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HILBERT'S PROGRAM AND THE OMEGA-RULE

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Abstract. In the first part of this paper we discuss some aspects of Detlefsen's attempt to save Hilbert's Program from the consequences of Gödel's Second Incompleteness Theorem. His arguments are based on his interpretation of the long standing and well-known controversy on what, exactly, finitistic means are. In his paper [1] Detlefsen takes the position that there is a form of the ω-rule which is a finitistically valid means of proof, sufficient to prove the consistency of elementary number theory Z. On the other hand, he claims that Z with its first-order logic is not strong enough to allow a formalization of such an ω-rule. This would explain why the unprovability of $\mathrm{Con}(Z)$ in Z does not imply that the consistency of Z cannot be proved by finitistic means. We show that Detlefsen's proposal is unacceptable as originally formulated in [1], but that a reasonable modification of the rule he suggest leads to a partial program already studied for many years. We investigate the scope of such a program in terms of proof-theoretic reducibilities. We also show that this partial program encompasses mathematically important theories studied in the "Reverse Mathematics" program. In order to investigate the provability with such a modified rule, we define new consistency and provability predicates which are weaker than the usual ones. We then investigate their properties, including a few that have no apparent philosophical significance but compare interestingly with the properties of the program based on the iteration of our ω-rule. We determine some of the limitations of such programs, pointing out that these limitations partly explain why partial programs that have been successfully carried out use quite different and substantially more radical extensions of finitistic methods with more general forms of restricted reasoning.

§**1. Introduction.** One of the most important parts of Hilbert's Program was proving the consistency of formal theories that correspond to the theories of mathematical practice using only restricted "finitistic" means. As Hilbert himself stated (see [10]), this was supposed to "establish once and for all the certitude of mathematical methods". Nevertheless, he never specified exactly what finitistic means are. Apparently, he believed that once such consistency proofs were achieved, everyone would recognize the means used in these proofs as finitistically valid. It is usually assumed that all finitistic means are formalizable in Peano's Arithmetic (PA), or even in Primitive Recursive Arithmetic (PRA), which is an equational theory with no quantifiers in the language. If we turn it into a first-order theory by adding the first order logic, then we get a conservative extension

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of (PRA) usually denoted[2] by $(QF - IA)$. Since there is a finitistic procedure[3] for transforming a proof in $(QF - IA)$ of a $\prod_1^0$ sentence into a proof in (PRA) of the corresponding open formula, provability in $(QF - IA)$ and in (PRA) (and consequently also consistency of these theories) are equivalent even for a finitist. Thus, we can use throughout this paper $(QF - IA)$ and $\prod_1$-sentences instead of (PRA) and open formulas. Tait [25] has argued that the finitistically acceptable functions are just those defined by primitive recursion as well as that a proposition of the form $f(x) = g(x)$, for f, g defined by primitive recursion, is finitistically provable if and only if it is provable in (PRA); we will call this claim "Tait's Thesis". Thus, according to Tait's Thesis, a proposition of the form $f(x) = g(x)$ is finitistically provable if and only if its universal closure is provable in $(QF - IA)$. We do not think that Tait's analysis delimits finitistic means beyond any doubt, but it is certainly a quite tenable working hypothesis. Henceforth, we will use the words "finitistic" and "finitistically" with this meaning.[4] By Gödel's Second Incompleteness Theorem we know that no consistent primitive recursively axiomatized theory T extending (or, more generally, interpreting a sufficiently strong fragment of) $(QF - IA)$ can prove the statement $\forall x \neg \mathrm{Prf}_T(x, \ulcorner 1 = 0 \urcorner)$ "saying that T is consistent, where $\mathrm{Prf}_T(x, y)$ is the standard, primitive recursive proof predicate for T. Thus, if we accept Tait's Thesis, the consistency of any such theory T is not provable by purely finitistic means, since otherwise the proof could be formalized in $(QF - IA)$ and would yield a proof p in $(QF - IA)$ of $\forall x \neg \mathrm{Prf}_T(x, \ulcorner 1 = 0 \urcorner)$ which is impossible. By Gödel's Theorem not only is this statement unprovable in $(QF - IA)$, but also in the (possibly stronger) theory T.

In order to overcome this difficulty, one could accept more powerful methods in consistency proofs as perhaps nonfinitistic, but still constructive: transfinite induction up to ε_0 for a primitive recursive well-ordering applied only to primitive recursive predicates suffices to show the consistency of (PA). Unfortunately, the epistemlogical value of such a consistency proof seems more doubtful than the value of a purely finitistic consistency proof; by Gödel's Second Incompleteness Theorem transfinite induction up to ε_0 is not justifiable on equal grounds as, say, truth of an instance of the induction schema $QF - IA$, and even truth of an instance of the induction schema for formulas of arbitrary complexity.[5]

Rather than look for stronger yet finitestically acceptable mathematical principles, to be employed in finitistic consistency proofs, another way out would be to accept more general and yet finitistically valid forms of logical inference.[6]. Hilbert

[2] See, for example, [21].

[3] This procedure is based on the cut elimination technique; for details see again [21].

[4] Note that we say "defined by primitive recursion" rather than just "primitive recursive". For a finitist it is important how a function is presented. For example, the function $y = f(x)$ defined by $(y = 0 \wedge \mathrm{Con}_{ZF}) \vee (y = 1 \wedge \neg \mathrm{Con}_{ZF})$ is primitive recursive since it is a constant function, but a finitist cannot compute its value at any x.

[5] Of course, one cannot rule out the possibility that any basis, sufficient to justify what is formalized by (PRA) and which satisfies some necessary closure properties in order to be acceptable as an epistemologically distinguished system of methods, is also sufficient to justify ε_0-transfinite induction.

[6] Note that when it comes to rules of inference such as the ω-rule, which involve numerical concepts, it is difficulty to say what is a logical inference and what is a mathematical principle.

himself took such a path in [11] with another aim: to obtain a complete and consistent system of arithmetic, most likely in response to Gödel's First Incompleteness Theorem (see [3] for a detailed discussion). In this paper Hilbert added to a standard form of the first-order arithmetic Z an informal rule of the following form:[7]

RULE ω^*. Whenever $A(z)$ is a quantifier free formula for which the following can be finitistically shown: *$A(z)$ is a correct numerical formula for each particular numerical instance z*, then its universal generalization can be taken as a new premise in all further proofs.

He denoted this semiformal system by Z^* and went on to show completeness properties of Z^* with respect to $\prod_1$ sentences.

As Feferman mentions in [3], the system Z^* is not at all in the spirit of Hilbert's original ideas: instead of having a precisely formalized system in which rules of inference are specified purely by their syntactical form, here not only an informal but also vaguely formulated rule of inference is used *within* the system Z^*. The vagueness comes from the fact that the rule depends on what is accepted as a valid finitistic proof.

Nevertheless, since we are mainly concerned with consistency proofs which are anyway informal mathematical proofs rather than formal proofs in a formal theory, using a rule of this sort is acceptable as long as we precisely define it and then justify it on a previously accepted epistemological basis. Thus, in the rest of this paper we consider various informal rules of this form, interpreting differently what "finitarily" in the above description of the ω^*-rule can mean. We will investigate on what grounds one can justify such a rule and which theories can be proved to be consistent using it. Since formal proofs (seen as sequences of symbols) and finitistic manipulations of formal proofs are reducible to (token) numbers seen as sequences of strokes and the corresponding finitistic manipulations of numbers, consistency statements for primitive recursively axiomatized theories can be taken to be statements about numbers of the form $\forall x F(x)$ with $F(x)$ a primitive recursive predicate.[8] This is why we are interested in means of proving formulas of such a form in general.

In light of Tait's Thesis, if a proposition of the above form is provable using only finitistic means, then there is a proof $p(x)$ in PRA of $F(x)$; thus, all instances $F(\underline{n})$ of $F(x)$ have uniform finitistic proofs: there exists a finite skeleton proof $p^*(x)$ which is the informal finitistic proof corresponding (as in Tait's [25]) to the formal proof PRA of $p(x)$, such that for each n, $p^*(\underline{n})$ is a finitistic proof of $F(\underline{n})$, obtained by mere substitution of the variable x by the numeral $\underline{n}$ in the proof $p^*(x)$.

One can argue that it is not necessary to require such strong uniformity. Perhaps just having an effective, explicitly given construction[9] producing for each individual

[7] See [3]; for the original formulation in German see [11, p. 491].

[8] As noted in [25], since the syntax of the first-order language is primitive recursively encodable, to prove finitistically that formulas or proofs of a primitive recursively axiomatized theory T have a (finitistically meaningful) property amounts to proving finitistically that the primitive recursive predicate which corresponds to that property is true of the corresponding codes.

[9] This term is used in Tait [25]; Detlefsen [1] uses the term "manual" to denote the same thing.

n a finitistic proof of $F(\underline{n})$ would suffice to accept $\forall x F(x)$ as finitistically proved on the basis of an ω-rule in which the only restriction to make it "finitistic" is that the finitistic proofs of each instance are produced using an effective construction. In particular, according to this view, to prove the consistency of a theory T it would be enough to describe an effective construction which produces for each n a finitistic proof of the fact that n does not code a proof of an inconsistency in T (with the standard first-order logic). Such a position is discussed in the following quotation from Detlefsen [1]; to make our discussion easier, we have added in square brackets an enumeration of his claims.

> "In this section I would like to sketch an argument against the claim that G2 implies the failure of the Hilbert's Program for finding a finitistic consistency proof for the various theories of classical mathematics. The central claim of the argument is that $\text{Con}(T)$, the consistency formula shown to be unprovable by G2, does not really "express" consistency in the sense of that term germane to an evaluation of Hilbert's Program.
>
> In order for a consistency formula to "express' consistency in the appropriate sense the quantifiers and operators in it must be construed finitistically, and *not* classically, since it is the finitistic consistency of a classical system that is at issue. But a finitistic interpretation of the universal quantifier would seem to differ drastically from a classical interpretation of it, as is clear from the following remark of Herbrand."

At this point Detlefsen gives the following quotation from Goldfarb's [8, pp. 288–289, footnote 5]:

> "· · · when we say that an argument (or theorem) is true for all (these) x, we mean that, for each x taken by itself, it is possible to repeat the general argument in question, which should be considered to be merely the prototype of these particular arguments."

Then Detlefsen continues:

> "And, again, he says that a proof of a universal claim is merely a description or manual of operations which are to be executed in each particular case [8, pp. 49–51]. [*Claim* 1] This view of the universal quantifier would seem to sponsor the following restricted ω-rule: if I have an effective procedure P (i.e., a manual of operations P) for showing of each individual n that '$F(\underline{n})$' is finitistically provable, then '$\forall x F(x)$' is also finitistically provable. [*Claim* 2] Indeed in a 1930 paper [11, pp. 49–51], Hilbert stated a rule something like this. [*Claim* 3] And at that time it was apparent to finitists that the rule did not give one the power to go beyond the means of some methods that had already been accepted as finitistic [8, p. 297].
>
> [*Claim* 4] Now one would not, in general, want to add the abovementioned ω-rule to a scheme designed to serve as the finitistic proof theory of the classical theory T, since that rule does not constitute a truth of the finitistic proof theory of the classical T! [*Claim* 5] Still, certain instances of the rule would seem to be called for; in particular, the one producing

> Con(T) from its instances. [*Claim* 6] This addition made, Con(Z) becomes provable in Z_{w^*} (= Z plus the above-mentioned instance of the restricted ω-rule).
>
> [*Claim* 7] Of course, if one adds instances of the restricted ω-rule to T, in order to get an adequate context in which to do the finitistic proof theory of the classical T, then one will not be able to formulate the finitistic proof theory of T as a proof system, but I see nothing in Hilbert's program which suggests that such formalizability is an essential or important feature of it."···

[*Claim* 8] "G2 [Gödel's Second Incompleteness Theorem] then only seems to imply the failure of Hilbert's program so long as one ignores the fact that the logic of the finitistic proof theory of the classical T and the logic of the classical T itself are two quite different logics!··· If the logic of T is expanded in a way that produces a scheme whose logic is in agreement with the logic of the finitistic proof theory of the classical T, then in at least some instances (e.g., for the case where T is the system Z), Con(T) becomes provable. The basic flaw of those using G2 to thwart Hilbert's Program is that they fail to recognize that the logic of the arithmetized proof theory of T in G2 (since the arithmetized proof theory is itself embedded in T) is the logic of T itself *not* the logic of the finitistic proof theory of T (which logic is *not* a subsystem of $T's$ logic)!"

First of all, the above quotation from Herbrand[10] states "··· to repeat the general argument in question, which should be considered to be merely the prototype of these particular arguments." This description fits much better our construction of proofs obtained from a single skeleton proof by replacement of the free variable by particular numerals than does Detlefsen's ω-rule from Claim 1 above. Indeed, the skeleton proof $p^*(x)$ can be seen as a general argument which is the prototype of all particular arguments obtained by simple substitution; regardless of the numeral being substituted, substitution leaves the structure of the argument the same. Also, Herbrand's claim ([8, pp. 49–51] given immediately after the quotation from Herbrand, refers in fact to an inductive argument about finitistically meaningful properties (p. 50) of "proofs put into signs···" (p. 49) i.e., formal proofs, and it is easy to see that this argument is formalizable in $(QF - IA)$. Thus, quotations from Herbrand do not offer any support to Claim 1.

Second, Hilbert's ω-rule mentioned in Detlefsen's Claim 2 is, in fact, the one which we presented as Rule ω^*, and is of substantially different nature (in the position of the universal quantifier) than the rule Detlefsen suggests in Claim 2. Hilbert's rule allows us to use $\forall x\varphi(x)$ as a new premise in proofs if the following *universal* statement has a finitistic proof: $A(x)$ *is a correct numerical formula for each particular numerical instance* z. Detlefsen's version of the ω-rule only requires that we have an effective procedure for showing that for each n *instance* $F(\underline{n})$ is finitistically provable.

[10] This quotation refers to the intuitionistic arguments, but at that time (and until Gödel-Gentzen double negation interpretation) finitism and intuitionism were thought to be coextensive.

Detlefsen seems to distinguish finitistic metamathematical arguments whose objects are formal proofs; these arguments he calls "*finitistic proof theory of the classical theory* T", on the other hand, he also considers formal proofs in T itself. Despite his Claim 3, he decides that the rule he proposes in his Claim 1 "$\cdots$ *does not constitute a truth of the finitistic proof theory of the classical* T!". This very odd statement presumably means that such a rule is not a finitistically acceptable inference in the metatheory, i.e., cannot be used in finitistically acceptable informal arguments about formal proofs in T. He offers no reason why this should be so. Yet, he says (Claim 5) that certain instances of such a rule "*are called for*", in particular, one producing $\mathrm{Con}(T)$ from its instances. Again, he does not say what is so peculiar about the consistency statements of Z compared to just any other statement which is of the same logical complexity, which makes the particular instance of his ω-rule a finitistically acceptable means of inference in case of the consistency statement for Z but not for the consistency statement for every other theory, or even more generally, every other statement of the same syntactical complexity. The addition of such a rule to the elementary number theory Z produces an "adequate context" in which we can carry arguments of finitistic proof theory of T, but now the predicate $\mathrm{Con}(Z)$ becomes provable using the additional rule (Claim 6) which is why the Second Incompleteness Theorem, according to Detlefsen, has no impact on Hilbert's Program. Those who believe that the Second Incompleteness Theorem does not an impact on Hilbert's Program. Those who believe that the Second Incompleteness Theorem does have an impact on Hilbert's Program just fail to see that finitistic arguments about formal proofs employ a different kind of logic (finitistically valid inferences) which are not formalizable in elementary number theory Z (Claim 8). Claim 7 implies that such finitistic arguments might not be formalizable by a formal system (in the usual sense) but this is really not required for a successful realization of Hilbert's program.

Thus, to summarize, Detlefsen's argument rests on two points. First, he mentions the old problem of delineating what constitutes a valid finitistic argument and whether there is a formal theory which captures exactly such arguments, but this is a well-known story. Second, he makes an unsuccessful attempt to show that a form of the ω-rule is an example of an argument which is finitistically valid but not formalizable in Z, since it suffices to prove consistency of Z. The quotations from Herbrand which Detlefsen cites do not support such a claim because, as we pointed out, they clearly *do not* refer to the kind of rule he mentions in Claim 5. On the other hand, he himself offers no explanations why certain very special instances of such a rule qualify as finitistically valid forms of inference.

To explain why the rule form Detlefsen's Claim 5 cannot be accepted as a finitistically valid inference, we show that using such a rule one can prove consistency of *any* consistent first-order theory T. This clearly makes his claim about the finitistic validity of such a rule implausible. On the other hand, we also show that there is a modification of this rule which, applied to consistency proofs, leads to a partial program of a reductive nature. Such a partial program is then shown to encompass theories of the "Reverse Mathematics" program. We also show that using such a modification of the rule mentioned in Claim 5 above, we can prove consistency of exactly the same theories as we could prove modification of the more general rule mentioned in Detlefsen's Claim 1.

For convenience, let us formulate the rule again.

RULE 1. Let T be a theory. If we have an effective procedure for producing for each individual n a finitistic proof of $\neg\mathrm{Prf}_T(\underline{n}, \ulcorner 1 = 0 \urcorner)$, then we derive $\mathrm{Con}(T)$.

To show that the Rule 1 cannot be plausibly held a finitistically valid inference, consider, for example, set theory (ZF) and the following procedure.

PROCEDURE. *Given n, find the least $i \leq lh(n)$ such that $(n)_i$ is a code of a formula*[11] *which is neither an axiom of (ZF) nor is it obtained by an application of a first-order rule of inference to some of the formulas $(n)_j$, for $j < i$, and prove this property of n.*

Assuming that (ZF) is consistent, for every sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ of formulas of (ZF) such that φ_k is of the form $\theta \wedge \neg\theta$, one can find a formula φ_i in that sequence that it is neither an axiom of (ZF) nor is it obtained from the previous ones using a first-order rule of inference. For every such sequence this fact has a proof which is unquestionably finitistically acceptable and easily formalizable in (PRA) because (ZF) is primitive recursively axiomatized and so it is easy to verify that φ_i is not an axiom of (ZF), while to show that φ_i is not derived from some of the previous formulas one has to check only a few rules of inference when applied to some of $\varphi_0, \ldots, \varphi_{i-1}$. Consequently, the procedure we described above is an effective procedure satisfying the conditions for the "manual" Detlefsen mentions. However, no finitist can take this effective procedure as a satisfactory proof of the consistency of (ZF) because he cannot realize that this effective procedure indeed produces for each n a finitistic proof of the corresponding instance of the consistency statement: in the above argument we *assumed* that (ZF) is consistent. Hence, Rule 1 cannot be seriously considered as a finitistic means of inference. In order to accept the consistency of a theory T on grounds that could be *reasonably* argued to be finitistic, we have at least to be able to give a *finitistic* proof that the function which is a suitable formalization of the effective procedure mentioned in the Rule 1 indeed has the property that for each n it produces a code of a proof of the formula $\neg\mathrm{Prf}_T(\underline{n}, \ulcorner 1 = 0 \urcorner)$ in a formal theory which we recognize as formalizing *only* finitistically acceptable principles (but not necessarily all such principles). Of course, this is just a necessary condition; in general, unless the procedure is extremely uniform (e.g., if it produces substitutional instances with the same skeleton proof) a finitist cannot recognize that such a procedure has the necessary property since he cannot realize that everything provable in a particular formal system is finitistically valid. A finitist can argue about formal proofs but not about the general properties of the informal finitistical proofs which he produces; they are not objects of his considerations. Thus, he either sees immediately that *all* proofs produced by an effective procedure are finitistic proofs from the very definition of the procedure, in which case, according to Tait's analysis, the statement is provable

[11] Here $lh(n)$ is the length of the sequence coded by n. If n is the code of a proof in a formal theory T, then each number $(n)_i$ is a code of a formula which is either an axiom of T or a formula which is derived form some of the previous formulas of the sequence coded by n using one of the rules of inference of the first order logic.

in (PRA), or he cannot accept such a means of inference as finitistically valid.[12] Nevertheless, it seems to us that a reasoning which employs a rule of the above form is sufficiently close to finitistical reasoning that it deserves a closer scrutiny. It seem uncontroversial that (PRA) is a theory which embodies only finitistic principles.[13] For convenience, we rather choose a first-order conservative extension of (PRA), i.e., $(QF - IA)$ for the following definition; recall that there is a primitive recursive function $h(x)$ such that for any open formula φ, $PRA \vdash \mathrm{Prf}_{(QF-IA)}(x, \varphi) \rightarrow \mathrm{Prf}_{(PRA)}(h(x), \varphi)$. As we have already mentioned, such a function is obtained by formalizing a cut elimination procedure.

DEFINITION 1. Consistency of a theory T can be proved *almost finitistically* if there exists a primitive recursive function f such that

$$QF - IA \vdash \forall x \mathrm{Prf}_{(QF-IA)}(f(x), \ulcorner \neg \mathrm{Prf}_T(\underline{x}, \ulcorner 1 = 0 \urcorner) \urcorner).$$

Here $\underline{x}$ denotes the term $S(S(\cdots(0))\cdots)$ whose value is x (such terms are defined by primitive recursion). To make our coding notation easier we introduce a "generic", "bar" notation for codes, and use it whenever there can be no ambiguity. Thus, instead of the above formula we will often write just $QF - IA \vdash \forall x \mathrm{Prf}_{(QF-IA)}(f(x), \overline{\neg \mathrm{Prf}_T(x, 1 = 0)})$, rather than spell out the details of the coding. Similarly,

$$\forall \varphi \in \Sigma_1 \mathrm{Thm}_{(QF-IA)}(\overline{\varphi \rightarrow \mathrm{Thm}_{(QF-IA)}(\varphi)})$$

is an abbreviation for

$$\forall \varphi \in \Sigma_1 \mathrm{Thm}_{(QF-IA)}(\mathrm{Imp}(\varphi, \mathrm{Sub}(\ulcorner \mathrm{Thm}_{(QF-IA)}(x) \urcorner, \mathrm{Num}(\varphi)))),$$

where φ is just a numerical variable, and Imp, Sub, Num are the usual functions associated with the chosen Gödel coding. For simpler formulas we will continue to use the standard notation which "spells out" the details of coding. In order to justify the concept introduced by Definition 1, we have to see whether there are any nontrivial theories whose consistency can be proved almost finitistically.

Question 1. What are the theories whose consistency can be proved almost finitistically?

Despite Detlefsen's claims, there is no reason to restrict such an ω-rule only to the consistency formulas; with equal justification we can apply it to any primitive recursive formula φ. Thus, we introduce the following (meta)definition.

[12] One could be tempted to try to bypass the above difficulty by introducing a "definition" of the following form. The consistency of a primitive recursively axiomatized theory T is provable *almost finitistically* if there exists a finitistically acceptable function f of which it can be finitistically proved that for each natural number n, $f(n)$ is a finitistic proof that n is not a code of a proof of a contradiction from the axioms of T with the standard first order logic. Unfortunately, the above is not a (mathematical) definition; the notions of "a finitistically acceptable function" and "a finitistic proof" have no rigorous definitions and are certainly not mathematical notions—informal mathematical proofs are not themselves objects of mathematical but only philosophical considerations. Thus, the above "definition" does not make sense.

[13] We are not claiming here that *all* finitistic principles are embodied in (PRA).

DEFINITION 2. A sentence of the form $\forall x \varphi(x)$, where φ is a primitive recursive formula,[14] is provable *almost finitistically* if there exists a primitive recursive function f such that

$$QF - IA \vdash \forall x \mathrm{Prf}_{(QF-IA)}(f(x), \ulcorner \varphi(\underline{x}) \urcorner).$$

We denote by S the set of all sentences which are provable almost finitistically.

Here again a *primitive recursive function* and a $(QF - IA)$ *proof* stand in place of the informal notions of *finitistically acceptable function* and a *finitistically acceptable proof*. As before, one can see S as the set of all sentences provable using one application of a restricted ω-rule. This rule can be seen as a strengthened combination of the rules mentioned in [3, pp. 212–213]. Not only do we restrict the complexity of the formula on which the rule is applied and require that the fact "*every instance of the formula is finitistically provable*" must be itself finitistically provable, but we also require that the proofs of all these instances must be generated by a finitistically acceptable function. If we accept Tait's Thesis, then these restrictions eliminate the vagueness from Hilbert's description of the informal ω-rule. We can adjoin to this form of the ω-rule the standard first-order logic to obtain a well-defined formal system.[15] Further, the form of such an ω-rule permits us to replace it with just the set S of its conclusions and get a first-order theory (S) with the usual rules of inference, axiomatized by S. It is now natural to ask the following question.

Question 2. What are the theories whose consistency can be proved in (S)?

Note that all induction axioms of $(QF - IA)$ can be written in the form $\psi(0) \wedge (\forall y < x)(\psi(y) \rightarrow \psi(y+1)) \rightarrow \psi(x)$, for ψ a primitive recursive formula, which is itself a primitive recursive formula. All other axioms of $(QF - IA)$ obviously are primitive recursive formulas. Since for any primitive recursive formula $\varphi(x)$ provable in $(QF - IA)$ with a proof $p(x)$, $f(x)$ given by $f(x) = \ulcorner p(\underline{x}) \urcorner$ satisfies the condition from the definition of the set S, the following Lemma immediately follows.

LEMMA 1. $(QF - IA) \subseteq S$.

Thus, S extends what Tait takes to be a correct formalization of the standard finitist reasoning.

In the next sections we answer the above two questions and consider some possible generalizations.

§**2. Answer to the first question.** As is well known, using the standard cut-elimination procedure one can show that the provably recursive functions in $(QF - IA)$ are precisely primitive recursive functions.[16] Thus, we get the following proposition.

[14] Detlefsen does not explicitly impose this complexity restriction; nevertheless, this restriction follows form what Hilbert accepted as finitistically meaningful sentences and is present in Hilbert's paper [11] that Detlefsen quotes.

[15] As it happens, as soon as we formalize precisely some informal notion one can question whether the formalization captures properly the informal notion being formalized, but our aim is anyway to investigate various plausible deliminations of the mentioned informal ω-rule.

[16] In fact, the procedure explicitly produces defining equations of a primitive recursive function.

PROPOSITION 2. *Let T be a primitive recursively axiomatized theory, then the consistency of T can be almost finitistically proved if and only if*

$$QF - IA \vdash \forall x \exists y \mathrm{Prf}_{(QF-IA)}(y, \overline{\neg \mathrm{Prf}_T(x, 1=0)}).$$

This is why we introduce the following definition.

DEFINITION 3. $\mathrm{Con}^*(T) \equiv \forall x \exists y \mathrm{Prf}_{(QF-IA)}(y, \overline{\neg \mathrm{Prf}_T(x, 1=0)})$.

It is easy to see that there are indeed theories whose consistency cannot be proved finitistically, but can be proved almost finitistically; $(QF - IA)$ is an example. To see if there are theories which are *mathematically* more interesting than $(QF - IA)$ and whose consistency can be proved almost finitistically, we need some further technical results.

Throughout the rest of this paper T denotes a consistent, primitive recursively axiomatized theory whose axioms are presented in such a way that (an extension by definition of) T provably extends $(QF - IA)$, i.e., such that

$$QF - IA \vdash \forall x (Ax_{(QF-IA)}(x) \rightarrow Ax_T(x)).$$

PROPOSITION 3. *Let T be as above, then*

(i) $QF - IA \vdash \mathrm{Con}(T) \leftrightarrow (\mathrm{Con}^*(T) \wedge \mathrm{Con}(QF - IA))$;

(ii) $QF - IA \vdash \mathrm{Con}^*(T) \leftrightarrow (\mathrm{Con}(QF - IA) - \mathrm{Con}(T))$.

PROOF. (i) We can assume $\mathrm{Con}(QF - IA)$, since $\neg\mathrm{Con}(QF - IA)$ implies $\neg\mathrm{Con}(T)$, and in this case (i) is obviously true. We first assume $\mathrm{Con}(T)$, then this implies $\forall x \neg \mathrm{Prf}_T(\underline{x}, \ulcorner 1 = 0 \urcorner)$, and so by demonstrable Σ_1 completeness[17] of $(QF - IA)$, we have $\forall x \exists y \mathrm{Prf}_{(QF-IA)}(y, \overline{\neg \mathrm{Prf}_T(x, 1=0)})$, i.e., $\mathrm{Con}^*(T)$. On the other hand, if we now assume $\neg\mathrm{Con}(T)$, then for some c, $\mathrm{Prf}_T(\underline{c}, \ulcorner 1 = 0 \urcorner)$ and so, as before, we get $\exists y \mathrm{Prf}_{(QF-IA)}(y, \overline{\mathrm{Prf}_T(\underline{c}, 1=0)})$. Since we assumed $\mathrm{Con}(QF - IA)$, we get $\neg \exists y \mathrm{Prf}_{(QF-IA)}(y, \overline{\neg \mathrm{Prf}_T(c, 1=0)})$. Consequently, $\neg\mathrm{Con}^*(T)$ which implies our claim.

(ii) This follows directly from (i) and the fact that

$$QF - IA \vdash \neg\mathrm{Con}(QF - IA) \rightarrow \mathrm{Con}^*(T). \qquad \square$$

COROLLARY 4. *Let T be any theory as before. Then*

(1) $QF - IA \perp \mathrm{Con}^*(T)$ *if and only if* $QF - IA \perp \mathrm{Con}(\mathrm{PRA}) \leftrightarrow \mathrm{Con}(T)$.

(2) *In particular,* $QF - IA \perp \mathrm{Con}^*(\mathrm{PRA})$.

Thus, one can prove consistency of a theory T almost finitistically if and only if one can show finitistically that consistency of T follows from the assumption that our finitistic number theory is consistent[18]

[17] This means that for arbitrary Σ_1 formula $\varphi(x)$, $QF - IA \vdash \forall x(\varphi(x) \rightarrow \mathrm{Thm}_{(QF-IA)}(\ulcorner \varphi(\underline{x}) \urcorner)$; here $\underline{x}$ is the term built only from the constant 0 and x iterations of the successor function. Formally, it is defined by primitive recursion; for more details about these points see [24].

[18] We note that $\mathrm{Con}(\mathrm{PRA}) \rightarrow \mathrm{Con}(T)$ is not a $\prod_1^0$-sentence, and consequently does not correspond directly to a finitistically meaningful statement. However, since $QF - IA \vdash \mathrm{Con}(\mathrm{PRA}) \rightarrow \mathrm{Con}(T)$ is equivalent to $\mathrm{PRA} \perp \forall d (\mathrm{Prf}_T(d, \ulcorner 1 = 0 \urcorner) \rightarrow \mathrm{Prf}_{\mathbf{PRA}}(f(d), \ulcorner 1 = 0 \urcorner)$ for a suitable primitive recursive function f, we can adopt a convention that the equiconsistency statement above has the meaning of the later, clearly finitistically meaningful statement.

This explains why our restricted ω-rule leads to a reductive program. But before proceeding with this point, we will first elaborate on a few more technical points. First of all, note that by Propositions 3(i) and (ii), $\mathrm{Con}^*(T)$ is the exact measure of "how much" $\mathrm{Con}(T)$ is stronger than $\mathrm{Con}(QF - IA)$ i.e., the consistency of the base theory, "measured" from that base theory $(QF - IA)$. More generally, we have the following proposition whose proof is similar to the proof of the previous proposition.

PROPOSITION 5. *Let S and T be any two theories provably extending* $(QF - IA)$, *then for* $\mathrm{Con}^*_S(T) \equiv \forall x \exists y \mathrm{Prf}_S(y, \overline{\neg \mathrm{Prf}_T(x, 1 = 0)})$ *we have*

$$QF - IA \vdash \mathrm{Con}^*_S(S), \qquad (QF - IA) \vdash \mathrm{Con}^*_S(T) \leftrightarrow (\mathrm{Con}(S) \rightarrow \mathrm{Con}(T)).$$

Also, observing that $\mathrm{Con}^*(S)$ corresponds to $\mathrm{Con}^*_{(QF-IA)}(S)$,

$$(QF - IA) \vdash \mathrm{Con}^*_{(QF-IA)}(S) \wedge \mathrm{Con}^*_S(T) \rightarrow \mathrm{Con}^*_{(QF-IA)}(T)$$

and if in $(QF - IA)$, S and T are "collinear" i.e., $QF - IA \subseteq S \subseteq T$ (provably so in $(QF - IA)$), then

$$(QF - IA) \vdash \mathrm{Con}^*_{(QF-IA)}(S) \wedge \mathrm{Con}^*_S(T) \leftrightarrow \mathrm{Con}^*_{(QF-IA)}(T).$$

Compare the last two formulas with the triangle inequality in a metric space, with $\wedge$ corresponding to $+$, $\rightarrow$ corresponding to $\geq$ and $\leftrightarrow$ corresponding to $=$. The above claims have easy proofs that we omit.

The notion of an almost finitistic consistency proof can be of interest only if there are significant theories whose consistency can be proved almost finitistically. The best example of such a theory is Friedman's WKL_0 which is a fragment of second-order arithmetic with induction only for Σ^0_1 formulas, comprehension for Δ^0_1 formulas, and König Lemma for binary trees.[19] In this theory it is possible to do a vast amount of classical analysis and algebra used in the sciences (for more details see [6, 7, 17, 18], and [19]). Yet, we can show almost finitistically the consistency of WKL_0 because Sieg's proof that WKL_0 is provably $\prod^0_2$ conservative over $(QF - IA)$ (see [21, Proposition 5.8]) is effective and can be formalized in $(QF - IA)$ yielding a primitive recursive function $h(p)$ such that

$$(1) \qquad QF - IA \vdash \forall \varphi \in \Pi^0_2 \forall p (\mathrm{Prf}_{WKL_0}(p, \varphi) \rightarrow \mathrm{Prf}_{(QF-IA)}(h(p), \varphi)).$$

In particular, taking $\varphi \equiv (1 = 0)$, we get $QF - IA \vdash \mathrm{Com}(QF - IA) \rightarrow \mathrm{Con}(WKL_0)$ and so $QF - IA \vdash \mathrm{Con}^*(WKL_0)$.

COROLLARY 6. *The consistency of* WKL_0 *can be proved almost finitistically.*

Yet as we noted, one can develop in WKL_0 a great deal of the classical mathematics needed for empirical sciences.

There can exist an almost finitistic proof of the consistency of a theory T even if T is not provably (in PRA) Π_1 conservative over PRA, even if T is not just Π_1 conservative over PRA. For example, let φ be a Rosser sentence for the theory $(QF - IA)$, and let T be $(QF - IA) + \varphi$. Then we have, just by formalizing the usual proof of the unprovability of Rosser's sentence (see for

[19] This form of König's Lemma is usually called the Weak König's Lemma; it asserts that every infinite binary tree has an infinite path; for more details see [18].

example [24]), $QF - IA \vdash \mathrm{Con}(QF - IA) \rightarrow \neg\mathrm{Thm}_{(QF-IA)}(\neg\varphi)$, i.e., $QF - IA \vdash \mathrm{Con}(QF - IA) \rightarrow \mathrm{Con}((QF - IA) + \varphi)$, but $(QF - IA) + \varphi$ is obviously not Π_1 conservative over $(QF - IA)$ because $QF - IA \nvdash \varphi$. On the other hand, we have the following Proposition.

PROPOSITION 7. *If T is a theory as above and $QF - IA \vdash \mathrm{Con}^*(T)$, then*

(2) $$QF - IA \vdash \forall\varphi \in \Pi_1(\mathrm{Thm}_T(\varphi) \rightarrow \mathrm{Thm}_{(QF-IA)+\mathrm{Con}(QF-IA)}(\varphi)).$$

We first prove the following Lemma.

LEMMA 8. *There are primitive recursive functions $v(x)$ and $w(x)$ such that*

(3)
$$QF - IA \vdash \forall\varphi \in \Sigma_1 \mathrm{Prf}_{(QF-IA)}(v(\varphi), \mathrm{Imp}(\varphi, \mathrm{Sub}(\ulcorner \mathrm{Prf}_{(QF-IA)}(x, y)\urcorner, w(\varphi), \mathrm{Num}(\varphi)))).$$

Notice that in the above formula the value of $w(x)$ is a closed term (denoting the code of a proof). We will see that this term is *not* uniform in φ, i.e., it is not of the form $t(\ulcorner\varphi\urcorner)$, where $t(x)$ is independent of φ. In our sloppy notation ("add codes where needed") the above formula becomes: $QF - IA \vdash \forall\varphi \in \Sigma_1 \mathrm{Prf}_{(QF-IA)}(v(\varphi), \overline{\varphi \rightarrow \mathrm{Prf}_{(QF-IA)}(w(\varphi), \varphi)})$. This clearly implies that $QF - IA \vdash \forall\varphi \in \Sigma_1 \mathrm{Thm}_{(QF-IA)}(\overline{\varphi \rightarrow \mathrm{Thm}_{(QF-IA)}(\varphi)})$.

As mentioned above, it is *not* the case that there is a primitive recursive function g such that for all Σ_1 formulas φ

(4) $$QF - IA \vdash \varphi \rightarrow \mathrm{Prf}_{(QF-IA)}(g[\ulcorner\varphi\urcorner), \ulcorner\varphi\urcorner)$$

because if (4) was true, then $\omega \models \varphi \leftrightarrow \mathrm{Prf}_{(QF-IA)}(g(\ulcorner\varphi\urcorner), \ulcorner\varphi\urcorner)$, since the other implication always holds on ω. This is impossible because, by Gödel's Diagonal Lemma, there is a formula $\psi \in \Sigma_0$ such that

$$QF - IA \vdash \psi \leftrightarrow \neg\mathrm{Prf}_{(QF-IA)}(g(\ulcorner\psi\urcorner), \ulcorner\psi\urcorner).$$

On the other hand, for all Σ_1 formulas φ,

(5) $$QF - IA \vdash \varphi \rightarrow \exists p \mathrm{Prf}_{(QF-IA)}(p, \ulcorner\varphi\urcorner).$$

Proof p is obtained from the computation which verifies the truth of φ; the reason why p cannot be obtained by a primitive recursive procedure from φ is that there is no primitive recursive procedure which given an arbitrary primitive recursive formula produces a computation testing the truth of the formula. Nevertheless, there is a primitive recursive procedure which for arbitrary n and arbitrary formula φ form the nth level $\mathscr{G}_n$ of Grzegorczyk's Hierarchy produces the code $\ulcorner p\urcorner$ of a proof p in $(QF_n - IA)$ and a proof in $QF - IA$ of $\varphi \rightarrow \mathrm{Prf}_{(QF_n-IA)}(\ulcorner p\urcorner, \ulcorner\varphi\urcorner)$. This is the basis of the proof of Lemma 8.

PROOF OF LEMMA 8. Let $(QF_n - IA)$ be the fragment of $(QF - IA)$ containing only functional symbols and defining equations for functions up to the nth leave $\mathscr{G}_n$ of Grzegorczyk's Hierarchy, and let L_n be the corresponding fragment of the language of $(QF - IA)$. By the well-known facts about Grzegorczyk's Hierarchy, there is a function $\mathscr{F}_{n+1} \in \mathscr{G}_{n+1}$ such that for any formula $\varphi \in L_n$, $(QF_{n+1} - IA) \nvdash \varphi \leftrightarrow \mathrm{Ver}_{(QF_n-IA)}(\mathscr{F}_{n+1}(\ulcorner\varphi\urcorner), \ulcorner\varphi\urcorner)$ where $\mathrm{Ver}_{(QF_n-IA)}(c, \varphi)$ is a formula of $(QF_n - IA)$ formalizing "c is a (code of a) computation testing the truth of φ". Also, it is easy to see that there is aprimitive recursive function Π such that one can prove by the usual metamathematical induction on n (not induction formalized in $(QF - IA)$) that $QF - IA \vdash \forall\varphi\forall c(\mathrm{Ver}_{QF-IA_n}(c,\varphi) \to \mathrm{Prf}_{(QF_n-IA)}(\Pi(\varphi),\varphi))$. Assuming that this is true for n, case $n+1$ is proved using the induction schema of $(QF - IA)$ (with complexity of φ as a variable); the atomic case $f(x,z) = y$, if $f \in \mathscr{G}_{n+1}$, and f is defined by primitive recursion on x from some $g(z)$, $h(x,z,t)$, is again proved by induction in $(QF - IA)$ with x as the induction variable. Here we use the fact that c contains computation $f(0,z) = g(z)$, $f(y+1,z) = h(y,z,f(y,z))$ for $y < x$, and that, by the inductive hypothesis, each equation is provable in $(QF_n - IA)$. Clearly, putting these proofs together we can get a proof of $f(x,z) = y$. Nonatomic formulas are handled in the usual inductive manner. Thus, if $\varphi \in \Sigma_1$ and $\varphi \in L_n$, then $(QF_{n+1} - IA) \vdash \varphi \to \mathrm{Prf}_{(QF_n-IA)}(\Pi(\mathscr{F}_{n+1}(\ulcorner\varphi\urcorner)),\ulcorner\varphi\urcorner)$. The above proof by mathematical induction is clearly finitistic and it can be formalized in $(QF - IA)$ yielding a primitive recursive function H such that

$$(6)\quad \begin{aligned}&QF - IA \vdash \forall n\forall\varphi\\ &\quad\in L_n \cap \Sigma_1 \mathrm{Prf}_{(QF_{n+1}-IA)}(H(n,\varphi), \overline{\varphi \to \mathrm{Prf}_{(QF_n-IA)}(\Pi(\mathscr{F}_{n+1}(\varphi)),\varphi)}).\end{aligned}$$

Since for each Σ_1 formula φ one can find in a primitive recursive way the least n such that $\varphi \in L_n$, we get the claim of our Lemma as an immediate consequence. □

Proof of Proposition 7. Essentially, we formalize the proof of Kreisel's Π_1 conservativeness theorem. Since by our assumption and Proposition 3(ii) $QF - IA \vdash \mathrm{Con}(QF - IA) \to \mathrm{Con}(T)$, we also have

$$QF - IA \vdash \mathrm{Thm}_{(QF-IA)+\mathrm{Con}(QF-IA)}(\ulcorner\mathrm{Con}(T)\urcorner).$$

By the provable Σ_1 completeness of $(QF - IA)$ we have

$$QF - IA \vdash \forall\varphi(\mathrm{Thm}_T(\varphi) \to \mathrm{Thm}_{(QF-IA)}(\overline{\mathrm{Thm}_T(\varphi)}));$$

thus,

$$QF - IA \vdash \forall\varphi(\mathrm{Thm}_T(\varphi) \to \mathrm{Thm}_{(QF-IA)+\mathrm{Con}(QF-IA)}(\overline{\neg\mathrm{Thm}_T(\neg\varphi)})).$$

This, together with the following direct consequence of Lemma 4,

$$QF - IA \vdash \forall\varphi \in \Pi_1 \mathrm{Thm}_{(QF-IA)}(\overline{\neg\mathrm{Thm}_{(QF-IA)}(\neg\varphi) \to \varphi}),$$

implies the claim of Proposition 7. □

The converse of Proposition 7 is false; just take for example $T = (QF - IA) + \mathrm{Con}(QF - IA)$. Then (2) is trivially true but $QF - IA \nvdash \mathrm{Con}^*((QF - IA) + \mathrm{Con}(QF - IA))$ because of Gödel's Second Incompleteness Theorem $(QF - IA) + \mathrm{Con}(QF - IA) \nvdash \mathrm{Con}((QF - IA) + \mathrm{Con}(QF - IA))$.

It is easy to see that (2) implies that there is a primitive recursive function $f(p)$ of which it can be proved in $(QF - IA)$ that for every quantifier free formula φ and every proof p of φ in T, $f(p)$ is a proof of φ in $(QF - IA) + \mathrm{Con}(QF - IA)$. This, in the terminology of [4], implies that T is proof-theoretically reducible to

$(QF - IA) + \mathrm{Con}(QF - IA)$ conservatively for Π_1 sentences; we denote this property by $T \leq ((QF - IA) + \mathrm{Con}(QF - IA))[\Pi_1]$. Thus, we get that

$$\{T : T \leq (QF - IA)[\Pi_1])\} \subset \{T : (QF - IA) \vdash \mathrm{Con}^*(T)\}$$
$$\subset \{T : T \leq ((QF - IA) + \mathrm{Con}(QF - IA))[\Pi_1]\}.$$

Since proof-theoretic reducibility of T_1 to T_2, conservatively for Π_1 formulas (in the above notation $T_1 \leq T_2[\Pi_1]$) implies that T_1 is Π_1 conservative over T_2 (see [4, pp. 368–9]), theories $(QF - IA) +$ *Rosser sentence for* $(QF - IA)$ and $(QF - IA + \mathrm{Con}(QF - IA))$ show that both (the first, and respectively, the second) inclusions are strict. It would be interesting to see if there are *mathematically significant* theories whose consistency can be almost finitistically proved and which are not Π_1 conservative over $(QF - IA)$.

As we observed, finding rich theories whose consistency can be almost finitistically proved is a reductive (partial) realization of Hilbert's Program. In such a program not only do we incorporate only a part of mathematics in our formal theories, but also we weaken the requirement of what is to be accepted as a proof of consistency of these theories: instead of producing an "absolute" consistency proof, we finitistically (in general, constructively) reduce the consistency of a theory to the consistency of the finitistic (constructive) number theory.[20]

Reductive partial programs represent one of the most important developments in proof theory and philosophy of mathematics since Gödel's destruction of Hilbert's Program in its original form.[21] The central accomplishment of a reductive program is a proof-theoretic reduction of a theory formalizing a significant part of mathematical practice done in the *classical* foundational framework to a suitable theory in a *constructive* (or otherwise restricted) foundational framework. Here the notion of a constructive framework consists of rules for generating objects of a constructive domain together with the appropriate notion of a constructive proof, which follows the construction of objects. The most fundamental example is the constructive definition of the notion of a natural number, where natural numbers are seen as generated by the construction of a successor.

A realization of a partial reductive program for a significant part of mathematical practice consists of formalizing this part of mathematical practice in a *classical*

[20] Nevertheless, having an almost finitistic proof of the consistency of a theory still seems like good grounds for a belief in the consistency of a theory, with an argument like this. Assume the consistency of T can be proved almost finitistically, with f supplying the necessary proofs. Yet, assume that T is inconsistent with k coding a proof of an inconsistency in T. Then f could not find the 'erroneous' place in the proof coded by k, and consequently, would not be able to produce a finitistic proof of the fact that k does not code an inconsistency in T. But we have a reliable (i.e., finitistic) proof of the fact that $f(n)$ is always a proof showing that n does not code a proof of an inconsistency in T. Thus, no number can code an inconsistency in T, and since all proofs in T are enumerated, T must be consistent. A careful reader comparing the above argument with Proposition 3(i) will notice immediately that we implicitly presuppose the consistency of our finitistic methods (actually we presuppose the soundness of finitistic methods for proving a variable free primitive recursive predicate, which is, by the completeness of finitistic methods for such predicates, equivalent to consistency).

[21] Here we extensively use Sieg's presentations from [23], [22], and [20] as well as Feferman's [4]. The reader is encouraged to read these papers for further details.

formal theory P^* and then finding a corresponding *constructive* (or otherwise restricted) theory F^* to which the classical theory P^* is to be reduced. This reduction consists in recognizing constructively that each P^* derivation d of a sentence φ from a class of formulas Φ is (constructively) sound. This means showing in F^*, which embodies constructively valid proofs, that for each P^* derivation d one can prove in F^* the following *partial* reflection principle for P^*:

$$\mathrm{Prf}_{P^*}(\ulcorner d \urcorner, \ulcorner \varphi \urcorner) \rightarrow \varphi.$$

The class Φ is a class of formulas which are formalizations of propositions meaningful from the constructive standpoint taken. Formally,

$$F^* \vdash \forall d \forall \varphi \in \Phi \mathrm{Prf}_{F^*}(f(d), \overline{\mathrm{Prf}_{P^*}(d, \varphi) \rightarrow \varphi}).$$

In this paper we consider various theories of the "Reverse Mathematics Program" as formalizing important parts of classical mathematics and choose finitism as the most basic constructive foundational standpoint. According to Tait's Thesis, we take $(QF - IA)$ as formalization of finitism, and we take the set of Π^0_1 sentences as (again according to Tait's Thesis) corresponding to the set of finitistically meaningful proposition.[22]

We now show that the above kind of reduction via proving the partial reflection principle for P^* is equivalent to proving the consistency of P^* almost finitistically. More precisely, we have the following proposition.

PROPOSITION 9. *Let P^* be a classical theory extending $(QF - IA)$. Then*

(1) *If Φ is a class of formulas containing all variable free formulas, and if*

$$QF - IA \perp \forall d \forall \varphi \in \Phi \mathrm{Prf}_{(QF-IA)}(f(d), \overline{\mathrm{Prf}_{P^*}(d, \varphi) \rightarrow \varphi}), \tag{7}$$

then one can prove in $(QF - IA)$ that P^* is equiconsistent to $(QF - IA)$, i.e., (7) *implies that*

$$QF - IA \vdash \mathrm{Con}(QF - IA) \rightarrow \mathrm{Con}(P^*). \tag{8}$$

(2) *If Φ consists of exactly Π^0_1 formulas, then* (7) *and* (8) *above are in fact equivalent and they imply that it is finitistically provable that P^* is Π^0_1-conservative over $QF - IA$. In the terminology we are using in this paper, this means that P^* is proof-theoretically reducible to $(QF - IA)$ conservatively for Π^0_1 sentences.*

PROOF. To prove (1) assume (7) and take for φ the formula $1 = 0$. Then

$$QF - IA \vdash \forall d \mathrm{Prf}_{(QF-IA)}(f(d), \overline{\neg \mathrm{Prf}_{P^*}(d, 1 = 0)}).$$

On the other hand, the provable Σ^0_1-completeness of $(QF - IA)$ implies that

$$QF - IA \vdash \forall d (\mathrm{Prf}_{P^*}(d, \ulcorner 1 = 0 \urcorner) \rightarrow \mathrm{Prf}_{(QF-IA)}(s(d), \overline{\mathrm{Prf}_{P^*}(d, 1 = 0)}))$$

for a primitive recursive function (p.r. function in the sequel) $s(d)$. Thus, $QF - IA \vdash \forall d(\mathrm{Con}(QF - IA) \rightarrow \neg \mathrm{Prf}_{P^*}(d, \ulcorner 1 = 0 \urcorner))$, i.e., $QF - IA \vdash \mathrm{Con}(QF - IA) \rightarrow \mathrm{Con}(P^*)$.

[22] Theories of "Reverse Mathematics Program" can also contain second-order variables, so we must specify that we consider Π^0_1 sentences rather than Π_1 sentences as corresponding to finitistically meaningful propositions.

To prove (2) assume that $QF - IA \vdash \mathrm{Con}(QF - IA) \to \mathrm{Con}(P^*)$. Since $QF - IA \vdash \mathrm{Con}(P^*) \to \forall p \neg \mathrm{Prf}_{P^*}(p, 1 = 0)$, provable Σ_1^0 completeness of $(QF - IA)$ implies

$$QF - IA \vdash \mathrm{Con}(P^*) \to \forall p \mathrm{Thm}_{(QF-IA)}(\overline{\neg \mathrm{Prf}_{P^*}(p, 1 = 0)}).$$

This and our assumption imply

$$QF - IA \vdash \mathrm{Con}(QF - IA) \to \forall p \mathrm{Thm}_{(QF-IA)}(\overline{\neg \mathrm{Prf}_{P^*}(p, 1 = 0)}).$$

Also, elementary properties of the proof predicate together with Σ_1 provable completeness of $(QF - IA)$ imply that for a p.r. function m, the following sentence is provable in $(QF - IA)$:

$$\forall d \forall d' \forall \varphi \mathrm{Thm}_{(QF-IA)}(\overline{\mathrm{Prf}_{P^*}(d, \varphi) \wedge \mathrm{Prf}_{(QF-IA)}(d', \neg\varphi) \to \mathrm{Prf}_{P^*}(m(d, d'), 1 = 0)}).$$

This implies that

$$QF - IA \vdash \mathrm{Con}(QF - IA) \\ \to \forall d \forall d' \forall \varphi \mathrm{Thm}_{(QF-IA)}(\overline{\mathrm{Prf}_{P^*}(d, \varphi) \to \neg \mathrm{Prf}_{(QF-IA)}(d', \neg\varphi)}).$$

On the other hand, Lemma 8 implies that there is a p.r. function l such that

$$QF - IA \vdash \forall \varphi \in \Pi_1 \mathrm{Thm}_{(QF-IA)}(\overline{\neg \mathrm{Prf}_{(QF-IA)}(l(\varphi), \neg\varphi) \to \varphi}).$$

Consequently, $QF - IA \vdash \mathrm{Con}(QF - IA) \to \forall d \forall \varphi \mathrm{Thm}_{(QF-IA)}(\overline{\mathrm{Prf}_{P^*}(d, \varphi) \to \varphi})$. Finally, if Sent denotes a formalization of the set of all sentences of the language of $(QF - IA)$, then clearly

$$QF - IA \vdash \forall \psi \in \mathrm{Sent}(\neg \mathrm{Con}(QF - IA) \to \mathrm{Thm}_{(QF-IA)}(\psi)).$$

Combining the above we get $QF - IA \vdash \forall d \mathrm{Thm}_{(QF-IA)}(\overline{\mathrm{Prf}_{P^*}(d, \varphi) \to \varphi})$ which implies (7). The second part of (2) is our Proposition 7. This completes the proof. □

Work on partial realizations of Hilbert's Program has produced truly impressive results in proof theory as well as in isolating fragments of the second-order arithmetic needed to formalize and develop significant parts of mathematics ("The Reverse Mathematics Program"). Proof-theoretic results of this kind are systematically presented in Feferman's paper [4], Sieg's papers [20], [22], and [23], while the accomplishments of the "Reverse Mathematics Program" are presented in Simpson's papers [18] and [17]. Other relevant reference can also be found in these papers.

Of course, we cannot do much set-theory or model-theory in theories whose consistency can be proved almost finitistically; standard model-theoretic arguments for the consistency of our number-theories like $(QF - IA)$ or PA cannot be formalized in these theories because they do not "know" that theories like PA or even $(QF - IA)$ are consistent: if $QF - IA \vdash \mathrm{Con}^*(T)$, then $(QF - IA) \vdash \mathrm{Con}(QF - IA) \to \mathrm{Con}(T)$, and so $T \nvdash \mathrm{Con}(QF - IA)$ since otherwise $T \vdash \mathrm{Con}(T)$ which is impossible. Moreover, the same argument can be used to prove the following analog to the standard Second Incompleteness Theorem.

Proposition 10. *Let T be a consistent primitively recursively axiomatized theory provably extending* (PRA). *If T is strong enough to prove the consistency of the finitistic reasoning about numbers, i.e., if $T \vdash \mathrm{Con}(\mathrm{PRA})$, then $T \nvdash \mathrm{Con}^*(T)$.*

COROLLARY 11. *We cannot prove almost finistically the consistency of strong theories such as PA.*

Interestingly, the predicate $\mathrm{Con}^*(T)$ has, if not some naturalness, then at least some properties similar to the properties of the "standard" $\mathrm{Con}(T)$ predicate. Unfortunately, to get a provability predicate for formulas of arbitrary complexity which would correspond to the consistency predicate $\mathrm{Con}^*(T)$, we must define it in a roundabout way, reducing provability to consistency. Bearing in mind that $T \vdash \varphi$ iff $T + \neg\varphi$ is inconsistent, we introduce the following provability predicate, which allows us to prove for it an analogue of the First Incompleteness Theorem.

DEFINITION 4. Let T be as before, then $\mathrm{Thm}^*_T(\varphi) \equiv \neg\mathrm{Con} * (T + \neg\varphi)$.

We list a few properties of $\mathrm{Con}^*(T)$ and Thm^*_T, comparing them with properties of the corresponding standard predicates. Since all the proofs are standard and easy, we omit them. We first relate the predicate Thm^*_T to the standard predicate Thm_T.

PROPOSITION 12. $QF - IA \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \leftrightarrow \mathrm{Thm}_T(\ulcorner\varphi\urcorner) \wedge \mathrm{Con}(QF - IA)$.

COROLLARY 13. *Let T be as before; then for all sentences φ and ψ we have*

(i) $QF - IA \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \rightarrow \mathrm{Thm}_T(\ulcorner\varphi\urcorner)$,
$QF - IA \nvdash \mathrm{Thm}_T(\ulcorner\varphi\urcorner) \rightarrow \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner)$;

(ii) $QF - IA \vdash \mathrm{Con}(T) \rightarrow \mathrm{Con}^*(T)$,
$QF - IA \nvdash \mathrm{Con}^*(T) \rightarrow \mathrm{Con}(T)$;

(iii) $\omega \Vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \Leftrightarrow T \vdash \varphi$,
$\omega \Vdash \mathrm{Con}^*(T) \Leftrightarrow T$ *is consistent*;

(iv) $QF - IA \nvdash \mathrm{Thm}^*_T(\ulcorner 0 = 0\urcorner)$;

(v) $QF - IA \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \leftrightarrow \mathrm{Thm}^*_T(\underline{0 = 0}) \wedge \mathrm{Thm}_T(\ulcorner\varphi\urcorner)$,

(vi) $QF - IA \vdash \mathrm{Con}^*(T) \leftrightarrow \neg\mathrm{Thm}^*_T(\ulcorner 1 = 0\urcorner)$;

(vii) $QF - IA \vdash \mathrm{Thm}^*_{T+\varphi}(\ulcorner\psi\urcorner) \Leftrightarrow \mathrm{Thm}^*_T(\ulcorner\varphi \rightarrow \psi\urcorner)$.

Despite having some unusual properties, $\mathrm{Thm}^*_T(\ulcorner\varphi\urcorner)$ still behaves in many respects as a provability predicate; for example, we have the following analogue of the First Incompleteness Theorem.

PROPOSITION 14. *Let T be as before, and let φ be a sentence asserting its own "star-unprovability" in T, i.e., let φ be obtained by applying Gödel's Diagonal Lemma on the predicate* $\neg\mathrm{Thm}^*_T(x)$,

$$QF - IA \vdash \varphi \leftrightarrow \neg\mathrm{Thm}^*_T(\ulcorner\varphi\urcorner),$$

then $T \vdash \varphi$ if and only if $T \vdash \neg\mathrm{Con}(QF - IA)$, and $T \vdash \neg\varphi$ if and only if $T \vdash \neg\mathrm{Con}^*(T)$. Thus, if T is an ω-consistent theory, then T neither proves nor refutes φ.

The proposition that would correspond to Löb's theorem, i.e.,

$$T \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \rightarrow \varphi \Leftrightarrow T \vdash \varphi$$

is true for theories having enough strength to prove $\mathrm{Con}(QF - IA)$, since $T \vdash \mathrm{Con}(QF - IA)$ implies $T \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \leftrightarrow \mathrm{Thm}_T(\ulcorner\varphi\urcorner)$, and so

$$T \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \rightarrow \varphi \Leftrightarrow T \vdash \mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \rightarrow \varphi \Leftrightarrow T \nvdash \varphi.$$

For weaker theories this need not be true; moreover, we can characterize the theories whose consistency can be proved almost finitistically in terms of the following Π_1 "star-soundness".

PROPOSITION 15. *Let T be as before, then the consistency of T is provable almost finitistically if and only if for all Π_1 formulas φ, $QF-IA \vdash \mathrm{Thm}^*_T\mathrm{Thm}^*_T(\ulcorner\varphi\urcorner) \rightarrow \varphi$.*

Taking for φ a Π_1 sentence independent of T, for example $\mathrm{Con}(T)$, we get a counterexample to the corresponding version of Löb's theorem, because $T \vdash \mathrm{Thm}^*_T(\ulcorner\mathrm{Con}(T)\urcorner) \rightarrow \mathrm{Con}(T)$ and, of course, $T \nvdash \mathrm{Con}(T)$.

§3. **Answer to the second question.** We now show that the theories T which are such that $S \vdash \mathrm{Con}(T)$ are exactly theories whose consistency can be proved almost finitistically.[23] By the same argument as before,

$$QF-IA \vdash \forall x \mathrm{Prf}_{(QF-IA)}(f(x), \ulcorner\varphi(\underline{x})\urcorner)$$

if and only if $QF-IA \vdash \forall x \exists y \mathrm{Prf}_{(QF-IA)}(y, \ulcorner\varphi(\underline{x})\urcorner)$, and so we introduce

DEFINITION 5. Let φ be of the form $\forall x \psi(x)$, where $\psi(x)$ is primitive recursive. Then

$$\mathrm{AFThm}(\ulcorner\varphi\urcorner) \equiv \forall x \exists y \mathrm{Prf}_{(QF-IA)}(y, \ulcorner\psi(\underline{x})\urcorner).$$

Here $\mathrm{AFThm}(x)$ stands for *almost finitistic theorem.*

Thus, $\varphi \in S$ if and only if $QF-IA \vdash \mathrm{AFThm}(\ulcorner\varphi\urcorner)$.

PROPOSITION 16. *For all Π_1 sentences*

$$QF-IA \vdash \mathrm{AFThm}(\varphi) \leftrightarrow (\mathrm{Con}(QF-IA) \rightarrow \varphi).$$

PROOF. The proof is similar to the proof of Proposition 3. □

From Proposition 15 of the previous section, taking $T = (QF-IA)$, we get that for all Π_1 sentences φ, $QF-IA \vdash \mathrm{Thm}^*_{(QF-IA)}(\ulcorner\varphi\urcorner) \rightarrow \varphi$; this, together with the previous proposition, implies that for all Π_1 sentences φ

$$QF-IA \vdash \mathrm{Thm}^*_{(QF-IA)}(\ulcorner\varphi\urcorner) \rightarrow \mathrm{AFThm}(\ulcorner\varphi\urcorner).$$

Also, just from the definitions, we have

$$(9) \qquad QF-IA \vdash \mathrm{Con}^*(T) \leftrightarrow \mathrm{AFThm}(\mathrm{Con}(T)).$$

To answer the second question we first prove the following proposition.

PROPOSITION 17. *S and $(QF-IA)+\mathrm{Con}(QF-IA)$ have the same set of consequences.*

PROOF. Since by Corollary 4 $QF-IA \vdash \mathrm{Con}^*(QF-IA)$, (9) implies $QF-IA \vdash \mathrm{AFThm}(\mathrm{Con}(QF-IA))$. Thus, $\mathrm{Con}(QF-IA) \in S$ and so, since by Lemma 1 $(QF-IA) \subseteq S$, we get $(QF-IA)+\mathrm{Con}(QF-IA) \subseteq S$. Conversely, let $\varphi \in S$, then $QF-IA \vdash \mathrm{AFThm}(\ulcorner\varphi\urcorner)$. By Proposition 16 we have $QF-IA \vdash \mathrm{AFThm}(\ulcorner\varphi\urcorner) \rightarrow (\mathrm{Con}(QF-IA) \rightarrow \varphi)$, and so $QF-IA \vdash \mathrm{Con}(QF-IA) \rightarrow \varphi$. Hence, $(QF-IA)+\mathrm{Con}(QF-IA) \vdash \varphi$ which shows that all sentences from S are provable in $(QF-IA)+\mathrm{Con}(QF-IA)$. □

COROLLARY 18. *Let T be as before, then $S \vdash \mathrm{Con}(T)$ if and only if the consistency of T can be proved almost finitistically.*

[23] Recall that by Lemma 1, $(QF-IA) \subseteq S$.

PROOF. By the previous proposition $S \vdash \mathrm{Con}(T)$ if and only if $(QF - IA) + \mathrm{Con}(QF - IA) \vdash \mathrm{Con}(T)$ which is the case if and only if $QF - IA \vdash \mathrm{Con}^*(T)$. $\square$

Thus, if we restrict the ω-rule from Detlefsen's first claim in the same manner (and with the same justification) as we restricted the ω-rule from Detlefsen's second claim, it turns out that this, at the first sight stronger rule, does not provide us with more power in proving consistency of theories than the previous one.

§4. **A generalization.** One could argue that once we accept the above finistically warranted ω-rule as a legitimate means of finitistic proof, then there is no reason why we could not iterate it, i.e., use it to justify some stronger ω-rules that are not justifiable in $(QF - IA)$. Thus, we could build a chain of theories S^i starting from $(QF - IA)$ by adding to S^{i+1} only those primitive recursive instances of the ω-rule applied to only finitistically meaningful formulas that are already justifiable in S^i.

DEFINITION 6. Let $S^0 = (QF - IA)$, and let S^{i+1} be the collection of all sentences of the form $\forall x \varphi(x)$, for $\varphi(x)$ primitive recursive, for which there exists a primitive recursive function $f(x)$ such that $S^i \vdash \forall x \mathrm{Prf}_{S^i}(f(x), \ulcorner \varphi(\underline{x}) \urcorner)$. Then we set $S^\omega = \bigcup_{i \in \omega} S^i$.

Note that the above definition allows us to remain committed to the same class of sentences as meaningful and the same class of functions as acceptable.

Now one could argue that the resulting theory S^ω is quite close to the standard finitistic reasoning, with an argument as follows. Assume $S^\omega \vdash \theta$. To prove θ, a finitist could start with some axioms of $(QF - IA)$ and then add in stages some instances of the ω-rule mentioned in Definition 6, always justifying, in an already obtained system, any new instance he wants to add. After finitely many steps, he gains enough power to prove θ.

It is easy to present applications of such rules in a more natural form. Consider the following sequence of rules and proof-systems defined simultaneously by induction. Let $\sigma_0 = (QF - IA)$, where $(QF - IA)$ is taken with a Tait style proof system (i.e., proofs are in a tree form with axioms and rules of inference). Let

$$R_{i+1} \colon \frac{\forall x \mathrm{Prf}_{\sigma_i}(f(x), \ulcorner \varphi(\underline{x}) \urcorner)}{\forall x \varphi(x)},$$

and let σ^{i+1} be obtained from σ^i by adding the rule R_{i+1} with the following restriction: for any application of the rule R_{i+1}, the immediate subderivation above the application of the rule is a σ^i derivation. It is easy to see that the sets of theorems of σ_i and S^i coincide.[24]

[24] Transfinite progressions of theories obtained by addition of reflection principles are considered in Feferman's paper [2] with the purpose of bridging the gap between the r.e. theories for which the incompleteness theorem holds and theories with nonconstructive set of axioms (e.g., all true sentences of arithmetic). One of them (2.16(iv)) is similar to the way the S^i's are built, except that the proofs need not be produced by primitive recursive functions and that there are no complexity restrictions put on formulas (i.e., we add to a theory A all formulas of the form $\forall x \varphi(x)$ for *all* φ such that $A \vdash \forall x \exists y \mathrm{Prf}_A(y, \ulcorner \varphi(\underline{x}) \urcorner)$). While the first restriction is inessential, the complexity restriction is important: without it, starting with $(QF - IA)$, we get in the very next step full PA (of course our complexity restriction is imposed by what we accept as finitistically meaningful sentences). The

But what are the theories whose consistency is provable in S^ω before, it is easy to see that $S^i \subseteq S^{i+1}$. Also, by Proposition 5 we have $QF - IA \vdash \mathrm{Con}^*_{S^i}(S^i)$ and so, again using the fact about the provably recursive functions of $(QF - IA)$, we get that for some primitive recursive function $f(x)$,

$$QF - IA \vdash \mathrm{Prf}_{S^i}(f(x), \ulcorner\neg\mathrm{Prf}_{S^i}(\underline{x}, \ulcorner 1 = 0\urcorner)\urcorner)$$

which implies $S^{i+1} \vdash \mathrm{Con}(S^i)$.

On the other hand, in the same way as in the proof of Proposition 17 we get that for any $\varphi \in S^{i+1}$, $S^i + \mathrm{Con}(S^i) \vdash \varphi$. Hence, we have the following proposition.

PROPOSITION 19. *S^{i+1} and $S^i + \mathrm{Con}(S^i)$ have the same set of consequences.*

Obviously in the theory $\bigcup_{i\in\omega} S^i$ one can prove consistency of more theories than in $(QF - IA)$ or S, but these theories do not seem to be mathematically much more interesting than theories whose consistency is already provable in S. For example, S proves the consistency of the fragment $(\Sigma_1 - IA)$ of Z (fragment with induction schema restricted to Σ_1 formulas), because $QF - IA \vdash \mathrm{Con}(QF - IA) \leftrightarrow \mathrm{Con}(\Sigma_1 - IA)$ and $S \vdash \mathrm{Con}(QF - IA)$. On the other hand, the strength of induction available in a theory whose consistency is provable in S^ω is below the full Σ_2-induction. This is a consequence of the following proposition.

PROPOSITION 20. *S^ω is a subtheory of $(\Sigma_2 - IA)$.*

PROOF. Let $\mathrm{Con}^i(QF - IA)$ be defined by induction for $i \geq 0$ as follows:

$$\mathrm{Con}^0(QF - IA) \equiv (0 = 0);$$
$$\mathrm{Con}^{i+1}(QF - IA) \equiv \mathrm{Con}((QF - IA) + \mathrm{Con}^i(QF - IA)).$$

These sentences are obviously Π_1 and for $i \geq 1$,

$$S^i \equiv ((QF - IA) + \mathrm{Con}^i(QF - IA)).$$

We prove by induction on i that $(\Sigma_2 - IA) \vdash \mathrm{Con}^i(QF - IA)$ which is enough to prove our claim, since $(\Sigma_2 - IA)$ by definition extends $(QF - IA)$. For $i = 0$ this is trivial. Assume that $(\Sigma_2 - IA) \vdash \mathrm{Con}^i(QF - IA)$. From the Propositions 3.1 and 1.6 of [21] it obviously follows that for all Σ_1 formulas φ, $\Sigma_2 - IA \vdash$

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iteraction is transfinite along a path through Kleene's O, and the theory obtained as the union of all these iterations is the complete theory of the structure of natural numbers. In [5] Feferman and Spector showed that there are paths through O (all Π^1_1 ones) along which the same procedure as above gives a theory incomplete even for Π_1 sentences. Proof systems with rules similar to the one we use are discussed in great detail in Schmerl's [15]. His systems are infinitary semiformal systems; be also discusses a proof system with an (infinitary) ω-rule which has a side condition that it must be provable in PRA that the rule is applied to some appropriate (infinite) collection of immediate subderivations. We, besides using only finitary proof-systems, are allowed to use σ_i rather than just PRA in proving $\forall x \mathrm{Prf}_{\sigma_i}(f(x), \ulcorner\varphi(\underline{x})\urcorner)$. Using the fact that for any recursively axiomatized extension T of $(QF - IA)$ and any Π_1 sentence φ, $T \vdash \varphi$ if and only if $\mathrm{PRA} + \mathrm{Con}(T) \vdash \varphi$ one can see that these conditions are actually equivalent. A relevant reference is also Rosser's paper [14] which, according to [3], contains the first published discussion of the "provable" ω-rule with no restriction on the complexity of formulas. In this paper we are mainly interested in applications of restricted forms of the omega rule in consistency proofs, connections of these proofs with Hilbert's Program, and related philosophical issues rather than in the ω-rule itself. Most of this paper I wrote during the third year of my graduate work (1988), without any knowledge of the above results. After reading this earlier version of the present paper David Libert called my attention to Feferman's paper [2]. Further references were provided by Professor Solomon Feferman. I am grateful to both of them for their help.

$\text{Thm}_{(QF-IA)}(\underline{\varphi}) \rightarrow \varphi$. Taking for φ formula $\neg\text{Con}^i(QF - IA)$, we get from our inductive assumption that $\Sigma_2 - IA \vdash \neg\text{Thm}_{(QF-IA)}(\ulcorner\neg\text{Con}^i(QF - IA)\urcorner)$ which by the basic property of (the formalized) proofs implies $\Sigma_2 - IA \vdash \text{Con}^{i+1}(QF - IA)$. This clearly implies the claim which was to be proved. □

COROLLARY 21. *If $S^\omega \vdash \text{Con}(T)$, then $(\Sigma_2 - IA) \nsubseteq T$.*

PROOF. Assume that $S^\omega \vdash \text{Con}(T)$ and that $\Sigma_2 - IA$ is a subtheory of T. By the previous proposition S^ω would also be a subtheory of T and consequently we would have that T proves $\text{Con}(T)$ which is not possible. □

From what we have seen it seems that by adding stronger forms of the ω-rule as above, we might in fact lose finitistic grounds faster than we gain power in proving consistency of theories. This indicates why partial realizations of Hilbert's Program that encompass stronger theories which have been accomplished so far are of rather different nature. They involve a "radical" replacement of finitism by other, more general and clearly nonfinitary but still restricted means. Finitistic methods are substituted by predicative, genetic ("inductive from below") or countably infinitary methods; for details we refer the reader to [4].

REFERENCES

[1] M. DETLEFSEN, *On interpreting Gödel's second theorem*, ***Journal of Philosphical Logic***, vol. 8 (1979), pp. 297–313.

[2] S. FEFERMAN, *Transfinite recursive progressions of axiomatic theories*, this JOURNAL, vol. 27 (1962), pp. 259–316.

[3] ———, *Introductory note to* 1931*c*, ***Kurt Gödel: Collected works, vol. I*** (S. Feferman editor-in-chief), Oxford University Press, London (1986).

[4] ———, *Hilbert's program relativized: proof-theoretical and foundational reductions*, this JOURNAL, vol. 53 (1988), pp. 354–384.

[5] S. FEFERMAN and C. SPECTOR, *Incompleteness along paths in progressions of theories*, this JOURNAL, vol. 27 (1962), pp. 383–390.

[6] H. FRIEDMAN, *Systems of second-order arithmetic with restricted induction*. I, II, this JOURNAL, vol. 41 (1976), pp. 557–559 (Abstracts).

[7] H. FRIEDMAN, S. G. SIMPSON, and R. L. SMITH, *Countable algebra and set existence axioms*, ***Annals of Pure and Applied Logic***, vol. 25 (1983), pp. 141–181; addendum, vol. 28 (1985), pp. 319–320.

[8] JACQUES HERBRAND, ***Logical Writings***, (W. Goldfarb, editor), Harvard University Press, Cambridge, Massachusetts, 1971, pp. 288–289.

[9] L. HARRINGTON, private communication to H. Friedman.

[10] D. HILBERT, *Über das Unendliche*, ***Mathematische Annalen***, vol. 95; English translation, ***Philosophy of mathematics, selected readings***, (P. Benacerrof and H. Putnam, editors), Cambridge University Press, London, (1983), pp. 161–190.

[11] DAVID HILBERT, *Die Grundlegung der elementaren Zahlenlehre*, ***Mathematische Annalen***, vol. 104 (1931), pp. 485–494.

[12] A. IGNJATOVIĆ, *Hilbert's program and the omega-rule*, unpublished manuscript, December 1988.

[13] ———, *Fragments of first- and second-order arithmetic and length of proofs*, Ph.D. Thesis, University of California at Berkeley, Berkeley, California (1990).

[14] B. ROSSER, *Gödel theorems for nonconstructive logics*, this JOURNAL, vol. 2 (1937), pp. 129–137.

[15] U. SCHMERL, *Iterated reflection principles and the ω-rule*, this JOURNAL, vol. 47 (1982), pp. 721–733.

[16] J. R. SHOENFIELD, *On a restricted ω-rule*, ***Bulletin De L'Academie Polonaise Des Sciences***, vol. VII, No. 7, 1959, pp. 405–407.

[17] S. G. Simpson, *Which set existence axioms are needed to prove the Cauchy-Peano theorem for ordinary differential equations?*, this Journal, vol. 49 (1984), pp. 783–802.

[18] ———, *Partial realization of Hilbert's program*, this Journal, vol. 53 (1988), pp. 349–363.

[19] ———, *Subsystems of second-order arithmetic* (in preparation).

[20] W. Sieg, *Foundations for analysis and proof theory*, ***Synthese***, vol. 60 (1984), pp. 159–200.

[21] ———, *Fragments of arithmetic*, ***Annals of Pure and Applied Logic***, vol. 28 (1985), pp. 33–71.

[22] ———, *Hilbert's program sixty years later*, this Journal, vol. 53 (1988), pp. 338–348.

[23] ———, *Relative consistency and accessible domains*, ***Synthese***, vol. 84 (1990), pp. 259–297.

[24] C. Smorynski, *The incompleteness theorems*, ***Handbook of mathematical logic*** (J. Barwise, editor), North-Holland, Amsterdam (1977), pp. 821–895.

[25] W. W. Tait, *Finitism*, ***Journal of Philosophy***, vol. 78 (1981), pp. 524–546.

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