).$$

Since $\mathcal{K}^n[\text{sinc}](t)$ is an even function for even n and odd for odd n , (5) implies

$$(14) \quad \mathcal{K}^k[f](0) = \sum_{n=-\infty}^{\infty} f(n) (-1)^k \mathcal{K}^k[\text{sinc}](n).$$

Equations (13) and (14) show that the coefficients of the Shannon expansion of a signal – the samples $f(n)$, and the coefficients of the chromatic expansion of the signal – the simultaneous samples of the chromatic derivatives $\mathcal{K}^n[f](0)$, are related by an orthonormal operator defined by the infinite matrix

$$\left[(-1)^k \mathcal{K}^k[\text{sinc}](n) : k \in \mathbb{N}, n \in \mathbb{Z} \right] = \left[\sqrt{2k+1} j_k(\pi n) : k \in \mathbb{N}, n \in \mathbb{Z} \right].$$

Secondly, let $\mathcal{S}_u[f(u)] = f(u+1)$ be the unit shift operator in the variable u (f might have other parameters). The Shannon expansion for the set of sampling points $\{u+n : n \in \mathbb{Z}\}$ can be written in a form analogous to the chromatic expansion, using operator polynomials $\mathcal{S}_u^n = \mathcal{S}_u \circ \dots \circ \mathcal{S}_u$, as

$$(15) \quad f(t) = \sum_{n=0}^{\infty} f(u+n) \text{sinc}(t - (u+n))$$

$$(16) \quad = \sum_{n=0}^{\infty} \mathcal{S}_u^n[f](u) \mathcal{S}_u^n[\text{sinc}(t-u)];$$

compare now (16) with (6). Note that the family of operator polynomials $\{\mathcal{S}_u^n\}_{n \in \mathbb{Z}}$ is also an orthonormal system, in the sense that their corresponding transfer functions $\{e^{i n \omega}\}_{n \in \mathbb{Z}}$ form an orthonormal system in $L^2[-\pi, \pi]$. Moreover, the transfer functions of the families of operators $\{\mathcal{K}^n\}_{n \in \mathbb{N}}$ and $\{\mathcal{S}^n\}_{n \in \mathbb{Z}}$, where \mathcal{K}^n are the chromatic derivatives associated with the Legendre polynomials, are orthogonal on $[-\pi, \pi]$ with respect to the same, constant weight $w(\omega) = 1/(2\pi)$.

In this paper we consider chromatic derivatives and chromatic expansions which correspond to some very general families of orthogonal polynomials, and prove generalizations of the above propositions, extending our previous work [12].⁴ However, having in mind the form of expansions (6) and (16), one can ask a more general (and somewhat vague) question.

Question 1. *What are the operators A for which there exists a family of operator polynomials $\{P_n(A)\}$, orthogonal under a suitably defined notion of orthogonality, such that for an associated function $\mathbf{m}_A(t)$,*

$$f(t) = \sum_n P_n(A)[f](u) P_n(A)[\mathbf{m}_A(t-u)]$$

for all functions from a corresponding (and significant) class \mathcal{C}_A ?

⁴Chromatic expansions corresponding to general families of orthogonal polynomials were first considered in [5]. However, Proposition 1 there is false; its attempted proof relies on an incorrect use of the Paley-Wiener Theorem. In fact, function F_g defined there need not be extendable to an entire function, as it can be shown using Example 4 in Section 4 of the present paper.

2. BASIC NOTIONS

2.1. Families of orthogonal polynomials. Let $\mathcal{M} : \mathcal{P}_\omega \rightarrow \mathbb{R}$ be a linear functional on the vector space \mathcal{P}_ω of real polynomials in the variable ω and let $\mu_n = \mathcal{M}(\omega^n)$. Such \mathcal{M} is a *moment functional*, μ_n is the *moment* of \mathcal{M} of order n and the *Hankel determinant* of order n is given by

$$\Delta_n = \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \mu_1 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots \\ \mu_n & \cdots & \mu_{2n} \end{vmatrix}.$$

The moment functionals \mathcal{M} which we consider are assumed to be:

- (i) positive definite, i.e., $\Delta_n > 0$ for all n ; such functionals also satisfy $\mu_{2n} > 0$;
- (ii) symmetric, i.e., $\mu_{2n+1} = 0$ for all n ;
- (iii) normalized, so that $\mathcal{M}(1) = \mu_0 = 1$.

For functionals \mathcal{M} which satisfy the above three conditions there exists a family $\{P_n^\mathcal{M}(\omega)\}_{n \in \mathbb{N}}$ of polynomials with real coefficients, such that

- (a) $\{P_n^\mathcal{M}(\omega)\}_{n \in \mathbb{N}}$ is an orthonormal system with respect to \mathcal{M} , i.e., for all m, n ,

$$\mathcal{M}(P_m^\mathcal{M}(\omega) P_n^\mathcal{M}(\omega)) = \delta(m - n);$$

- (b) each polynomial $P_n^\mathcal{M}(\omega)$ contains only powers of ω of the same parity as n ;
- (c) $P_0^\mathcal{M}(\omega) = 1$.

A family of polynomials is the family of orthonormal polynomials corresponding to a symmetric positive definite moment functional \mathcal{M} just in case there exists a sequence of reals $\gamma_n > 0$ such that for all $n > 0$,

$$(17) \quad P_{n+1}^\mathcal{M}(\omega) = \frac{\omega}{\gamma_n} P_n^\mathcal{M}(\omega) - \frac{\gamma_{n-1}}{\gamma_n} P_{n-1}^\mathcal{M}(\omega).$$

If we set $\gamma_{-1} = 1$ and $P_{-1}^\mathcal{M}(\omega) = 0$, then (17) holds for $n = 0$ as well.

We will make use of the Christoffel-Darboux equality for orthogonal polynomials,

$$(18) \quad (\omega - \sigma) \sum_{k=0}^n P_k^\mathcal{M}(\omega) P_k^\mathcal{M}(\sigma) = \gamma_n (P_{n+1}^\mathcal{M}(\omega) P_n^\mathcal{M}(\sigma) - P_{n+1}^\mathcal{M}(\sigma) P_n^\mathcal{M}(\omega)),$$

and of its consequences obtained by setting $\sigma = -\omega$ in (18) to get

$$(19) \quad \omega \left(\sum_{k=0}^n P_{2k+1}^\mathcal{M}(\omega)^2 - \sum_{k=0}^n P_{2k}^\mathcal{M}(\omega)^2 \right) = \gamma_{2n+1} P_{2n+2}^\mathcal{M}(\omega) P_{2n+1}^\mathcal{M}(\omega),$$

and by letting $\sigma \rightarrow \omega$ in (18) to get

$$(20) \quad \sum_{k=0}^n P_k^\mathcal{M}(\omega)^2 = \gamma_n (P_{n+1}^\mathcal{M}(\omega)' P_n^\mathcal{M}(\omega) - P_{n+1}^\mathcal{M}(\omega) P_n^\mathcal{M}(\omega)').$$

For every positive definite moment functional \mathcal{M} there exists a non-decreasing bounded function $a(\omega)$, called a *moment distribution function*, such that for the associated Stieltjes integral we have

$$(21) \quad \int_{-\infty}^{\infty} \omega^n da(\omega) = \mu_n$$

and such that for the corresponding family of polynomials $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$

$$(22) \quad \int_{-\infty}^{\infty} P_n^{\mathcal{M}}(\omega) P_m^{\mathcal{M}}(\omega) da(\omega) = \delta(m - n).$$

We denote by $L_{a(\omega)}^2$ the Hilbert space of functions $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ for which the Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} |\varphi(\omega)|^2 da(\omega)$ is finite, with the scalar product defined by $\langle \alpha, \beta \rangle_{a(\omega)} = \int_{-\infty}^{\infty} \alpha(\omega) \overline{\beta(\omega)} da(\omega)$, and with the corresponding norm denoted by $\|\varphi\|_{a(\omega)}$.

We define a function $\mathbf{m} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$(23) \quad \mathbf{m}(t) = \int_{-\infty}^{\infty} e^{i\omega t} da(\omega).$$

Since

$$\int_{-\infty}^{\infty} |(i\omega)^n e^{i\omega t}| da(\omega) \leq \left(\int_{-\infty}^{\infty} \omega^{2n} da(\omega) \int_{-\infty}^{\infty} da(\omega) \right)^{1/2} = \sqrt{\mu_{2n}} < \infty,$$

we can differentiate (23) under the integral sign any number of times, and obtain that for all sn ,

$$(24) \quad \mathbf{m}^{(2n)}(0) = (-1)^n \mu_{2n};$$

$$(25) \quad \mathbf{m}^{(2n+1)}(0) = 0.$$

2.2. The chromatic derivatives. Given a moment functional \mathcal{M} satisfying conditions (i) – (iii) above, we associate with \mathcal{M} a family of linear differential operators $\{\mathcal{K}^n\}_{n \in \mathbb{N}}$ defined by the operator polynomial⁵

$$\mathcal{K}_t^n = \frac{1}{i^n} P_n^{\mathcal{M}}(iD_t),$$

and call them *the chromatic derivatives associated with \mathcal{M}* . Since \mathcal{M} is symmetric, such operators have real coefficients and satisfy the recurrence

$$(26) \quad \mathcal{K}^{n+1} = \frac{1}{\gamma_n} (D \circ \mathcal{K}^n) + \frac{\gamma_{n-1}}{\gamma_n} \mathcal{K}^{n-1},$$

with the same coefficients $\gamma_n > 0$ as in (17). Thus,

$$(27) \quad \mathcal{K}_t^n [e^{i\omega t}] = i^n P_n^{\mathcal{M}}(\omega) e^{i\omega t}$$

and

$$(28) \quad \mathcal{K}^n[\mathbf{m}](t) = \int_{-\infty}^{\infty} i^n P_n^{\mathcal{M}}(\omega) e^{i\omega t} da(\omega).$$

The basic properties of orthogonal polynomials imply that for all m, n ,

$$(29) \quad (\mathcal{K}^n \circ \mathcal{K}^m)[\mathbf{m}](0) = (-1)^n \delta(m - n),$$

and, if $m < n$ or if $m - n$ is odd, then

$$(30) \quad (D^m \circ \mathcal{K}^n)[\mathbf{m}](0) = 0.$$

⁵Thus, to obtain \mathcal{K}^n , one replaces ω^k in $P_n^{\mathcal{M}}(\omega)$ by $i^k D_t^k$, where $D_t^k[f] = \frac{d^k}{dt^k} f(t)$. We use the square brackets to indicate the arguments of operators acting on various function spaces. If A is a linear differential operator, and if a function $f(t, \vec{w})$ has parameters \vec{w} , we write $A_t[f]$ to distinguish the variable t of differentiation; if $f(t)$ contains only variable t , we write $A[f(t)]$ for $A_t[f(t)]$ and $D^k[f(t)]$ for $D_t^k[f(t)]$.

The following Lemma corresponds to the Christoffel-Darboux equality for orthogonal polynomials and has a similar proof which uses (26) to represent the left hand side of (31) as a telescoping sum.

Lemma 2.1 ([12]). *Let $\{\mathcal{K}^n\}_{n \in \mathbb{N}}^{\mathcal{M}}$ be the family of chromatic derivatives associated with a moment functional \mathcal{M} , and let $f, g \in C^\infty$; then*

$$(31) \quad D \left[\sum_{m=0}^n \mathcal{K}^m[f] \mathcal{K}^m[g] \right] = \gamma_n (\mathcal{K}^{n+1}[f] \mathcal{K}^n[g] + \mathcal{K}^n[f] \mathcal{K}^{n+1}[g]).$$

2.3. Chromatic expansions. Let f be infinitely differentiable at a real or complex u ; the formal series

$$(32) \quad \begin{aligned} \text{CE}^{\mathcal{M}}[f, u](t) &= \sum_{k=0}^{\infty} \mathcal{K}^k[f](u) \mathcal{K}_u^k[\mathbf{m}(t-u)] \\ &= \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](u) \mathcal{K}^k[\mathbf{m}](t-u) \end{aligned}$$

is called the *chromatic expansion* of f associated with \mathcal{M} , centered at u , and

$$\text{CA}^{\mathcal{M}}[f, n, u](t) = \sum_{k=0}^n (-1)^k \mathcal{K}^k[f](u) \mathcal{K}^k[\mathbf{m}](t-u)$$

is the *chromatic approximation* of f of order n .

From (29) it follows that the chromatic approximation $\text{CA}^{\mathcal{M}}[f, n, u](t)$ of order n of $f(t)$ for all $m \leq n$ satisfies

$$\mathcal{K}_t^m[\text{CA}^{\mathcal{M}}[f, n, u](t)]|_{t=u} = \sum_{k=0}^n (-1)^k \mathcal{K}^k[f](u) (\mathcal{K}^m \circ \mathcal{K}^k)[\mathbf{m}](0) = \mathcal{K}^m[f](u).$$

Since \mathcal{K}^m is a linear combination of derivatives D^k for $k \leq m$, also $f^{(m)}(u) = D_t^m[\text{CA}^{\mathcal{M}}[f, n, u](t)]|_{t=u}$ for all $m \leq n$. In this sense, just like the Taylor approximation, a chromatic approximation is a local approximation. Thus, for all $m \leq n$,

$$(33) \quad f^{(m)}(u) = D_t^m[\text{CA}^{\mathcal{M}}[f, n, u](t)]|_{t=u} = \sum_{k=0}^n (-1)^k \mathcal{K}^k[f](u) (D^m \circ \mathcal{K}^k)[\mathbf{m}](0).$$

Similarly, since $D_t^m[\sum_{k=0}^n f^{(k)}(u)(t-u)^k/k!]|_{t=u} = f^{(m)}(u)$ for $m \leq n$, we also have

$$(34) \quad \mathcal{K}^m[f](u) = \mathcal{K}_t^m \left[\sum_{k=0}^n f^{(k)}(u)(t-u)^k/k! \right] |_{t=u} = \sum_{k=0}^n f^{(k)}(u) \mathcal{K}^m[t^k/k!](0).$$

Equations (33) and (34) for $m = n$ relate the standard and the chromatic bases of the vector space space of linear differential operators,

$$(35) \quad D^n = \sum_{k=0}^n (-1)^k (D^n \circ \mathcal{K}^k)[\mathbf{m}](0) \mathcal{K}^k;$$

$$(36) \quad \mathcal{K}^n = \sum_{k=0}^n \mathcal{K}^n[t^k/k!](0) D^k.$$

Note that, since for $j > k$ all powers of t in $\mathcal{K}^k[t^j/j!]$ are positive, we have

$$(37) \quad j > k \Rightarrow \mathcal{K}^k[t^j/j!](0) = 0.$$

3. CHROMATIC MOMENT FUNCTIONALS

3.1. We now introduce the broadest class of moment functionals which we study.

Definition 3.1. *Chromatic moment functionals are symmetric positive definite moment functionals for which the sequence $\{\mu_n^{1/n}/n\}_{n \in \mathbb{N}}$ is bounded.*

If \mathcal{M} is chromatic, we set

$$(38) \quad \rho = \limsup_{n \rightarrow \infty} \left(\frac{\mu_n}{n!} \right)^{1/n} = e \limsup_{n \rightarrow \infty} \frac{\mu_n^{1/n}}{n} < \infty.$$

Lemma 3.2. *Let \mathcal{M} be a chromatic moment functional and ρ such that (38) holds. Then for every α such that $0 \leq \alpha < 1/\rho$ the corresponding moment distribution $a(\omega)$ satisfies*

$$(39) \quad \int_{-\infty}^{\infty} e^{\alpha|\omega|} da(\omega) < \infty.$$

Proof. For all $b > 0$, $\int_{-b}^b e^{\alpha|\omega|} da(\omega) = \sum_{n=0}^{\infty} \alpha^n/n! \int_{-b}^b |\omega|^n da(\omega)$. For even n we have $\int_{-b}^b \omega^n da(\omega) \leq \mu_n$. For odd n we have $|\omega|^n < 1 + \omega^{n+1}$ for all ω , and thus $\int_{-b}^b |\omega|^n da(\omega) < \int_{-b}^b da(\omega) + \int_{-b}^b \omega^{n+1} da(\omega) \leq 1 + \mu_{n+1}$. Let $\beta_n = \mu_n$ if n is even, and $\beta_n = 1 + \mu_{n+1}$ if n is odd. Then also $\int_{-\infty}^{\infty} e^{\alpha|\omega|} da(\omega) \leq \sum_{n=0}^{\infty} \alpha^n \beta_n/n!$. Since $\limsup_{n \rightarrow \infty} (\beta_n/n!)^{1/n} = \rho$ and $0 \leq \alpha < 1/\rho$, the last sum converges to a finite limit. \square

On the other hand, the proof of Theorem 5.2 in §II.5 of [7] shows that if (39) holds for some $\alpha > 0$, then (38) also holds for some $\rho \leq 1/\alpha$. Thus, we get the following Corollary.

Corollary 3.3. *A symmetric positive definite moment functional is chromatic just in case for some $\alpha > 0$ the corresponding moment distribution function $a(\omega)$ satisfies (39).*

Note: *For the remaining part of this section we assume that \mathcal{M} is a chromatic moment functional.*

For every $a > 0$, we let $S(a) = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < a\}$. The following Corollary directly follows from Lemma 3.2.

Corollary 3.4. *If $u \in S(\frac{1}{2\rho})$, then $e^{i u \omega} \in L^2_{a(\omega)}$.*

We now extend function $\mathbf{m}(t)$ given by (23) from \mathbb{R} to the complex strip $S(\frac{1}{\rho})$.

Proposition 3.5. *Let for $z \in S(\frac{1}{\rho})$,*

$$(40) \quad \mathbf{m}(z) = \int_{-\infty}^{\infty} e^{i\omega z} da(\omega).$$

Then $\mathbf{m}(z)$ is analytic on the strip $S(\frac{1}{\rho})$.

Proof. Fix n and let $z = x + iy$ with $|y| < 1/\rho$; then for every $b > 0$,

$$\int_{-b}^b |(i\omega)^n e^{i\omega z}| da(\omega) \leq \int_{-b}^b |\omega|^n e^{|\omega y|} da(\omega) = \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \int_{-b}^b |\omega|^{n+k} da(\omega).$$

As in the proof of Lemma 3.2, we let $\beta_k = \mu_{n+k}$ for even values of $n+k$, and $\beta_k = 1 + \mu_{n+k+1}$ for odd values of $n+k$; then the above inequality implies

$$\int_{-\infty}^{\infty} |(i\omega)^n e^{i\omega z}| da(\omega) \leq \sum_{k=0}^{\infty} \frac{|y|^k \beta_k}{k!},$$

and it is easy to see that for every fixed n , $\limsup_{k \rightarrow \infty} (\beta_k/k!)^{1/k} = \rho$. \square

Proposition 3.6. *Let $\varphi(\omega) \in L^2_{a(\omega)}$; we can define a corresponding function $f_\varphi : S(\frac{1}{2\rho}) \rightarrow \mathbb{C}$ by*

$$(41) \quad f_\varphi(z) = \int_{-\infty}^{\infty} \varphi(\omega) e^{i\omega z} da(\omega).$$

Such $f_\varphi(z)$ is analytic on $S(\frac{1}{2\rho})$ and for all n and $z \in S(\frac{1}{2\rho})$,

$$(42) \quad \mathcal{K}^n[f_\varphi](z) = \int_{-\infty}^{\infty} i^n P_n^{\mathcal{M}}(\omega) \varphi(\omega) e^{i\omega z} da(\omega).$$

Proof. Let $z = x + iy$, with $|y| < 1/(2\rho)$. For every n and $b > 0$ we have

$$\begin{aligned} & \int_{-b}^b |(i\omega)^n \varphi(\omega) e^{i\omega z}| da(\omega) \\ & \leq \int_{-b}^b |\omega|^n |\varphi(\omega)| e^{|\omega y|} da(\omega) \\ & \leq \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \int_{-b}^b |\omega|^{n+k} |\varphi(\omega)| da(\omega) \\ & \leq \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \left(\int_{-b}^b \omega^{2n+2k} da(\omega) \int_{-b}^b |\varphi(\omega)|^2 da(\omega) \right)^{1/2}. \end{aligned}$$

Thus, also

$$\int_{-\infty}^{\infty} |(i\omega)^n \varphi(\omega) e^{i\omega z}| da(\omega) \leq \|\varphi(\omega)\|_{a(\omega)} \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \sqrt{\mu_{2n+2k}}.$$

The claim now follows from the fact that $\limsup_{k \rightarrow \infty} \sqrt[2k]{\mu_{2n+2k}} / \sqrt[k]{k!} = 2\rho$ for every fixed n . \square

Lemma 3.7. *If \mathcal{M} is chromatic, then $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$ is a complete system in $L^2_{a(\omega)}$.*

Proof. Follows from a theorem of Riesz (see, for example, Theorem 5.1 in §II.5 of [7]) which asserts that if $\liminf_{n \rightarrow \infty} (\mu_n/n!)^{1/n} < \infty$, then $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$ is a complete system in $L^2_{a(\omega)}$.⁶ \square

Proposition 3.8. *Let $\varphi(\omega) \in L^2_{a(\omega)}$; if for some fixed $u \in S(\frac{1}{2\rho})$ the function $\varphi(\omega) e^{i\omega u}$ also belongs to $L^2_{a(\omega)}$, then in $L^2_{a(\omega)}$ we have*

$$(43) \quad \varphi(\omega) e^{i\omega u} = \sum_{n=0}^{\infty} (-i)^n \mathcal{K}^n[f_\varphi](u) P_n^{\mathcal{M}}(\omega),$$

⁶Note that we need the stronger condition $\limsup_{n \rightarrow \infty} (\mu_n/n!)^{1/n} < \infty$ to insure that function $m(z)$ defined by (40) is analytic on a strip (Proposition 3.5).

and for $f_\varphi(z)$ given by (41) we have

$$(44) \quad \sum_{n=0}^{\infty} |\mathcal{K}^n[f_\varphi](u)|^2 = \|\varphi(\omega)e^{i\omega u}\|_{a(\omega)}^2 < \infty.$$

Proof. By Proposition 3.6, if $u \in S(\frac{1}{2\rho})$, then equation (42) holds for the corresponding f_φ given by (41). If also $\varphi(\omega)e^{i\omega u} \in L_{a(\omega)}^2$, then (42) implies that

$$(45) \quad \langle \varphi(\omega)e^{i\omega u}, P_n^{\mathcal{M}}(\omega) \rangle_{a(\omega)} = (-i)^n \mathcal{K}^n[f_\varphi](u).$$

Since $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L_{a(\omega)}^2$, (45) implies (43), and Parseval's Theorem implies (44). \square

Corollary 3.9. *For every $\varphi(\omega) \in L_{a(\omega)}^2$ and every $u \in \mathbb{R}$, equality (43) holds and*

$$(46) \quad \sum_{n=0}^{\infty} |\mathcal{K}^n[f_\varphi](u)|^2 = \|\varphi(\omega)\|_{a(\omega)}^2.$$

Thus, the sum $\sum_{n=0}^{\infty} |\mathcal{K}^n[f_\varphi](u)|^2$ is independent of $u \in \mathbb{R}$.

Proof. If $u \in \mathbb{R}$, then $\varphi(\omega)e^{i\omega u} \in L_{a(\omega)}^2$ and $\|\varphi(\omega)e^{i\omega u}\|_{a(\omega)}^2 = \|\varphi(\omega)\|_{a(\omega)}^2$. \square

Corollary 3.10. *Let $\varepsilon > 0$; then for all $u \in S(\frac{1}{2\rho} - \varepsilon)$*

$$(47) \quad \sum_{n=0}^{\infty} |\mathcal{K}^n[\mathbf{m}](u)|^2 < \left\| e^{(\frac{1}{2\rho} - \varepsilon)|\omega|} \right\|_{a(\omega)}^2 < \infty.$$

Proof. Corollary 3.4 implies that we can apply Proposition 3.8 with $\varphi(\omega) = 1$, in which case $f_\varphi(z) = \mathbf{m}(z)$, and, using Lemma 3.2, obtain

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[\mathbf{m}](u)|^2 = \|e^{i\omega u}\|_{a(\omega)}^2 \leq \|e^{|\operatorname{Im}(u)\omega|}\|_{a(\omega)}^2 < \left\| e^{(\frac{1}{2\rho} - \varepsilon)|\omega|} \right\|_{a(\omega)}^2 < \infty.$$

\square

Definition 3.11. $\mathbf{L}_{\mathcal{M}}^2$ is the vector space of functions $f : S(\frac{1}{2\rho}) \rightarrow \mathbb{C}$ which are analytic on $S(\frac{1}{2\rho})$ and satisfy $\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty$.

Proposition 3.12. *The mapping*

$$(48) \quad f(z) \mapsto \varphi_f(\omega) = \sum_{n=0}^{\infty} (-i)^n \mathcal{K}^n[f](0) P_n^{\mathcal{M}}(\omega)$$

is an isomorphism between the vector spaces $\mathbf{L}_{\mathcal{M}}^2$ and $L_{a(\omega)}^2$, and its inverse is given by (41).

Proof. Let $f \in \mathbf{L}_{\mathcal{M}}^2$; since $\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty$, the function $\varphi_f(\omega)$ defined by (48) belongs to $L_{a(\omega)}^2$. By Proposition 3.6, f_{φ_f} defined from φ_f by (41) is analytic on $S(\frac{1}{2\rho})$ and by Proposition 3.8 it satisfies $\varphi_f(\omega) = \sum_{n=0}^{\infty} (-i)^n \mathcal{K}^n[f_{\varphi_f}](0) P_n^{\mathcal{M}}(\omega)$. By the uniqueness of the Fourier expansion of $\varphi_f(\omega)$ with respect to the system $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$ we have $\mathcal{K}^n[f](0) = \mathcal{K}^n[f_{\varphi_f}](0)$ for all n . Thus, $f(z) = f_{\varphi_f}(z)$ for all $z \in S(\frac{1}{2\rho})$. \square

Proposition 3.12 and Corollary 3.9 imply the following Corollary.

Corollary 3.13. *For all $f \in \mathbf{L}_{\mathcal{M}}^2$ and all $t \in \mathbb{R}$ the sum $\sum_{n=0}^{\infty} |\mathcal{K}^n[f](t)|^2$ converges and is independent of t .*

Definition 3.14. *For every $f(z) \in \mathbf{L}_{\mathcal{M}}^2$ we call the corresponding $\varphi_f(\omega) \in L_{a(\omega)}^2$ given by equation (48) the \mathcal{M} -Fourier-Stieltjes transform of $f(z)$ and denote it by $\mathcal{F}^{\mathcal{M}}[f](\omega)$.*

Assume that $a(\omega)$ is absolutely continuous; then $a'(\omega) = w(\omega)$ almost everywhere for some non-negative weight function $w(\omega)$. Then (41) implies

$$f(z) = \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[f](\omega) e^{i\omega z} w(\omega) d\omega.$$

This implies the following Proposition.

Proposition 3.15 ([12]). *Assume that a function $f(z)$ is analytic on the strip $S(\frac{1}{2\rho})$ and that it has a Fourier transform $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ such that $f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega z} d\omega$ for all $z \in S(\frac{1}{2\rho})$; then $f(z) \in \mathbf{L}_{\mathcal{M}}^2$ if and only if $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 w(\omega)^{-1} d\omega < \infty$, in which case $\hat{f}(\omega) = 2\pi \mathcal{F}^{\mathcal{M}}[f](\omega) w(\omega)$.*

3.2. Uniform convergence of chromatic expansions. The Shannon expansion of an $f \in \mathbf{BL}(\pi)$ is obtained by representing its Fourier transform $\hat{f}(\omega)$ as series of the trigonometric polynomials; similarly, the chromatic expansion of an $f \in \mathbf{L}_{\mathcal{M}}^2$ is obtained by representing $\mathcal{F}^{\mathcal{M}}[f](\omega)$ as a series of orthogonal polynomials $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$.

Proposition 3.16. *Assume $f \in \mathbf{L}_{\mathcal{M}}^2$; then for all $u \in \mathbb{R}$ and $\varepsilon > 0$, the chromatic series $\text{CE}^{\mathcal{M}}[f, u](z)$ of $f(z)$ converges to $f(z)$ uniformly on the strip $S(\frac{1}{2\rho} - \varepsilon)$.*

Proof. Assume $u \in \mathbb{R}$; by applying (42) to $\mathbf{m}(z - u)$ we get that for all $z \in S(\frac{1}{2\rho})$,

$$(49) \quad \text{CA}^{\mathcal{M}}[f, n, u](z) = \int_{-\infty}^{\infty} \sum_{k=0}^n (-i)^k \mathcal{K}^k[f](u) P_k^{\mathcal{M}}(\omega) e^{i\omega(z-u)} da(\omega).$$

Since $f \in \mathbf{L}_{\mathcal{M}}^2$, Proposition 3.12 implies $\mathcal{F}^{\mathcal{M}}[f](\omega) \in L_{a(\omega)}^2$. Corollary 3.9 and equation (43) imply that in $L_{a(\omega)}^2$

$$\mathcal{F}^{\mathcal{M}}[f](\omega) e^{i\omega u} = \sum_{k=0}^{\infty} (-i)^k \mathcal{K}^k[f](u) P_k^{\mathcal{M}}(\omega).$$

Thus,

$$(50) \quad f(z) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} (-i)^k \mathcal{K}^k[f](u) P_k^{\mathcal{M}}(\omega) e^{i\omega(z-u)} da(\omega).$$

Consequently, from (49) and (50),

$$\begin{aligned} & |f(z) - \text{CA}^{\mathcal{M}}[f, n, u](z)| \\ & \leq \int_{-\infty}^{\infty} \left| \sum_{k=n+1}^{\infty} (-i)^k \mathcal{K}^k[f](u) P_k^{\mathcal{M}}(\omega) e^{i\omega(z-u)} \right| da(\omega) \\ & \leq \left(\int_{-\infty}^{\infty} \left| \sum_{k=n+1}^{\infty} (-i)^k \mathcal{K}^k[f](u) P_k^{\mathcal{M}}(\omega) \right|^2 da(\omega) \int_{-\infty}^{\infty} |e^{i\omega(z-u)}|^2 da(\omega) \right)^{1/2} \end{aligned}$$

For $z \in S(\frac{1}{2\rho} - \varepsilon)$ we have

$$(51) \quad |f(z) - \text{CA}^{\mathcal{M}}[f, n, u](z)| \leq \left(\sum_{k=n+1}^{\infty} |\mathcal{K}^k[f](u)|^2 \int_{-\infty}^{\infty} e^{(\frac{1}{\rho} - 2\varepsilon)|\omega|} d\mathbf{a}(\omega) \right)^{1/2}.$$

Consequently, Lemma 3.2 and Corollary 3.13 imply that $\text{CE}^{\mathcal{M}}[f, u](z)$ converges to $f(z)$ uniformly on $S(\frac{1}{2\rho} - \varepsilon)$. \square

Proposition 3.17. *Space $\mathbf{L}_{\mathcal{M}}^2$ consists precisely of functions of the form $f(z) = \sum_{n=0}^{\infty} a_n \mathcal{K}^n[\mathbf{m}](z)$ where $a = \langle\langle a_n \rangle\rangle_{n \in \mathbb{N}}$ is a complex sequence in l^2 .*

Proof. Assume $a \in l^2$; by Proposition 3.10, for every $\varepsilon > 0$, if $z \in S(\frac{1}{2\rho} - \varepsilon)$ then

$$\begin{aligned} \sum_{n=k}^{\infty} |a_n \mathcal{K}^n[\mathbf{m}](z)| &\leq \left(\sum_{n=k}^{\infty} |a_n|^2 \sum_{n=k}^{\infty} |\mathcal{K}^n[\mathbf{m}](z)|^2 \right)^{1/2} \\ &\leq \left(\sum_{n=k}^{\infty} |a_n|^2 \right)^{1/2} \left\| e^{(\frac{1}{2\rho} - \varepsilon)|\omega|} \right\|_{a(\omega)}, \end{aligned}$$

which implies that the series converges absolutely and uniformly on $S(\frac{1}{2\rho} - \varepsilon)$. Consequently, $f(z) = \sum_{n=0}^{\infty} a_n \mathcal{K}^n[\mathbf{m}](z)$ is analytic on $S(\frac{1}{2\rho})$, and

$$\mathcal{K}^m[f](0) = \sum_{n=0}^{\infty} a_n (\mathcal{K}^m \circ \mathcal{K}^n)[\mathbf{m}](0) = (-1)^m a_m.$$

Thus, $\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$ and so $f(z) \in \mathbf{L}_{\mathcal{M}}^2$. Proposition 3.16 provides the opposite direction. \square

Note that Proposition 3.17 implies that for every $\varepsilon > 0$ functions $f(z) \in \mathbf{L}_{\mathcal{M}}^2$ are bounded on the strip $S(\frac{1}{2\rho} - \varepsilon)$ because

$$|f(z)| \leq \left(\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 \right)^{1/2} \left\| e^{(\frac{1}{2\rho} - \varepsilon)|\omega|} \right\|_{a(\omega)}.$$

3.3. A function space with a locally defined scalar product.

Definition 3.18. $L_{\mathcal{M}}^2$ is the space of functions $f(t) : \mathbb{R} \mapsto \mathbb{C}$ obtained from functions in $\mathbf{L}_{\mathcal{M}}^2$ by restricting their domain to \mathbb{R} .

Assume that $f, g \in L_{\mathcal{M}}^2$; then (45) implies that for all $u \in \mathbb{R}$,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{K}^n[f](u) \overline{\mathcal{K}^n[g](u)} &= \langle \mathcal{F}^{\mathcal{M}}[f](\omega) e^{i\omega u}, \mathcal{F}^{\mathcal{M}}[g](\omega) e^{i\omega u} \rangle_{a(\omega)} \\ &= \langle \mathcal{F}^{\mathcal{M}}[f](\omega), \mathcal{F}^{\mathcal{M}}[g](\omega) \rangle_{a(\omega)}. \end{aligned}$$

Note that for all $t, u \in \mathbb{R}$,

$$\begin{aligned} &\sum_{n=0}^n \mathcal{K}^k[f](u) \mathcal{K}_u^k[g](t-u) \\ &= \int_{-\infty}^{\infty} \sum_{k=0}^n \mathcal{K}^k[f](u) (-i)^k P_k^{\mathcal{M}}(\omega) \mathcal{F}^{\mathcal{M}}[g](\omega) e^{i\omega(t-u)} d\mathbf{a}(\omega). \end{aligned}$$

By (43), the sum $\sum_{k=0}^n (-i)^k \mathcal{K}^k[f](u) P_k^{\mathcal{M}}(\omega)$ converges in $L_{a(\omega)}^2$ to $\mathcal{F}^{\mathcal{M}}[f](\omega) e^{i\omega u}$. Since $\mathcal{F}^{\mathcal{M}}[g](\omega) e^{i\omega(t-u)} \in L_{a(\omega)}^2$, and since

$$\left| \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[g](\omega) \mathcal{F}^{\mathcal{M}}[f](\omega) e^{i\omega t} da(\omega) \right| < \|\mathcal{F}^{\mathcal{M}}[f](\omega)\|_{a(\omega)} \|\mathcal{F}^{\mathcal{M}}[g](\omega)\|_{a(\omega)},$$

we have that for all $t, u \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} \mathcal{K}^k[f](u) \mathcal{K}_u^k[g](t-u) = \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[g](\omega) \mathcal{F}^{\mathcal{M}}[f](\omega) e^{i\omega t} da(\omega) < \infty.$$

Proposition 3.19 ([12]). *We can introduce locally defined scalar product, an associated norm and a convolution of functions in $L_{\mathcal{M}}^2$ by the following sums which are independent of $u \in \mathbb{R}$:*

$$(52) \quad \|f\|_{\mathcal{M}}^2 = \sum_{n=0}^{\infty} |K^n[f](u)|^2 = \|\mathcal{F}^{\mathcal{M}}[f](\omega)\|_{a(\omega)}^2;$$

$$(53) \quad \begin{aligned} \langle f, g \rangle_{\mathcal{M}} &= \sum_{n=0}^{\infty} K^n[f](u) \overline{K^n[g](u)} \\ &= \langle \mathcal{F}^{\mathcal{M}}[f](\omega), \mathcal{F}^{\mathcal{M}}[g](\omega) \rangle_{a(\omega)}; \end{aligned}$$

$$(54) \quad \begin{aligned} (f *_{\mathcal{M}} g)(t) &= \sum_{n=0}^{\infty} K^n[f](u) K_u^n[g](t-u) \\ &= \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[f](\omega) \mathcal{F}^{\mathcal{M}}[g](\omega) e^{i\omega t} da(\omega). \end{aligned}$$

Letting $g(t) \equiv \mathbf{m}(t)$ in (54), we get $(f *_{\mathcal{M}} \mathbf{m})(t) = \text{CE}^{\mathcal{M}}[f, u](t) = f(t)$ for all $f(t) \in L_{\mathcal{M}}^2$, while by setting $u = 0$, $u = t$ and $u = t/2$ in (54), we get the following lemma.

Lemma 3.20 ([12]). *For every $f, g \in L_{\mathcal{M}}^2$ and for every $t \in \mathbb{R}$,*

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](t) \mathcal{K}^k[g](0) &= \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](0) \mathcal{K}^k[g](t) \\ &= \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](t/2) \mathcal{K}^k[g](t/2). \end{aligned}$$

Since $\mathbf{m}(z)$ is analytic on $S(\frac{1}{\rho})$, so are $\mathcal{K}^n[\mathbf{m}](z)$ for all n ; thus, since by (29) $\sum_{m=0}^{\infty} (\mathcal{K}^n \circ \mathcal{K}^m)[\mathbf{m}](0)^2 = 1$, we have $\mathcal{K}^n[\mathbf{m}](t) \in L_{\mathcal{M}}^2$ for all n . Let u be a fixed real parameter; consider functions $B_u^n(t) = \mathcal{K}_u^n[\mathbf{m}(t-u)] = (-1)^n \mathcal{K}^n[\mathbf{m}](t-u) \in L_{\mathcal{M}}^2$. Since

$$(55) \quad \begin{aligned} \langle B_u^n(t), B_u^m(t) \rangle_{\mathcal{M}} &= \sum_{k=0}^{\infty} (\mathcal{K}_t^k \circ \mathcal{K}_u^n)[\mathbf{m}(t-u)] (\mathcal{K}_t^k \circ \mathcal{K}_u^m)[\mathbf{m}(t-u)] \\ &= \sum_{k=0}^{\infty} (-1)^{m+n} (\mathcal{K}^k \circ \mathcal{K}^n)[\mathbf{m}](t-u) (\mathcal{K}^k \circ \mathcal{K}^m)[\mathbf{m}](t-u) \Big|_{t=u} \\ &= \delta(m-n), \end{aligned}$$

the family $\{B_u^n(t)\}_{n \in \mathbb{N}}$ is orthonormal in $L_{\mathcal{M}}^2$ and for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$,

$$(56) \quad \sum_{k=0}^{\infty} (\mathcal{K}^k \circ \mathcal{K}^n)[\mathbf{m}](t)^2 = 1.$$

By (29), for $f \in L_{\mathcal{M}}^2$,

$$(57) \quad \begin{aligned} \langle f, \mathcal{K}_u^n[\mathbf{m}(t-u)] \rangle_{\mathcal{M}} &= \sum_{k=0}^{\infty} \mathcal{K}^k[f](t) (\mathcal{K}_t^k \circ \mathcal{K}_u^n)[\mathbf{m}(t-u)] \Big|_{t=u} \\ &= \sum_{k=0}^{\infty} (-1)^n \mathcal{K}^k[f](u) (\mathcal{K}^k \circ \mathcal{K}^n)[\mathbf{m}](0) \\ &= \mathcal{K}^n[f](u). \end{aligned}$$

Proposition 3.21 ([12]). *The chromatic expansion $\text{CE}^{\mathcal{M}}[f, u](t)$ of $f(t) \in L_{\mathcal{M}}^2$ is the Fourier series of $f(t)$ with respect to the orthonormal system $\{\mathcal{K}_u^n[\mathbf{m}(t-u)]\}_{n \in \mathbb{N}}$. The chromatic expansion converges to $f(t)$ in $L_{\mathcal{M}}^2$; thus, $\{\mathcal{K}_u^n[\mathbf{m}(t-u)]\}_{n \in \mathbb{N}}$ is a complete orthonormal base of $L_{\mathcal{M}}^2$.*

Proof. Since $\mathcal{K}_t^k[f(t) - \text{CE}^{\mathcal{M}}[f, n, u](t)]|_{t=u}$ equals 0 for $k \leq n$ and equals $\mathcal{K}^k[f](u)$ for $k > n$, $\|f - \text{CE}^{\mathcal{M}}[f, n, u]\|_{\mathcal{M}} = \sum_{k=n+1}^{\infty} \mathcal{K}^k[f](u)^2 \rightarrow 0$. \square

Note that using (56) with $n = 0$ we get

$$(58) \quad \begin{aligned} |f(t) - \text{CE}^{\mathcal{M}}[f, n, u](t)| &\leq \sum_{k=n+1}^{\infty} |\mathcal{K}^k[f](u) \mathcal{K}^k[\mathbf{m}](t-u)| \\ &\leq \left(\sum_{k=n+1}^{\infty} \mathcal{K}^k[f](u)^2 \sum_{k=n+1}^{\infty} \mathcal{K}^k[\mathbf{m}](t-u)^2 \right)^{1/2} \\ &= \left(\sum_{k=n+1}^{\infty} \mathcal{K}^k[f](u)^2 \right)^{1/2} \left(1 - \sum_{k=0}^n \mathcal{K}^k[\mathbf{m}](t-u)^2 \right)^{1/2}. \end{aligned}$$

Let

$$E_n(t) = \left(1 - \sum_{k=0}^n \mathcal{K}^k[\mathbf{m}](t)^2 \right)^{1/2};$$

then, using Lemma 2.1, we have

$$E'_n(t) = \gamma_n \mathcal{K}^{n+1}[\mathbf{m}](t) \mathcal{K}^n[\mathbf{m}](t) \left(1 - \sum_{k=0}^n \mathcal{K}^k[\mathbf{m}](t)^2 \right)^{-1/2}.$$

Since $(D^k \circ \mathcal{K}^n)[\mathbf{m}](0) = 0$ for all $0 \leq k \leq n-1$, we get that $E_n^{(k)}(t) = 0$ for all $k \leq 2n+1$. Thus, $E_n(0) = 0$ and $E_n(t)$ is very flat around $t = 0$, as the following graph of $E_{15}(t)$ shows, for the particular case of the chromatic derivatives associated with the Legendre polynomials. This explains why chromatic expansions provide excellent local approximations of signals $f \in \mathbf{BL}(\pi)$.

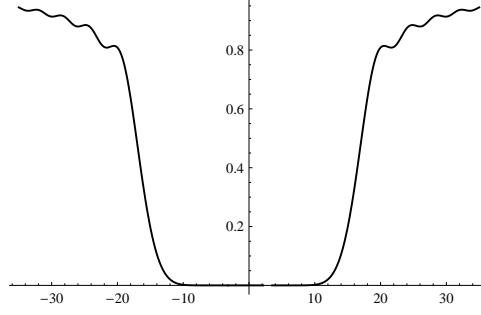


FIGURE 4. Error bound $E_{15}(t)$ for the chromatic approximation of order 15 associated with the Legendre polynomials.

3.4. Chromatic expansions and linear operators. Let A be a linear operator on $L_{\mathcal{M}}^2$ which is continuous with respect to the norm $\|f\|_{\mathcal{M}}$. If A is shift invariant, i.e., if for every fixed h , $A[f(t+h)] = A[f](t+h)$ for all $f \in L_{\mathcal{M}}^2$, then A commutes with differentiation on $L_{\mathcal{M}}^2$ and

$$A[f](t) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](u) \mathcal{K}^n[A[\mathbf{m}]](t-u) = (f *_{\mathcal{M}} A[\mathbf{m}])(t).$$

Consequently, the action of such A on any function in $L_{\mathcal{M}}^2$ is uniquely determined by $A[\mathbf{m}]$, which plays the role of the *impulse response* $A[\text{sinc}]$ of a *continuous time invariant linear system* in the standard signal processing paradigm based on Shannon's expansion.

Note that if A is a continuous linear operator A on $L_{\mathcal{M}}^2$ such that $(A \circ D^n)[\mathbf{m}](t) = (D^n \circ A)[\mathbf{m}](t)$ for all n , then Lemma 3.20 implies that for every $f \in L_{\mathcal{M}}^2$,

$$\begin{aligned} A[f](t) &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0) \mathcal{K}^n[A[\mathbf{m}]](t) \\ &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[A[\mathbf{m}]](0) \mathcal{K}^n[f](t). \end{aligned}$$

Since operators $\mathcal{K}^n[f](t)$ are shift invariant, such A must be also shift invariant.

3.5. A geometric interpretation. For every particular value of $t \in \mathbb{R}$ the mapping of $L_{\mathcal{M}}^2$ into l^2 given by $f \mapsto f_t = \langle \mathcal{K}^n[f](t) \rangle_{n \in \mathbb{N}}$ is unitary isomorphism which maps the base of $L_{\mathcal{M}}^2$, consisting of vectors $B^k(t) = (-1)^k \mathcal{K}^k[\mathbf{m}(t)]$, into vectors $B_t^k = \langle (-1)^k (\mathcal{K}^n \circ \mathcal{K}^k)[\mathbf{m}(t)] \rangle_{n \in \mathbb{N}}$. Since the first sum in (55) is independent of t , we have $\langle B_t^k, B_t^m \rangle = \delta(m-k)$, and (57) implies $\langle f_t, B_t^k \rangle = \mathcal{K}^k[f](0)$. Thus, since $\sum_{k=0}^{\infty} \mathcal{K}^k[f](0)^2 < \infty$, we have $\sum_{k=0}^{\infty} \mathcal{K}^k[f](0) B_t^k \in l^2$ and

$$\begin{aligned} \sum_{k=0}^{\infty} \langle f_t, B_t^k \rangle B_t^k &= \sum_{k=0}^{\infty} \mathcal{K}^k[f](0) B_t^k \\ &= \left\langle \left\langle \sum_{k=0}^{\infty} \mathcal{K}^k[f](0) (-1)^k (\mathcal{K}^n \circ \mathcal{K}^k)[\mathbf{m}(t)] \right\rangle_{n \in \mathbb{N}} \right\rangle. \end{aligned}$$

Since for $f \in L_{\mathcal{M}}^2$ the chromatic series of f converges uniformly on \mathbb{R} , we have $K^n[f](t) = \sum_{k=0}^{\infty} K^k[f](0)(-1)^k(\mathcal{K}^n \circ \mathcal{K}^k)[\mathbf{m}(t)]$. Thus,

$$\sum_{k=0}^{\infty} \langle f_t, B_t^k \rangle B_t^k = \sum_{k=0}^{\infty} K^k[f](0) B_t^k = \langle \mathcal{K}^n[f](t) \rangle_{n \in \mathbb{N}} = f_t.$$

Thus, while the coordinates of $f_t = \langle \mathcal{K}^n[f](t) \rangle_{n \in \mathbb{N}}$ in the usual base of l^2 vary with t , the coordinates of f_t in the bases $\{B_t^k\}_{k \in \mathbb{N}}$ remain the same as t varies.

We now show that $\{B_t^n\}_{n \in \mathbb{N}}$ is the moving frame of a helix $H : \mathbb{R} \mapsto l_2$.

Lemma 3.22 ([12]). *Let $f \in L_{\mathcal{M}}^2$ and let $t \in \mathbb{R}$ vary; then $\vec{f}(t) = \langle \mathcal{K}^n[f](t) \rangle_{n \in \mathbb{N}}$ is a continuous curve in l_2 .*

Proof. Let $f \in L_{\mathcal{M}}^2$; then, since $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ converges to a continuous (constant) function, by Dini's theorem, it converges uniformly on every finite interval I . Thus, the last two sums on the right side of inequality $\|f(t) - f(t+h)\|_{\mathcal{M}}^2 \leq \sum_{n=0}^N (\mathcal{K}^n[f](t) - \mathcal{K}^n[f](t+h))^2 + 2 \sum_{n=N+1}^{\infty} \mathcal{K}^n[f](t)^2 + 2 \sum_{n=N+1}^{\infty} \mathcal{K}^n[f](t+h)^2$ can be made arbitrarily small on I if N is sufficiently large. Since functions $\mathcal{K}^n[f](t)$ have continuous derivatives, they are uniformly continuous on I . Thus, $\sum_{n=0}^N (\mathcal{K}^n[f](t) - \mathcal{K}^n[f](t+h))^2$ can also be made arbitrarily small on I by taking $|h|$ sufficiently small. \square

Lemma 3.23 ([12]). *If $g' \in L_{\mathcal{M}}^2$, then $\lim_{|h| \rightarrow 0} \left\| \frac{g(t) - g(t+h)}{h} - g'(t) \right\|_{\mathcal{M}} = 0$; thus, the curve $\vec{g}(t) = \langle \mathcal{K}^n[g](t) \rangle_{n \in \mathbb{N}}$ is differentiable, and $(\vec{g})'(t) = \langle \mathcal{K}^n[g'](t) \rangle_{n \in \mathbb{N}}$.*

Proof. Let I be any finite interval; since $g' \in L_{\mathcal{M}}^2$, for every $\varepsilon > 0$ there exists N such that $\sum_{n=N+1}^{\infty} \mathcal{K}^n[g'](u)^2 < \varepsilon/8$ for all $u \in I$. Since functions $\mathcal{K}^n[g'](u)$ are uniformly continuous on I , there exists a $\delta > 0$ such that for all $t_1, t_2 \in I$, if $|t_1 - t_2| < \delta$ then $\sum_{n=0}^N (\mathcal{K}^n[g'](t_1) - \mathcal{K}^n[g'](t_2))^2 < \varepsilon/2$. Let h be an arbitrary number such that $|h| < \delta$; then for every t there exists a sequence of numbers ξ_n^t that lie between t and $t-h$, and such that $(\mathcal{K}^n[g](t) - \mathcal{K}^n[g](t-h))/h = \mathcal{K}^n[g'](\xi_n^t)$. Thus, for all $t \in I$,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{K}^n \left[\frac{g(t) - g(t-h)}{h} - g'(t) \right]^2 &= \sum_{n=0}^{\infty} (\mathcal{K}^n[g'](\xi_n^t) - \mathcal{K}^n[g'](t))^2 \\ &< \sum_{n=0}^N (\mathcal{K}^n[g'](\xi_n^t) - \mathcal{K}^n[g'](t))^2 + 2 \sum_{n=N+1}^{\infty} \mathcal{K}^n[g'](\xi_n^t)^2 \\ &\quad + 2 \sum_{n=N+1}^{\infty} \mathcal{K}^n[g'](t)^2 < \varepsilon/2 + 4\varepsilon/8 = \varepsilon. \end{aligned}$$

\square

Since $\mathcal{K}^n[\mathbf{m}](t) \in L_{\mathcal{M}}^2$ for all n , if we let $\vec{e}_{k+1}(t) = \langle (\mathcal{K}^k \circ \mathcal{K}^n)[\mathbf{m}](t) \rangle_{n \in \mathbb{N}}$ for $k \geq 0$, then by Lemma 3.23, $\vec{e}_k(t)$ are differentiable for all k . Since l_2 is complete and $\vec{e}_1(t)$

is continuous, $\vec{e}_1(t)$ has an antiderivative $\vec{H}(t)$. Using (26), we have

$$\begin{aligned}\vec{e}_1(t) &= \vec{H}'(t); \\ \vec{e}_1'(t) &= \langle\langle (D \circ \mathcal{K}^0 \circ \mathcal{K}^n)[\mathbf{m}](t) \rangle\rangle_{n \in \mathbb{N}} = \gamma_0 \langle\langle (\mathcal{K}^1 \circ \mathcal{K}^n)[\mathbf{m}](t) \rangle\rangle_{n \in \mathbb{N}} \\ &= \gamma_0 \vec{e}_2(t); \\ \vec{e}_k'(t) &= -\gamma_{k-2} \langle\langle (\mathcal{K}^{k-2} \circ \mathcal{K}^n)[\mathbf{m}](t) \rangle\rangle_{n \in \mathbb{N}} + \gamma_{k-1} \langle\langle (\mathcal{K}^k \circ \mathcal{K}^n)[\mathbf{m}](t) \rangle\rangle_{n \in \mathbb{N}} \\ &= -\gamma_{k-2} \vec{e}_{k-1}(t) + \gamma_{k-1} \vec{e}_{k+1}(t), \quad \text{for } k \geq 2.\end{aligned}$$

This means that the curve $\vec{H}(t)$ is a helix in l_2 because it has constant curvatures $\kappa_k = \gamma_{k-1}$ for all $k \geq 1$; the above equations are the corresponding Frenet–Serret formulas and $\vec{e}_{k+1}(t) = \langle\langle (\mathcal{K}^k \circ \mathcal{K}^n)[\mathbf{m}](t) \rangle\rangle_{n \in \mathbb{N}}$ for $k \geq 0$ form the orthonormal moving frame of the helix $\vec{H}(t)$.

4. EXAMPLES

We now present a few examples of chromatic derivatives and chromatic expansions, associated with several classical families of orthogonal polynomials. More details and more examples can be found in [9].

4.1. Example 1: Legendre polynomials/Spherical Bessel functions. Let $L_n(\omega)$ be the Legendre polynomials; if we set $P_n^L(\omega) = \sqrt{2n+1} L_n(\omega/\pi)$ then

$$\int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) \frac{d\omega}{2\pi} = \delta(m-n).$$

The corresponding recursion coefficients in equation (17) are given by the formula $\gamma_n = \pi(n+1)/\sqrt{4(n+1)^2-1}$; the corresponding space $L_{a(\omega)}^2$ is $L^2[-\pi, \pi]$. The space $\mathbf{L}_{\mathcal{M}}^2$ for this particular example consists of all entire functions whose restrictions to \mathbb{R} belong to L^2 and which have a Fourier transform supported in $[-\pi, \pi]$. Proposition 3.19 implies that in this case our locally defined scalar product $\langle f, g \rangle_{\mathcal{M}}$, norm $\|f\|_{\mathcal{M}}$ and convolution $(f *_{\mathcal{M}} g)(t)$ coincide with the usual scalar product, norm and convolution on L_2 .

4.2. Example 2: Chebyshev polynomials of the first kind/Bessel functions. Let $P_n^T(\omega)$ be the family of orthonormal polynomials obtained by normalizing and rescaling the Chebyshev polynomials of the first kind, $T_n(\omega)$, by setting $P_0^T(\omega) = 1$ and $P_n^T(\omega) = \sqrt{2} T_n(\omega/\pi)$ for $n > 0$. In this case

$$\int_{-\pi}^{\pi} P_n^T(\omega) P_m^T(\omega) \frac{d\omega}{\pi\sqrt{\pi^2-\omega^2}} = \delta(n-m).$$

By Proposition 3.15, the corresponding space $\mathbf{L}_{\mathcal{M}}^2$ contains all entire functions $f(t)$ which have a Fourier transform $\hat{f}(\omega)$ supported in $[-\pi, \pi]$ that also satisfies $\int_{-\pi}^{\pi} \sqrt{\pi^2-\omega^2} |\hat{f}(\omega)|^2 d\omega < \infty$. In this case the corresponding space $\mathbf{L}_{\mathcal{M}}^2$ contains functions which do not belong to L^2 ; the corresponding function (40) is $\mathbf{m}(z) = J_0(\pi z)$ and for $n > 0$, $\mathcal{K}^n[\mathbf{m}](z) = (-1)^n \sqrt{2} J_n(\pi z)$, where $J_n(z)$ is the Bessel function of the first kind of order n . In the recurrence relation (26) the coefficients are given by $\gamma_0 = \pi/\sqrt{2}$ and $\gamma_n = \pi/2$ for $n > 0$.

The chromatic expansion of a function $f(z)$ is the Neumann series of $f(z)$ (see [23]),

$$f(t) = f(u)J_0(\pi(z-u)) + \sqrt{2} \sum_{n=1}^{\infty} \mathcal{K}^n[f](u)J_n(\pi(z-u)).$$

Thus, the chromatic expansions corresponding to various families of orthogonal polynomials can be seen as generalizations of the Neumann series, while the families of corresponding functions $\{\mathcal{K}^n[\mathbf{m}](z)\}_{n \in \mathbb{N}}$ can be seen as generalizations and a uniform representation of some familiar families of special functions.

4.3. Example 3: Hermite polynomials/Gaussian monomial functions.

Let $H_n(\omega)$ be the Hermite polynomials; then polynomials $P_n^H(\omega) = (2^n n!)^{-1/2} H_n(\omega)$ satisfy

$$\int_{-\infty}^{\infty} P_n^H(\omega) P_m^H(\omega) e^{-\omega^2} \frac{d\omega}{\sqrt{\pi}} = \delta(n-m).$$

The corresponding space $\mathbf{L}_{\mathcal{M}}^2$ contains entire functions whose Fourier transform $\widehat{f}(\omega)$ satisfies $\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2 e^{\omega^2} d\omega < \infty$. In this case the space $L_{\mathcal{M}}^2$ contains non-bandlimited signals; the corresponding function defined by (40) is $\mathbf{m}(z) = e^{-z^2/4}$ and $\mathcal{K}^n[\mathbf{m}](z) = (-1)^n (2^n n!)^{-1/2} z^n e^{-z^2/4}$. The corresponding recursion coefficients are given by $\gamma_n = \sqrt{(n+1)/2}$. The chromatic expansion of $f(z)$ is just the Taylor expansion of $f(z) e^{z^2/4}$, multiplied by $e^{-z^2/4}$.

4.4. Example 4: Herron family. This example is a slight modification of an example from [9]. Let the family of orthonormal polynomials be given by the recursion $L_0(\omega) = 1$, $L_1(\omega) = \omega$, and $L_{n+1}(\omega) = \omega/(n+1)L_n(\omega) - n/(n+1)L_{n-1}(\omega)$. Then

$$\frac{1}{2} \int_{-\infty}^{\infty} L(m, \omega) L(n, \omega) \operatorname{sech}\left(\frac{\pi\omega}{2}\right) d\omega = \delta(m-n).$$

In this case $\mathbf{m}(z) = \operatorname{sech} z$ and $\mathcal{K}^n[\mathbf{m}](z) = (-1)^n \operatorname{sech} z \tanh^n z$. The recursion coefficients are given by $\gamma_n = n+1$ for all $n \geq 0$. If E_n are the Euler numbers, then $\operatorname{sech} z = \sum_{n=0}^{\infty} E_{2n} z^{2n} / (2n)!$, with the series converging only in the disc of radius $\pi/2$. Thus, in this case $\mathbf{m}(z)$ is not an entire function.

5. WEAKLY BOUNDED MOMENT FUNCTIONALS

5.1. To study local convergence of chromatic expansions of functions which are not in $L_{\mathcal{M}}^2$ we found it necessary to restrict the class of chromatic moment functionals. The restricted class, introduced in [12], is still very broad and contains functionals that correspond to many classical families of orthogonal polynomials. It consists of functionals such that the corresponding recursion coefficients $\gamma_n > 0$ appearing in (26) are such that sequences $\{\gamma_n\}_{n \in \mathbb{N}}$ and $\{\gamma_{n+1}/\gamma_n\}_{n \in \mathbb{N}}$ are bounded from below by a positive constant, and such that the growth rate of the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ is sub-linear in n . For technical simplicity in the definition below these conditions are formulated using a single constant M in all of the bounds.

Definition 5.1 ([12]). *Let \mathcal{M} be a moment functional such that for some $\gamma_n > 0$ (26) holds.*

(1) \mathcal{M} is weakly bounded if there exist some $M \geq 1$, some $0 \leq p < 1$ and some integer $r \geq 0$, such that for all $n \geq 0$,

$$(59) \quad \frac{1}{M} \leq \gamma_n \leq M(n+r)^p,$$

$$(60) \quad \frac{\gamma_n}{\gamma_{n+1}} \leq M^2.$$

(2) \mathcal{M} is bounded if there exists some $M \geq 1$ such that for all $n \geq 0$,

$$(61) \quad \frac{1}{M} \leq \gamma_n \leq M.$$

Since (26) is assumed to hold for some $\gamma_n > 0$, weakly bounded moment functionals are positive definite and symmetric. Every bounded functional \mathcal{M} is also weakly bounded with $p = 0$. Functionals in our Example 1 and Example 2 are bounded. For bounded moment functionals \mathcal{M} the corresponding moment distribution $a(\omega)$ has a finite support [2] and consequently $\mathbf{m}(t)$ is a band-limited signal. However, $\mathbf{m}(t)$ can be of infinite energy (i.e., not in L^2) as is the case in our Example 2. Moment functional in Example 3 is weakly bounded but not bounded ($p = 1/2$); the moment functional in Example 4 is not weakly bounded ($p = 1$). We note that important examples of classical orthogonal polynomials which correspond to weakly bounded moment functionals in fact satisfy a stronger condition from the following simple Lemma.

Lemma 5.2. *Let \mathcal{M} be such that (26) holds for some $\gamma_n > 0$. If for some $0 \leq p < 1$ the sequence γ_n/n^p converges to a finite positive limit, then \mathcal{M} is weakly bounded.*

Weakly bounded moment functionals allow a useful estimation of the coefficients in the corresponding equations (35) and (36) relating the chromatic and the “standard” derivatives.

Lemma 5.3 ([12]). *Assume that \mathcal{M} is such that for some $M \geq 1$, $r \geq 0$ and $p \geq 0$ the corresponding recursion coefficients γ_n for all n satisfy inequalities (59). Then the following inequalities hold for all k and n :*

$$(62) \quad |(\mathcal{K}^n \circ D^k)[\mathbf{m}](0)| \leq (2M)^k (k+r)!^p;$$

$$(63) \quad \left| \mathcal{K}^n \left[\frac{t^k}{k!} \right] (0) \right| \leq (2M)^n.$$

Proof. By (30), it is enough to prove (62) for all n, k such that $n \leq k$. We proceed by induction on k , assuming the statement holds for all $n \leq k$. Applying (26) to $D^k[\mathbf{m}](t)$ we get

$$|(\mathcal{K}^n \circ D^{k+1})[\mathbf{m}](t)| \leq \gamma_n |(\mathcal{K}^{n+1} \circ D^k)[\mathbf{m}](t)| + \gamma_{n-1} |(\mathcal{K}^{n-1} \circ D^k)[\mathbf{m}](t)|.$$

Using the induction hypothesis and (30) again, we get for all $n \leq k+1$,

$$\begin{aligned} |(\mathcal{K}^n \circ D^{k+1})[\mathbf{m}](0)| &< (M(k+1+r)^p + M(k+r)^p)(2M)^k (k+r)!^p \\ &< (2M)^{k+1} (k+1+r)!^p. \end{aligned}$$

Similarly, by (37), it is enough to prove (63) for all $k \leq n$. This time we proceed by induction on n and use (26), (59) and (60) to get

$$\left| \mathcal{K}^{n+1} \left[\frac{t^k}{k!} \right] \right| \leq M \left| \mathcal{K}^n \left[\frac{t^{k-1}}{(k-1)!} \right] \right| + M^2 \left| \mathcal{K}^{n-1} \left[\frac{t^k}{k!} \right] \right|.$$

By induction hypothesis and using (37) again, we get that for all $k \leq n + 1$, $\left| \mathcal{K}^{n+1} \left[\frac{\mu^k}{k!} \right] (0) \right| < M (2M)^n + M^2 (2M)^{n-1} < (2M)^{n+1}$. \square

Corollary 5.4 ([12]). *Let \mathcal{M} be weakly bounded; then for every fixed n*

$$\lim_{k \rightarrow \infty} \left| (\mathcal{K}^n \circ D^k) [\mathbf{m}] (0) / k! \right|^{1/k} = 0,$$

and the convergence is uniform in n .

Proof. Let $R(k) = (k + r)! / k!$; then $R(k)$ is a polynomial of degree r , and, by (62),

$$(64) \quad \left| \frac{(\mathcal{K}^n \circ D^k) [\mathbf{m}] (0)}{k!} \right|^{1/k} \leq \frac{2MR(k)^{p/k}}{k^{!(1-p)/k}} < \frac{2Me^{1-p} R(k)^{p/k}}{k^{1-p}}.$$

\square

Corollary 5.5 ([12]). *Let $\mathbf{m}(z)$ correspond to a weakly bounded moment functional \mathcal{M} ; then*

$$(65) \quad \lim_{k \rightarrow \infty} \left(\frac{\mu_k}{k!} \right)^{1/k} = \lim_{k \rightarrow \infty} \left| \frac{\mathbf{m}^k(0)}{k!} \right|^{1/k} = 0.$$

Thus, since (38) is satisfied with $\rho = 0$, every weakly bounded moment functional is chromatic.

Note that this and Proposition 3.5 imply that $\mathbf{m}(z)$ is an entire function. If (59) holds with $p = 1$, then Lemma 5.3 implies only

$$(66) \quad \limsup_{k \rightarrow \infty} \left| \frac{(\mathcal{K}^n \circ D^k) [\mathbf{m}] (0)}{k!} \right|^{1/k} \leq 2M.$$

Example 4 shows that in this case the corresponding function $\mathbf{m}(z)$ need not be entire. Thus, if we are interested in chromatic expansions of entire functions, the upper bound in (59) of the definition of a weakly bounded moment functional is sharp.

Lemma 5.3 and Proposition 3.7 imply the following Corollary.

Corollary 5.6 ([12]). *If \mathcal{M} is weakly bounded, then the corresponding family of polynomials $\{P_n^{\mathcal{M}}(\omega)\}_{n \in \mathbb{N}}$ is a complete system in $L^2_{a(\omega)}$.*

Thus, we get that the Chebyshev, Legendre, Hermite and similar classical families of orthogonal polynomials are complete in their corresponding spaces $L^2_{a(\omega)}$.

To simplify our estimates, we choose $K \geq 1$ such that for p , M and r as in Definition 5.1 for all $k > 0$, we have

$$(67) \quad \frac{(2M)^k (k + r)!^p}{k!^p} < K^k.$$

The following Lemma slightly improves a result from [12].

Lemma 5.7. *Let \mathcal{M} be weakly bounded and $p < 1$ and $K \geq 1$ such that (59) and (67) hold. Let also k be an integer such that $k \geq 1/(1 - p)$. Then there exists a polynomial $P(x)$ of degree $k - 1$ such that for every n and every $z \in \mathbb{C}$,*

$$(68) \quad |\mathcal{K}^n [\mathbf{m}] (z)| < \frac{|Kz|^n}{n!^{1-p}} P(|z|) e^{|Kz|^k}.$$

Proof. Using the Taylor series for $\mathcal{K}^n[\mathbf{m}](z)$, (30), (62) and (67), we get that for z such that $|Kz| \geq 1$,

$$\begin{aligned} |\mathcal{K}^n[\mathbf{m}](z)| &< \sum_{m=0}^{\infty} \frac{|Kz|^{n+m}}{(n+m)!^{1-p}} \leq \frac{|Kz|^n}{n!^{1-p}} \sum_{m=0}^{\infty} \frac{|Kz|^m}{m!^{1/k}} \\ &< \frac{|Kz|^n}{n!^{1-p}} \sum_{m=0}^{\infty} \frac{|Kz|^{k\lfloor m/k \rfloor + k-1}}{\lfloor m/k \rfloor!} \\ &= k \frac{|Kz|^{n+k-1}}{n!^{1-p}} \sum_{j=0}^{\infty} \frac{|Kz|^{kj}}{j!} \\ &= k \frac{|Kz|^{n+k-1}}{n!^{1-p}} e^{|Kz|^k}. \end{aligned}$$

If $|Kz| < 1$, then a similar calculation shows that for such z we have $|\mathcal{K}^n[\mathbf{m}](z)| < k e |Kz|^n / n!^{1-p}$. The claim now follows with $P(|z|) = k(|Kz|^{k-1} + e)$. \square

5.2. Local convergence of chromatic expansions.

Proposition 5.8. *Let \mathcal{M} be weakly bounded, $p < 1$ as in (59), $f(z)$ a function analytic on a domain $G \subseteq \mathbb{C}$ and $u \in G$.*

- (1) *If the sequence $|\mathcal{K}^n(u)/n!^{1-p}|^{1/n}$ is bounded, then the chromatic expansion $\text{CE}^{\mathcal{M}}[f, u](z)$ of $f(z)$ converges uniformly to $f(z)$ on a disc $D \subseteq G$, centered at u .*
- (2) *In particular, if $|\mathcal{K}^n(u)/n!^{1-p}|^{1/n}$ converges to zero, then the chromatic expansion $\text{CE}^{\mathcal{M}}[f, u](z)$ of $f(z)$ converges for every $z \in G$ and the convergence is uniform on every finite closed disc around u , contained in G .*

Proof. Assume that R is such that $\limsup_{n \rightarrow \infty} |\mathcal{K}^n[f](u)/n!^{1-p}|^{1/n} < R$. Then $|\mathcal{K}^n[f](u)| < R^n n!^{1-p}$ for all sufficiently large n . Let K and k be such that (68) holds; then Lemma 5.7 implies that for all sufficiently large n ,

$$|\mathcal{K}^n[f](u) \mathcal{K}^n[\mathbf{m}](z-u)|^{1/n} < RK(P(|z-u|)e^{K(z-u)^k})^{1/n} |z-u|.$$

Thus, the chromatic series converges uniformly inside every disc $D \subseteq G$, centered at u , of radius less than $1/(RK)$. Since

$$\mathcal{K}^j[\text{CA}^{\mathcal{M}}[f, u](z)] \Big|_{z=u} = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](u) (\mathcal{K}^j \circ \mathcal{K}^n)[\mathbf{m}](0) = \mathcal{K}^j[f](u),$$

$\text{CA}^{\mathcal{M}}[f, u](z)$ converges to $f(z)$ on D . \square

Lemma 5.9. *Let M be as in Definition 5.1. Then*

$$\limsup_{n \rightarrow \infty} \left| \frac{\mathcal{K}^n[f](u)}{n!^{1-p}} \right|^{1/n} \leq 2M \limsup_{n \rightarrow \infty} \left| \frac{f^{(n)}(u)}{n!^{1-p}} \right|^{1/n}.$$

Proof. Let $\beta > 0$ be any number such that $\limsup_{n \rightarrow \infty} |f^{(n)}(u)/n!^{1-p}|^{1/n} < \beta$; then there exists $B_\beta \geq 1$ such that $|f^{(k)}(u)| \leq B_\beta k!^{1-p} \beta^k$ for all k . Using (36)

and (63) we get

$$\begin{aligned} |\mathcal{K}^n[f](u)| &\leq \sum_{k=0}^n \left| \mathcal{K}^n \left[\frac{t^k}{k!} \right] (0) \right| |f^{(k)}(u)| \leq (2M)^n B_\beta \sum_{k=0}^n k!^{1-p} \beta^k \\ &< (2M)^n B_\beta \sum_{k=0}^n \left(\frac{n}{e} \right)^{(k+1)(1-p)} \beta^k. \end{aligned}$$

Summation of the last series shows that $|\mathcal{K}^n[f](u)| < 2B_\beta(2M\beta)^n n!^{1-p}$ for sufficiently large n . \square

Corollary 5.10. *Let \mathcal{M} be weakly bounded, $p < 1$ as in (59), $f(z)$ a function analytic on a domain $G \subseteq \mathbb{C}$ and $u \in G$.*

- (1) *If the sequence $|f^{(n)}(u)/n!^{1-p}|^{1/n}$ is bounded, then the chromatic expansion $\text{CE}^{\mathcal{M}}[f, u](z)$ of $f(z)$ converges uniformly to $f(z)$ on a disc $D \subseteq G$ centered at u .*
- (2) *In particular, if $|f^{(n)}(u)/n!^{1-p}|^{1/n}$ converges to zero, then the chromatic expansion $\text{CE}^{\mathcal{M}}[f, u](z)$ of $f(z)$ converges for all $z \in G$ and the convergence is uniform on every closed disc around u , contained in G .*

Corollary 5.11 ([12]). *If \mathcal{M} is bounded, then for every entire function f and all $u, z \in \mathbb{C}$, the chromatic expansion $\text{CE}[f, u](z)$ converges to $f(z)$ for all z , and the convergence is uniform on every disc around u of finite radius.*

Proof. If $f(z)$ is entire, then for every u , $\lim_{n \rightarrow \infty} |f^{(n)}(u)/n!|^{1/n} = 0$. The Corollary now follows from Corollary 5.10 with $p = 0$. \square

Proposition 5.12. *Assume \mathcal{M} is weakly bounded and let $0 \leq p < 1$ be such that (59) holds, and k such that $k \geq 1/(1-p)$. Then there exists $C, L > 0$ such that $|f(z)| \leq C \|f\|_{\mathcal{M}} e^{L|z|^k}$ for all $f(z) \in \mathbf{L}_{\mathcal{M}}^2$.*

Proof. Since $f(z) \in \mathbf{L}_{\mathcal{M}}^2$, the chromatic expansion of $f(z)$ and (68) yield

$$\begin{aligned} |f(z)| &\leq \left(\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 \sum_{n=0}^{\infty} |\mathcal{K}^n[\mathbf{m}](z)|^2 \right)^{1/2} \\ &\leq \|f\|_{\mathcal{M}} P(|z|) e^{K|z|^k} \left(\sum_{n=0}^{\infty} \frac{|Kz|^{2n}}{n!^{2(1-p)}} \right)^{1/2}, \end{aligned}$$

which, using the method from the proof of Lemma 5.7, can easily be shown to imply our claim. \square

Note that for bounded moment functionals, such as those corresponding to the Legendre or the Chebyshev polynomials, we have $p = 0$; thus, Proposition 5.12 implies that functions which are in $\mathbf{L}_{\mathcal{M}}^2$ are of exponential type. For \mathcal{M} corresponding to the Hermite polynomials $p=1/2$ (see Example 3); thus, we get that there exists $C, L > 0$ such that $|f(z)| \leq C \|f\|_{\mathcal{M}} e^{L|z|^2}$ for all $f \in \mathbf{L}_{\mathcal{M}}^2$. It would be interesting to establish when the reverse implication is true and thus obtain a generalization of the Paley-Wiener Theorem for functions satisfying $|f(z)| < Ce^{L|z|^k}$ for $k > 1$.

5.3. Generalizations of some classical equalities for the Bessel functions.

Corollaries 5.11 and 5.10 generalize the classic result that every entire function can be expressed as a Neumann series of Bessel functions [23], by replacing the Neumann series with a chromatic expansion that corresponds to any (weakly) bounded moment functional. Thus, many classical results on Bessel functions from [23] immediately follow from Corollary 5.11, and, using Corollary 5.10, generalize to functions $\mathcal{K}^n[\mathbf{m}](z)$ corresponding to any weakly bounded moment functional \mathcal{M} . Below we give a few illustrative examples.

Corollary 5.13. *Let $P_n^{\mathcal{M}}(\omega)$ be the orthonormal polynomials associated with a weakly bounded moment functional \mathcal{M} ; then for every $z \in \mathbb{C}$,*

$$(69) \quad e^{i\omega z} = \sum_{n=0}^{\infty} i^n P_n^{\mathcal{M}}(\omega) \mathcal{K}^n[\mathbf{m}](z).$$

Proof. If $p < 1$ then

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{\left| \frac{d^n}{dz^n} e^{i\omega z} \right|_{z=0}}}{n^{1-p}} = \lim_{n \rightarrow \infty} \frac{|\omega|}{n^{1-p}} = 0,$$

and the claim follows from Proposition 5.10 and (27). \square

Corollary 5.13 generalizes the well known equality for the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions $J_n(z)$, i.e.,

$$e^{i\omega z} = J_0(z) + 2 \sum_{n=1}^{\infty} i^n T_n(\omega) J_n(z).$$

In Example 3, 5.13 becomes the equality for the Hermite polynomials $H_n(\omega)$:

$$e^{i\omega z} = \sum_{n=0}^{\infty} \frac{H_n(\omega)}{n!} \left(\frac{iz}{2} \right)^n e^{-\frac{z^2}{4}}.$$

By applying Corollary 5.10 to the constant function $f(t) \equiv 1$, we get that its chromatic expansion yields that for all $z \in \mathbb{C}$

$$\mathbf{m}(z) + \sum_{n=1}^{\infty} \left(\prod_{k=1}^n \frac{\gamma_{2k-2}}{\gamma_{2k-1}} \right) \mathcal{K}^{2n}[\mathbf{m}](z) = 1,$$

with γ_n the recursion coefficients from (17). This equality generalizes the equality

$$J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) = 1.$$

Using Proposition 3.16 to expand $\mathbf{m}(z+u) \in \mathbf{L}_{\mathcal{M}}^2$ into chromatic series around $z=0$, we get that for all $z, u \in \mathbb{C}$

$$\mathbf{m}(z+u) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[\mathbf{m}](u) \mathcal{K}^n[\mathbf{m}](z),$$

which generalizes the equality

$$J_0(z+u) = J_0(u) J_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n J_n(u) J_n(z).$$

6. SOME NON-SEPARABLE SPACES

6.1. Let \mathcal{M} be weakly bounded; then periodic functions do not belong to $L_{\mathcal{M}}^2$ because $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ diverges. We now introduce some nonseparable inner product spaces in which pure harmonic oscillations have finite norm and are pairwise orthogonal.

Note: *In the remainder of this paper we consider only weakly bounded moment functionals and real functions which are restrictions of entire functions.*

Definition 6.1. *Let let $0 \leq p < 1$ be as in (59). We denote by $\mathcal{C}^{\mathcal{M}}$ the vector space of functions such that the sequence*

$$(70) \quad \nu_n^f(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^n \mathcal{K}^k[f](t)^2$$

converges uniformly on every finite interval $I \subset \mathbb{R}$.

Proposition 6.2. *Let $f, g \in \mathcal{C}^{\mathcal{M}}$ and*

$$(71) \quad \sigma_n^{fg}(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^n \mathcal{K}^k[f](t) \mathcal{K}^k[g](t);$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n \in \mathbb{N}}$ converges to a constant function. In particular, $\{\nu_n^f(t)\}_{n \in \mathbb{N}}$ also converges to a constant function.

Proof. Since $\nu_n^f(t)$ and $\nu_n^g(t)$ given by (70) converge uniformly on every finite interval, the same holds for the sequence $\sigma_n^{fg}(t)$. Consequently, it is enough to show that for all t , the derivative $\sigma_n^{fg}(t)'$ of $\sigma_n^{fg}(t)$ satisfies $\lim_{n \rightarrow \infty} \sigma_n^{fg}(t)' = 0$. Let

$$S_k(t) = \mathcal{K}^k[f](t)^2 + \mathcal{K}^{k+1}[f](t)^2 + \mathcal{K}^k[g](t)^2 + \mathcal{K}^{k+1}[g](t)^2;$$

then, since $f, g \in \mathcal{C}^{\mathcal{M}}$, the sequence $1/(n+1)^{1-p} \sum_{k=0}^n S_k(t)$ converges everywhere to some $\alpha(t)$. We now show that if t is such that $\alpha(t) > 0$, then there are infinitely many k such that $S_k(t) < 2\alpha(t)k^{-p}$. Assume opposite, and let K be such that $S_k(t) \geq 2\alpha(t)k^{-p}$ for all $k \geq K$. Then, since

$$\sum_{k=K}^n k^{-p} > \int_K^{n+1} x^{-p} dx = \frac{(n+1)^{1-p} - K^{1-p}}{1-p},$$

we would have that for all $n > K$,

$$\frac{\sum_{k=K}^n S_k(t)}{(n+1)^{1-p}} \geq \frac{2\alpha(t) \sum_{k=K}^n k^{-p}}{(n+1)^{1-p}} > \frac{2\alpha(t)((n+1)^{1-p} - K^{1-p})}{(n+1)^{1-p}(1-p)}.$$

However, since $0 \leq p < 1$, this would imply $\sum_{k=0}^n S_k(t)/(n+1)^{1-p} > \alpha(t)$ for all sufficiently large n , which contradicts the definition of $\alpha(t)$. Consequently, for infinitely many n all four summands in $S_n(t)$ must be smaller than $2\alpha(t)n^{-p}$. For those values of n we have

$$|\mathcal{K}^{n+1}[f](t) \mathcal{K}^n[g](t)| + |\mathcal{K}^n[f](t) \mathcal{K}^{n+1}[g](t)| < 4\alpha(t)n^{-p}.$$

Since \mathcal{M} is weakly bounded, (31) and (59) imply that for some $M \geq 1$ and an integer r ,

$$|\sigma_n^{fg}(t)'| < \frac{M(n+r)^p}{(n+1)^{1-p}} (|\mathcal{K}^{n+1}[f](t) \mathcal{K}^n[g](t)| + |\mathcal{K}^n[f](t) \mathcal{K}^{n+1}[g](t)|).$$

Thus, for infinitely many n we have

$$|\sigma_n^{fg}(t)'| < \frac{4M(n+r)^p n^{-p} \alpha(t)}{(n+1)^{1-p}}.$$

Consequently, $\liminf_{n \rightarrow \infty} |\sigma_n^{fg}(t)'| = 0$ and since $\lim_{n \rightarrow \infty} \sigma_n^{fg}(t)'$ exists, it must be equal to zero. \square

Corollary 6.3. *Let $\mathcal{C}_0^{\mathcal{M}}$ be the vector space consisting of functions $f(t)$ such that $\lim_{n \rightarrow \infty} \nu_n^f(t) = 0$; then in the quotient space $\mathcal{C}_2^{\mathcal{M}} = \mathcal{C}^{\mathcal{M}}/\mathcal{C}_0^{\mathcal{M}}$ we can introduce a scalar product by the following formula whose right hand side is independent of t :*

$$(72) \quad \langle f, g \rangle^{\mathcal{M}} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^n \mathcal{K}^k[f](t) \mathcal{K}^k[g](t).$$

The corresponding norm on $\mathcal{C}_2^{\mathcal{M}}$ is denoted by $\|\cdot\|^{\mathcal{M}}$. Clearly, all real valued functions from $L_{\mathcal{M}}^2$ belong to $\mathcal{C}_0^{\mathcal{M}}$.

Proposition 6.4. *If $f \in \mathcal{C}_2^{\mathcal{M}}$, then the chromatic expansion of $f(t)$ converges to $f(t)$ for every t and the convergence is uniform on every finite interval.*

Proof. Since $1/(n+1)^{1-p} \sum_{k=0}^n \mathcal{K}^k[f](t)^2$ converges to $0 < (\|f\|^{\mathcal{M}})^2 < \infty$, for all sufficiently large n ,

$$\begin{aligned} |\mathcal{K}^n[f](t)|^{1/n} &\leq \left(\sum_{k=0}^n \mathcal{K}^k[f](t)^2 \right)^{1/(2n)} \\ &\leq (2\|f\|^{\mathcal{M}})^{1/n} (n+1)^{(1-p)/(2n)}. \end{aligned}$$

Thus, $|\mathcal{K}^n[f](t)/n!^{1-p}|^{1/n} \rightarrow 0$, and the claim follows from Proposition 5.8. \square

Since $\mathcal{K}^n[\mathbf{m}](t) \in L_{\mathcal{M}}^2$, we have $\left\| \sum_{j=0}^n (-1)^j \mathcal{K}^j[f](0) \mathcal{K}^j[\mathbf{m}](t) \right\|^{\mathcal{M}} = 0$ for all n . Thus, the chromatic expansion of $f \in \mathcal{C}_2^{\mathcal{M}}$ does not converge to $f(t)$ in $\mathcal{C}_2^{\mathcal{M}}$. Moreover, there can be no such series representation of functions $f \in \mathcal{C}_2^{\mathcal{M}}$, converging in $\mathcal{C}_2^{\mathcal{M}}$, because the space $\mathcal{C}_2^{\mathcal{M}}$ is in general nonseparable, as the remaining part of this paper shows.

6.2. Space $\mathcal{C}_2^{\mathcal{M}}$ associated with the Chebyshev polynomials (Example 2).

For this case the corresponding space $\mathcal{C}_2^{\mathcal{M}}$ will be denoted by \mathcal{C}_2^T , and in (59) we have $p = 0$. Thus, the scalar product on \mathcal{C}_2^T is defined by

$$\langle f, g \rangle^T = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n \mathcal{K}^k[f](t) \mathcal{K}^k[g](t).$$

Proposition 6.5. *Functions $f_\omega(t) = \sqrt{2} \sin \omega t$ and $g_\omega(t) = \sqrt{2} \cos \omega t$ for $0 < \omega < \pi$ form an orthonormal system of vectors in \mathcal{C}_2^T .*

Proof. From (27) we get

$$(73) \quad \begin{aligned} \langle f_\omega, f_\sigma \rangle^{\mathcal{M}} &= \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n P_{2k}^T(\omega) P_{2k}^T(\sigma) \sin \omega t \sin \sigma t}{2n+1} \\ &+ \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} P_{2k+1}^T(\omega) P_{2k+1}^T(\sigma) \cos \omega t \cos \sigma t}{2n+1}. \end{aligned}$$

Since $P_n^T(\omega) \leq \sqrt{2}$ on $(0, \pi)$, (18) implies that for $\omega, \sigma \in (0, \pi)$ and $\omega \neq \sigma$,

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n P_k^T(\omega) P_k^T(\sigma)}{n+1} = \lim_{n \rightarrow \infty} \frac{P_{n+1}^T(\omega) P_n^T(\sigma) - P_{n+1}^T(\sigma) P_n^T(\omega)}{2(n+1)} = 0.$$

Since $P_{2n}^T(\omega)$ are even functions and $P_{2n+1}^T(\omega)$ odd, this also implies that

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n P_{2k}^T(\omega) P_{2k}^T(\sigma)}{2n+1} = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} P_{2k+1}^T(\omega) P_{2k+1}^T(\sigma)}{2n+1} = 0.$$

Thus, by (73), $\langle f_\omega, f_\sigma \rangle^{\mathcal{M}} = 0$. Using (20), one can verify that for $0 < \omega < \pi$

$$\frac{1}{n+1} \sum_{k=0}^n P_k^T(\omega)^2 = \frac{1+2n}{2n+2} + \frac{\sin((2n+1) \arccos \omega)}{(2n+2)\sqrt{1-\omega^2}} \rightarrow 1.$$

Thus, (19) implies that for $0 < \omega < \pi$

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{k=0}^n P_{2k}^T(\omega)^2 = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{k=0}^n P_{2k+1}^T(\omega)^2 = \frac{1}{2}$$

Consequently, $\|\sqrt{2} \sin \omega t\|^{\mathcal{M}} = 1$. \square

6.3. Space $\mathcal{C}_2^{\mathcal{M}}$ associated with the Hermite polynomials (Example 3).

The corresponding space $\mathcal{C}_2^{\mathcal{M}}$ in this case is denoted by \mathcal{C}_2^H , and in (59) we have $p = 1/2$. Thus, the scalar product in \mathcal{C}_2^H is defined by

$$\langle f, g \rangle^H = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^n \mathcal{K}^k[f](t) \mathcal{K}^k[g](t).$$

Proposition 6.6. *For all $\omega > 0$ functions $f_\omega(t) = \sin \omega t$ and $g_\omega(t) = \cos \omega t$ form an orthogonal system in \mathcal{C}_2^H , and $\|f_\omega\|^{\mathcal{M}} = \|g_\omega\|^{\mathcal{M}} = e^{\omega^2/2}/\sqrt[4]{2\pi}$.*

Proof. For all ω and for $n \rightarrow \infty$,

$$P_n^H(\omega) - \frac{\Gamma(n+1)^{\frac{1}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)} e^{\frac{\omega^2}{2}} \cos\left(\sqrt{2n+1} \omega - \frac{n\pi}{2}\right) \rightarrow 0;$$

see, for example, 8.22.8 in [18]. Using the Stirling formula we get

$$(74) \quad P_n^H(\omega) - \left(\frac{2}{\pi}\right)^{\frac{1}{4}} n^{-\frac{1}{4}} e^{\frac{\omega^2}{2}} \cos\left(\sqrt{2n+1} \omega - \frac{n\pi}{2}\right) \rightarrow 0.$$

This fact and (18) are easily seen to imply that $\langle f_\omega, f_\sigma \rangle^{\mathcal{M}} = 0$ for all distinct $\omega, \sigma > 0$, while (74) and (19) imply that

$$(75) \quad \lim_{n \rightarrow \infty} \left(\frac{\sum_{k=0}^n P_{2k}^H(\omega)^2}{\sqrt{2n+1}} - \frac{\sum_{k=0}^{n-1} P_{2k+1}^H(\omega)^2}{\sqrt{2n+1}} \right) = 0.$$

Since $H'_n(\omega) = 2n H_{n-1}(\omega)$, we have $P'_n(\omega) = \sqrt{2n} P_{n-1}^H(\omega)$. Using this, (74) and (20) one can verify that

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n P_k^H(\omega)^2}{\sqrt{n+1}} = \sqrt{\frac{2}{\pi}} e^{\omega^2},$$

which, together with (75), implies that $\|f_\omega\|^{\mathcal{M}} = e^{\omega^2/2}/\sqrt[4]{2\pi}$. \square

Note that in this case, unlike the case of the family associated with the Chebyshev polynomials, the norm of a pure harmonic oscillation of unit amplitude depends on its frequency.

One can verify that propositions similar to Proposition 6.5 and Proposition 6.6 hold for other classical families of orthogonal polynomials, such as the Legendre polynomials. Our numerical tests indicate that the following conjecture is true.⁷

Conjecture 6.7. *Assume that for some $p < 1$ the recursion coefficients γ_n in (17) are such that γ_n/n^p converges to a finite positive limit. Then, for the corresponding family of orthogonal polynomials we have*

$$(76) \quad 0 < \lim_{n \rightarrow \infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^n P_k^{\mathcal{M}}(\omega)^2 < \infty$$

for all ω in the support $sp(a)$ of the corresponding moment distribution function $a(\omega)$. Thus, in the corresponding space $\mathcal{C}_2^{\mathcal{M}}$ all pure harmonic oscillations with positive frequencies ω belonging to the support of the moment distribution $a(\omega)$ have finite positive norm and are mutually orthogonal.

Note that (20) implies that (76) is equivalent to

$$0 < \lim_{n \rightarrow \infty} \frac{P_{n+1}^{\mathcal{M}}(\omega)' P_n^{\mathcal{M}}(\omega) - P_{n+1}^{\mathcal{M}}(\omega) P_n^{\mathcal{M}}(\omega)'}{(n+1)^{1-2p}} < \infty.$$

7. REMARKS

The special case of the chromatic derivatives presented in Example 2 were first introduced in [11]; the corresponding chromatic expansions were subsequently introduced in [14]. These concepts emerged in the course of the author's design of a pulse width modulation power amplifier. The research team of the author's startup, *Kromos Technology Inc.*, extended these notions to various systems corresponding to several classical families of orthogonal polynomials [5, 9]. We also designed and implemented a channel equalizer [8] and a digital transceiver (unpublished), based on chromatic expansions. A novel image compression method motivated by chromatic expansions was developed in [3, 4]. In [6] chromatic expansions were related to the work of Papoulis [17] and Vaidyanathan [19]. In [15] and [20] the theory was cast in the framework commonly used in signal processing. Chromatic expansions were also studied in [5], [1] and [22]. Local convergence of chromatic expansions was studied in [12]; local approximations based on trigonometric functions were introduced in [13]. A generalization of chromatic derivatives, with the prolate spheroidal wave functions replacing orthogonal polynomials, was introduced in [21]; the theory was also extended to the prolate spheroidal wavelet series that combine chromatic series with sampling series.

Note: Some Kromos technical reports and some manuscripts can be found at the author's web site <http://www.cse.unsw.edu.au/~ignjat/diff/>.

⁷We have tested this Conjecture numerically, by setting $\gamma_n = n^p$ for several values of $p < 1$, and in all cases a finite limit appeared to exist. Paul Nevai has informed us that the special case of this Conjecture for $p = 0$ is known as Nevai-Totik Conjecture, and is still open.

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SCHOOL OF COMPUTER SCIENCE AND ENGINEERING, UNIVERSITY OF NEW SOUTH WALES and NATIONAL ICT AUSTRALIA (NICTA), SYDNEY, NSW 2052, AUSTRALIA

E-mail address: ignjat@cse.unsw.edu.au

URL: www.cse.unsw.edu.au/~ignjat