Numerical Differentiation And Signal Processing*

Aleksandar Ignjatovic
Kromos Technology Inc.
4984 El Camino Real, Suite 210
Los Altos, CA 94022

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1 Introduction

1.1 Motivation And Preliminaries

The aim of this work is to study *local approximations* of band-limited signals as a signal representation and processing paradigm *complementary* to the standard signal representation and processing paradigm based on global approximations of band-limited signals. Nyquist's interpolation formula[1]

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

requires infinitely many samples, equidistant in time, and consequently it is *global in nature.* On the other hand, Taylor's formula

$$f(t_0 + t) = \sum_{j=0}^{\infty} f^{(j)}(t_0) \frac{t^j}{j!}$$

requires infinitely many derivatives at a single point in time. Since derivatives can be (theoretically) obtained from a continuous signal over an arbitrarily small interval in time, Taylor's formula is *local in nature.* The above observations have practical consequences, because the truncation errors of the finitary approximations of Nyquist's expansion and of Taylor's expansion reflect the nature of convergence of their infinitary counterparts.

Taylor's formula is of little use for signal processing. In general, despite the fact that a vast number of physical processes are characterized by equations which involve either partial or "standard" derivatives of physical parameters, the sampled values of these parameters are seldom used for numerical differentiation. It is generally held that in most cases the problem of the impact of the measurement errors on numerical evaluation of differential operators is insurmountable. Many sources on numerical analysis caution that numerical differentiation should be avoided. As Hildebrand mentions (see [1], page 85), "These expectations are borne out in practice. In particular, numerical differentiation should be avoided whenever possible, particularly when the data are empirical and subject to appreciable errors of observation."

We will show that for important classes of signals, for example the band-limited signals (not necessarily of finite energy), one can evaluate essentially all differential operators of practical significance with high accuracy, including such operators of a high degree, 16 and higher, based on measured values of the signal on the oversampled scale, or on its continuous waveform, processed in the analog domain.

The values of these differential operators will be used in a "Taylor-type" approximation formula, which, unlike the real Taylor's formula, is useful for signal processing.

We now show that the main problem of stable numerical evaluation of differential operators of practical significance is the poor choice of the basis of the vector space of linear differential operators with real coefficients, namely the basis consisting of the standard derivatives $f(t)$, $f'(t)$,

*This paper is an abridged version of the Kromos Technology technical report number 1 "Numerical differentiation and signal processing". The paper can be found at www.kromos.com. It contains the proofs of all statements which are not proved in this paper.

[1]We normalize the frequency bound to π, so that the integers are the Nyquist rate sampling moments.

$f''(t), \ldots, f^{(k)}(t), \ldots$

Let $f(t)$ be a band-limited signal whose Fourier transform is $\widehat{f}(\omega)$, i.e. such that

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega \tag{1}$$

Then the k^{th} derivative of such a signal is given by

$$f^{(k)}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (i\omega)^k \widehat{f}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega \tag{2}$$

Thus, the transfer function of differentiation of order k is $(i\omega)^k$ and so its power spectrum is ω^{2k}. However, operators with such a transfer function destroy any information about the spectrum of a band limited signal, and this makes derivatives useless for the purpose of signal processing.

To appreciate this fact, we first normalize the standard derivatives so that the power spectra of their transfer functions all have the same range. This is accomplished by by replacing the sequence $f(t), f'(t), f''(t), \ldots, f^{(k)}(t), \ldots$ by the sequence $f(t), \frac{f'(t)}{\pi}, \frac{f''(t)}{\pi^2}, \ldots, \frac{f^{(k)}(t)}{\pi^k}, \ldots$. The transfer functions of such a normalized sequence and the corresponding power spectra respectively, are $\mathcal{D}(k,\omega) = (\frac{i\omega}{\pi})^k$, and $\wp(k,\omega) = \left(\frac{\omega}{\pi}\right)^{2k}$. Thus $\wp(k,\omega) \leq 1$ for all k and all $\omega \in [-\pi, \pi]$.

Figure 1 represents the graphs of $\wp(16,\omega)$ and $\wp(17,\omega)$, showing that they are essentially indistinguishable from each other and that they both attain extremely small values at frequencies just below the band limit. Thus, any application of the standard or normalized standard derivatives to signal processing is numerically impossible; even if evaluation of derivatives was not hampered by their sensitivity to noise, they would still be useless for the methods which are based on spectral analysis. Since the absolute values of their transfer functions are so small everywhere except near the very end of the spectrum, *the values of derivatives do not encode the spectral content of the signal.* Fortunately, this problem is *not* inherent in differentiation, *but it is due to the poor choice of the basis of the vector space of the linear differential operators with real coefficients.* We will show that with a proper choice of the base for the vector space of linear differential operators, the above problems disappear.

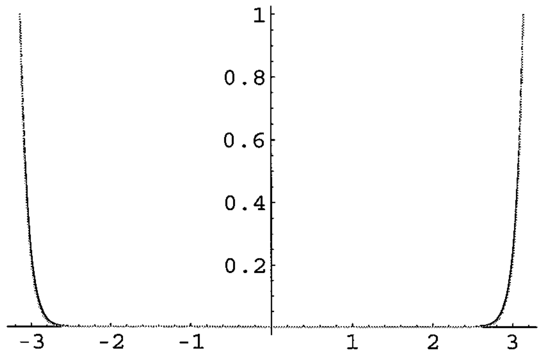

Figure 1: Power spectra of differentiation of orders 16 (black) and 17 (gray)

In general, a "good basis" for a vector space is an orthogonal basis. Such orthogonal operators should be "spectrum friendly", *i.e.*, instead of destroying information about the spectrum of the signal, they should *encode* such information in a way that is usable for signal processing; for this reason we will call them "chromatic derivatives". Since the values of such differential operators will be obtained from relatively small sections of the signal, they will provide a form of "spectral analysis" of transients. This will provide the basis of signal processing based on local signal behavior, as encoded by the values of families of specially chosen differential operators.

1.2 Signal Representation

The standard method of signal processing based on harmonic analysis relies on representation of the π band-limited signals by the Nyquist interpolation formula $f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$, which is a linear combination of all integer shifts $\frac{\sin(\pi(t-k))}{\pi(t-k)}$ of a single function $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$, with coefficients equal to the values $f(k)$ of the input signal f at the integers. The method of wavelets, on the other hand, represents the input signal as a linear combination of integer shifts and binary scalings $\psi\left(2^j t - k\right)$ of a single "mother wavelet" $\psi(t)$.

The signal processing method presented here is based on a representation of the signal obtained using only a single base function $B_0(t)$ as well as a single operator $\mathbf{K} : \mathbb{N} \times C^{\infty} \to C^{\infty}$ defined by recursion, satisfying that for each k, $\mathbf{K}[k, \cdot\ ]$ is a linear differential operator of order k. The members of the corresponding family $\{\mathbf{K}[k, \cdot\ ]\}_{k \in N}$ of linear differential operators are called "chromatic derivatives". More precisely, the signal is represented in the form

$$f(t) = \sum_{k=0}^{\infty} \mathbf{K}[k, f](t_0)\, \mathbf{K}[k, B_0(t - t_0)]$$

where $B_0(t)$ is a single base function, associated with the operator $\mathbf{K}$ and suitably normalized. Thus, in the case of *ideal* reconstruction with infinitely many parameters we do not use shifts or scaling of the base function B_0, but only its chromatic derivatives. In the finite approximation case the truncation of the above formula up to the N^{th} term results in a "Taylor-type", local expansion, with $\mathbf{K}[k, B_0(t - t_0)]$ replacing polynomials $\frac{(t-t_0)^k}{k!}$, and with $\mathbf{K}[k, f][t_0]$ replacing the values $f^{(k)}(t_0)$ of the standard operator of differentiation appearing in the Taylor formula.

In a sense polynomials are the worst interpolants for band limited signals: they themselves are not band limited and grow very rapidly. However, B_0 is a band-limited function, as well as all of the interpolants $\mathbf{K}[k, B_0]$, and they tend to zero for large values of the argument. Thus, an approximation based on the interpolants $\mathbf{K}[k, B_0]$ has the error term with fundamentally different features than the error term of the standard Taylor's expansion. As seen on the figure 2, the error accumulates gently and approximation stays bounded. This makes such an approximation a versatile tool for signal processing purposes.

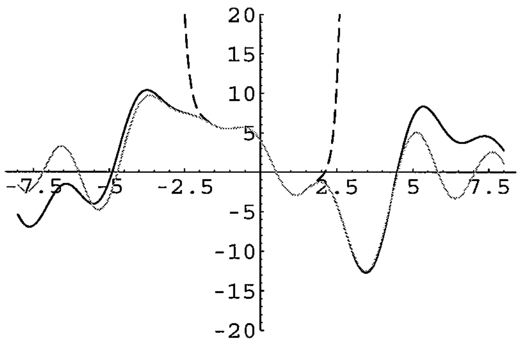

Figure 2: Black - signal; dashed - Taylor's approximation of order 12; gray - Chromatic approximation of order 12

2 Orthogonal Bases For Differential Operators

We now introduce a notion of orthogonality on the vector space of linear differential operators with real coefficients, and the corresponding inner product.

2.1 Orthogonality of linear differential operators

Let Λ be the vector space consisting of linear differential operators with real coefficients, *i.e.*, operators of the form:

$$\Delta[f] = a_0 f(t) + a_1 f'(t) + \ldots + a_n f^{(n)}(t),$$

with $a_i \in \mathbb{R}$, (n is called the order of Δ) with usual addition of functions and usual scalar multiplication:

$$(f + g)(t) = f(t) + g(t) \text{ and } (c \cdot f)(t) = c \cdot f(t).$$

When we say for short *a (differential) operator* we actually mean linear differential operator with constant and real coefficients.

For simplicity of notation, application of an operator to a function will be denoted by square brackets, *i.e.*, if D is an operator and $f(t)$ a function, then $D[f]$ is the value of the operator D applied to f, and $(D[f])(t_0)$ stands for the value of the function $D[f]$ at the point t_0.

We now introduce a family of inner products in the vector space Λ.

Definition 1 *Let B be an analytic function, $t_0 \in \mathbb{R}$ and let $D^{(0)}, D^{(1)}, \ldots, D^{(n)}, \ldots \in \Lambda$ be a sequence such that for every n, $D^{(n)}$ is an operator of order exactly n.*
We say that the sequence $\{D^{(n)}\}_{n\in N}$ is (B, t_0)–orthogonal if

$$(D^{(m)}[D^{(n)}[B]])(t_0) = (-1)^n c_n \delta_{m,n} \tag{3}$$

for some $c_n > 0$, and $\delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

Clearly, since for every n the operator $D^{(n)}$ is of order exactly n, $\{D^{(n)}\}_{n\in N}$ forms a basis of Λ.

We note that the sequence of the standard derivatives is *not* orthogonal with respect to any pair (B, t_0) because otherwise, for example, $D^{(4)}[B](t_0) = D^{(2)}[D^{(2)}][B](t_0) = c_2\delta_{2,2} = c_2 \neq 0$ and also $D^{(4)}[B](t_0) = D^{(3)}[D^{(1)}][B](t_0) = -c_1\delta_{3,1} = 0$ which is a contradiction.
If a family of differential operators is orthogonal with respect to the pair $(B, 0)$, it is also orthogonal with respect to the pair $(B(t - t_0), t_0)$ for every t_0.

Let for $\lambda_1, \lambda_2 \in \Lambda$ the operation $\langle\lambda_1, \lambda_2\rangle$ be defined as the linear extension of

$$\langle D^{(m)}, D^{(n)}\rangle = (-1)^n (D^{(m)}[D^{(n)}[B(t)]])(t_0)$$

i.e.,

$$\begin{aligned}
\langle\lambda_1, \lambda_2\rangle &= \left\langle \sum_{i=0}^{k} \mu_i D^{(i)}, \sum_{j=0}^{m} \rho_j D^{(j)} \right\rangle \\
&= \sum_{i,j=0}^{i=k,j=m} \mu_i\rho_j \langle D^{(i)}, D^{(j)}\rangle \\
&= \sum_{i,j=0}^{i=k,j=m} \mu_i\rho_j (-1)^j D^{(i)}[D^{(j)}[B]](t_0) \\
&= \sum_{i=0}^{min\{k,m\}} \mu_i\rho_i c_i
\end{aligned}$$

Lemma 1 *$\langle\lambda_1, \lambda_2\rangle$ is a scalar product.*

Proof. All properties of a scalar product are easy to verify; for example if $\lambda \neq 0$ and $\lambda = \sum_{i=0}^{k} \mu_i D^{(i)}$ then from the above

$$\langle\lambda, \lambda\rangle = \sum_{i=0}^{k} \mu_i^2 c_i > 0.$$

□

Theorem 1 *Every sequence of orthogonal differential operators satisfies a recursion relation of the form*

$$D^{(m+2)} = \alpha_{m+1}(D^{(m+1)})' + \beta_{m+1}(D^{(m+1)}) + \delta_m D^{(m)} \tag{4}$$

where $(D^{(m+1)})'$ denotes composition of the "standard" operator of differentiation and $D^{(m+1)}$.

Proof. The proof is analogous to the similar statement for orthogonal polynomials; for details see [3]. □

We denote functions $(D^{(k)}[B])(t)$ by $\overline{B_k}(t)$.

2.2 Isomorphic images of Λ into subspaces of analytic functions

Fix a family of linear differential operators $\{D^{(k)}\}$ orthogonal with respect to a pair (B, t_0). Let $\Psi : \Lambda \to \mathbb{C}^\infty$ be defined as the linear extension of the mapping

$$\Psi_0 : D^{(k)} \mapsto D^{(k)}[B].$$

We now consider the set L_B associated with the pair $\langle B, t_0\rangle$ consisting of all analytic functions f such that

$$\sum_{k=0}^{\infty} \frac{1}{c_k} \left((D^{(k)}[f])(t_0)\right)^2 < \infty \tag{5}$$

where c_k are the coefficients appearing in the condition of orthogonality of the family $D^{(k)}$.
We define a scalar product $\langle f, g\rangle_\Psi$ by

$$\langle f, g\rangle_\Psi = \sum_{k=0}^{\infty} \frac{1}{c_k} \, (D^{(k)}[f])(t_0) \, (D^{(k)}[g])(t_0) \tag{6}$$

Then, for $f \equiv g$, we get $||f||^2_\Psi = \sum_{k=0}^{\infty} \frac{1}{c_k} \left(D^{(k)}[f](t_0)\right)^2 < \infty$ by our assumption.

Also, for $f = \overline{B}_n = D^{(n)}[B]$, we get

$$\langle \overline{B}_n, \overline{B}_m \rangle_\Psi =$$
$$\sum_{k=0}^{\infty} \frac{1}{c_k} (D^{(k)}[\overline{B}_n])(t_0) \cdot (D^{(k)}[\overline{B}_m])(t_0) =$$
$$\sum_{k=0}^{\infty} \frac{1}{c_k} D^{(k)}[D^{(n)}[B]](t_0) \cdot D^{(k)}[D^{(m)}[B]](t_0) =$$
$$\begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{c_n}\left((-1)^n c_n\right)^2 = c_n & \text{if } m = n. \end{cases}$$

That is, $\langle \overline{B}_n, \overline{B}_m \rangle_\Psi = c_n \delta_{m,n} \geq 0$. This implies that

$$\Psi : \sum_{i=0}^{k} \lambda_i D^{(i)} \to \sum_{i=0}^{k} \lambda_i \overline{B}_i$$

is an isomorphic embedding of Λ into $\mathbb{C}^\infty$ with respect to the scalar products as above and with the set of finite linear combinations of $\overline{B}_k$'s as the range of Ψ.

We now consider functions $B_n(t) = \frac{\overline{B}_n(t)}{\sqrt{c_n}}$. Then,

$$\begin{aligned} \langle B_n, B_n \rangle_\Psi &= \langle \frac{\overline{B_n}}{\sqrt{c_n}}, \frac{\overline{B_n}}{\sqrt{c_n}} \rangle_\Psi \\ &= \frac{1}{c_n} \langle \overline{B_n}, \overline{B_n} \rangle_\Psi = \frac{1}{c_n} c_n = 1 \end{aligned}$$

Thus,

$$\{B_n(t)\}_{n \in \mathbb{N}} = \left\{ \frac{\overline{B}_n(t)}{\sqrt{c_n}} \right\}_{n \in \mathbb{N}} = \left\{ \frac{D^{(n)}[B](t)}{\sqrt{c_n}} \right\}_{n \in \mathbb{N}}$$

is an orthonormal system with respect to the above product. Also,

$$\begin{aligned} &\langle f, B_n \rangle_\Psi && (7) \\ &= \langle f, \frac{D^{(n)}[B]}{\sqrt{c_n}} \rangle_\Psi \\ &= \sum_{k=0}^{\infty} \frac{1}{c_k} (D^{(k)}[f])(t_0) \left(D^{(k)} \left[\frac{D^{(n)}[B]}{\sqrt{c_n}} \right] \right)(t_0) \\ &= \frac{1}{c_n} (D^{(n)}[f])(t_0) \frac{1}{\sqrt{c_n}} \left(D^{(n)}[D^{(n)}[B]] \right)(t_0) \\ &= \frac{1}{c_n} \frac{1}{\sqrt{c_n}} (D^{(n)}[f])(t_0)(-1)^n c_n \\ &= \frac{(-1)^n}{\sqrt{c_n}} (D^{(n)}[f])(t_0) && (8) \end{aligned}$$

Thus, letting

$$\begin{aligned} &A_P[f, M, t] = && (9) \\ &\sum_{n=0}^{M} < f, B_n > B_n(t) = \\ &\sum_{n=0}^{M} \frac{(-1)^n}{\sqrt{c_n}} (D^{(n)}[f])(t_0) \cdot B_n(t) && (10) \end{aligned}$$

and by substituting $B_n(t)$ by $\frac{\overline{B}_n(t)}{\sqrt{c_n}}$,

$$\begin{aligned} &A_P[f, M, t] = && (11) \\ &\sum_{n=0}^{M} \frac{(D^{(n)}[f])(t_0)}{\sqrt{c_n}} (-1)^n \frac{\overline{B}_n(t)}{\sqrt{c_n}} = \\ &\sum_{n=0}^{M} \frac{(D^{(n)}[f])(t_0)}{\sqrt{c_n}} (-1)^n \frac{(D^{(n)}[B])(t)}{\sqrt{c_n}} = \\ &\sum_{n=0}^{M} \frac{(-1)^n}{c_n} (D^{(n)}[f])(t_0)(D^{(n)}[B])(t) && (12) \end{aligned}$$

If the family of operators $\{D_n\}_{n \in N}$ is $(B(t), 0)$ orthogonal, then it is easy to see that it is also $(B(t - t_0), t_0)$ orthogonal for every t_0. In this case one obtains

$$\begin{aligned} &A_P[f, M, t_0, t] = \\ &\sum_{n=0}^{M} (D^{(n)}[f])(t_0) \frac{(-1)^n}{c_n} (D^{(n)}[B])(t - t_0) && (13) \end{aligned}$$

and equation 10 becomes

$$A_P[f, M, t_0, t] = \sum_{n=0}^{M} \frac{(-1)^n}{\sqrt{c_n}} (D^{(n)}[f])(t_0) \cdot B_n(t - t_0) \qquad (14)$$

From the above equation we get that the sum of the squares of the coefficients of the expansion $A_P[f, M, t]$ is equal to

$$\sum_{n=0}^{M} \left(\frac{(-1)^n (D^{(n)}[f])(t_0)}{\sqrt{c_n}} \right)^2 = \sum_{n=0}^{M} \frac{1}{c_n} \left((D^{(n)}[f])(t_0) \right)^2$$

which for $M \to \infty$ converges to

$$\sum_{k=0}^{\infty} \frac{1}{c_k} \left[(D^{(k)}[f])(t_0) \right]^2 = \|f\|_\Psi^2$$

We are interested in cases when $A_P[f, M, t_0, t]$ is a *local approximation* of $f(t)$.

Before proceeding, we examine briefly another isomorphism. Let Φ be given by

$$\Phi : f \rightarrow < (D^{(0)}[f])(t_0), \ldots, (D^{(n)}[f])(t_0), \ldots >$$

This is an isomorphism from the set of all functions in L_B into the set S_B of all sequences $S = \langle a_n \rangle_{n \in w}$ such that $\sum_{n \in N} \frac{a_n^2}{c_n} < \infty$.

With $\langle S, P \rangle_\Phi = \sum_{n \in N} \frac{1}{c_n} a_n b_n$, for $S = \langle a_n \rangle_{n \in \omega}$, $P = \langle b_n \rangle_{n \in \omega}$, we get an isomorphism between L_B and S_B with corresponding scalar products. Note that

$$\Phi(\overline{B}_n) = \\ \langle D^{(0)}[D^{(n)}[B]](t_0), \ldots, D^{(n)}[D^{(n)}[B]](t_0), \ldots \rangle = \\ \langle 0, \ldots, 0, (-1)^n c_n, 0, \ldots \rangle;$$

and so

$$\Phi(B_n) \;=\; \Phi\left(\frac{\overline{B}_n}{\sqrt{c_n}}\right) = \langle 0, \ldots, 0, (-1)^n \sqrt{c_n}, 0, \ldots \rangle.$$

Thus,

$$\begin{aligned} \langle \Phi(B_n), \Phi(B_n) \rangle_\Phi &= \frac{1}{c_k}((-1)^n \sqrt{c_k}))^2 \\ &= 1 \end{aligned}$$

We note that under this isomorphism a function is identified with the sequence of values of the images of this function under orthogonal differential operators, evaluated at a *single* point in time.

The sequence of chromatic derivatives can be seen as values at natural numbers of a particular function $\mathcal{D}$ associated with the family of orthogonal differential operators $D^{(k)}$, and we will later provide a such a function for one of the most important cases of orthogonal differential operators.

Problem: Determine what is the class of all real analytic functions B_0 such that the pair $(B_0, 0)$ defines a family of orthogonal linear differential operators with real coefficients.

3 Differential Operators Associated With Chebyshev Polynomials

We now present one of the main cases of orthogonal linear differential operators associated with pairs (B, t_0), and resulting approximations of functions, examining the nature of the convergence of these approximations. We restrict our attention to the cases when B is a $[-\pi, \pi]$ band limited function. However, similar arguments hold for other choices of B with convergent Fourier transforms which are not necessarily band limited. Among $[-\pi, \pi]$ band limited functions B such that $(B, 0)$ defines a family orthogonal differential operators, we will examine in detail one of the most important cases: the pair $(J_0(\pi t), 0)$, where $J_0(\pi t)$ stands for the Bessel function of the first kind and of order 0.

Sequences of orthogonal differential operators which are associated with pairs $(B, 0)$ where B is a specially chosen π band limited function (signal) have property that they encode information about the spectrum of the signal and for that reason we call operators from such families "Chromatic derivatives". It will be shown that the transfer functions of such orthogonal operators can be expressed using families of polynomials orthogonal over the interval $[-\pi, \pi]$. We first present the case of operators orthogonal with respect to $(J_0(\pi t), 0)$.

3.1 Chebyshev Polynomials

Chebyshev polynomials can be defined by recursion:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

It is easy to see that Chebyshev polynomials with even index contain only even powers of x and those with odd index contain only odd powers of x. We will use Chebyshev polynomials in the frequency domain. The Fourier transform of a real valued band-limited function of finite energy has continuous and even real part and continuous and odd imaginary part. Every continuous function can be approximated by polynomials on any finite interval. Thus, the Fourier transform of a real π-band limited function can be approximated using only Chebyshev polynomials of even index for the real part and only Chebyshev polynomials of odd index for the imaginary part. Thus, $i^k T_k\left(\frac{\omega}{\pi}\right)$ is *an approximating base* for all complex functions defined on the interval $[-\pi, \pi]$, with continuous and even real part and continuous and odd imaginary part.

3.2 Chromatic Derivatives Associated With Chebyshev Polynomials

Let the operator $\mathbf{K}_{\mathrm{T}}[\,\cdot\,,\,\cdot\,] : \mathbb{N} \times C^{\infty} \to C^{\infty}$, associated with Chebyshev polynomials T_n, be defined by recursion as follows:

$$\begin{aligned}
\mathbf{K}_{\mathrm{T}}[0,f] &= f;\\
\mathbf{K}_{\mathrm{T}}[1,f] &= \frac{1}{\pi}\frac{\mathrm{d}f}{\mathrm{d}t};\\
\mathbf{K}_{\mathrm{T}}[i,f] &= \frac{2}{\pi}\frac{\mathrm{d}(\mathbf{K}_{\mathrm{T}}[i-1,f])}{\mathrm{d}t} + \mathbf{K}_{\mathrm{T}}[i-2,f]
\end{aligned}$$

We denote $\mathbf{K}_{\mathrm{T}}[i,f](t)$ by $\mathbf{K}_{\mathrm{T}}[i,f,t]$. If it is clear from the context what family of orthogonal operators we are referring to, then $\mathbf{K}_{\mathrm{T}}[i,f]$ is abbreviated by $\mathbf{K}[i,f]$ and $\mathbf{K}_{\mathrm{T}}[i,f,t]$ by $\mathbf{K}[i,f,t]$. Due to the properties of the operators $\mathbf{K}_{\mathrm{T}}[i,\,\cdot\,]$ we call this family of differential operators *a family of chromatic derivatives associated with Chebyshev polynomials.* The association with Chebyshev polynomials is clear from the proposition below.

Proposition 1 *Let* $f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega$. *Then*

$$\mathbf{K}[n,f,t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} i^n T_n\left(\frac{\omega}{\pi}\right)\widehat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

where T_n *is the* n^{th} *Chebyshev polynomial.*

Proof. By induction using recursive definitions of Chebyshev polynomials and chromatic derivatives; see [3]. □

The power spectrum of the transfer function of $\mathbf{K}[n,\,\cdot\,]$ is given by $\left|i^n T_n\left(\frac{\omega}{\pi}\right)\right|^2$; recall that for $|\omega| \leq \pi$, $T_n\left(\frac{\omega}{\pi}\right)^2 \leq 1$. Thus, the transfer functions of Chromatic derivatives (Chebyshev polynomials) form an increasingly refined (unevenly spaced) sequence of comb filters. The power spectra $\left|T_{16}\left(\frac{\omega}{\pi}\right)\right|^2$ and $\left|T_{17}\left(\frac{\omega}{\pi}\right)\right|^2$ of the transfer functions of $\mathbf{K}[16,f]$ and $\mathbf{K}[17,f]$ are shown on the

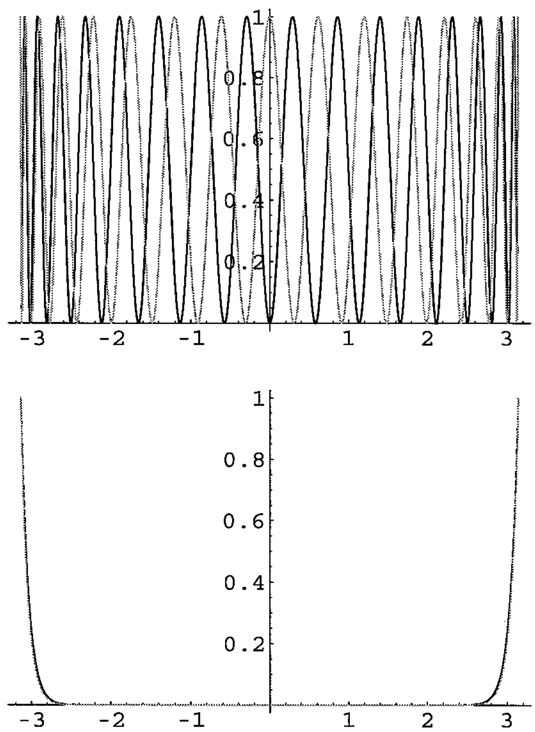

Figure 3: Comparison of the power spectra of the chromatic derivatives of order 16 and 17 (top) and power spectra of the standard derivatives of the same respective orders

above graph and compared with the power spectra $\left(\frac{\omega}{\pi}\right)^{32}$ and $\left(\frac{\omega}{\pi}\right)^{34}$ of the normalized standard derivatives $\frac{f^{(16)}(t)}{\pi^{16}}$ and $\frac{f^{(17)}(t)}{\pi^{17}}$. The above features of chromatic derivatives, illustrated on the figure, justify the name "chromatic derivatives". The sequence of chromatic derivatives encodes the spectral content of the signal in a "divide and conquer" manner, by corresponding to interleaved and increasingly refined comb filters.

We now present basic properties of chromatic derivatives associated with Chebyshev polynomials. Similar theorems hold for chromatic derivatives associated with other families of orthogonal polynomials, but are somewhat more cumbersome.

3.3 Composition Of Chromatic Derivatives Associated With Chebyshev Polynomials

We first show how to iterate taking chromatic derivatives of a function f.

Proposition 2

$$\mathbf{K}[m, \mathbf{K}[n, f]] = \frac{1}{2}(\mathbf{K}[m+n, f] + (-1)^{\min(m,n)}\mathbf{K}[|m-n|, f])$$

Proof. Using basic properties of Chebyshev polynomials and chromatic derivatives associated with them; see [3]. □

3.4 Chromatic Derivatives of Bessel Functions

Chromatic derivatives associated with Chebyshev polynomials act in a particularly simple way on the family of Bessel functions.

Proposition 3 $\mathbf{K}[m, J_0(\pi t)] = (-1)^m J_m(\pi t)$.

Proof. Using the following well known recurrence formula for Bessel functions of the first kind $J_{m-2}(x) - J_m(x) = 2\frac{\mathrm{d}(J_{m-1}(x))}{\mathrm{d}x}$ we get:

$$J_m(\pi t) = \frac{2}{\pi}\frac{\mathrm{d}(J_{m-1}(\pi t))}{\mathrm{d}t} - J_{m-2}(\pi t).$$

This, together with $J_1(\pi t) = -\frac{1}{\pi}\frac{\mathrm{d}(J_0(\pi t))}{\mathrm{d}t}$ and the recurrence formula defining chromatic derivatives, imply the statement of the Proposition. □

The previous proposition and the fact that the Fourier transform of $J_0(\pi t)$ is $\frac{2}{\pi}\frac{1}{\sqrt{1-(\frac{\omega}{\pi})^2}}$ for $-\pi < \omega < \pi$ (and 0 outside $[-\pi, \pi]$) implies that the Fourier transform of $J_m(\pi t)$ is $\frac{2(-i)^m T_m(\frac{\omega}{\pi})}{\pi\sqrt{1-(\frac{\omega}{\pi})^2}}$ for $-\pi < \omega < \pi$.

Proposition 4

$$\mathbf{K}[m, J_n(\pi t)] = \frac{(-1)^m}{2}\left(J_{m+n}(\pi t) + (-1)^{\min(m,n)} J_{|m-n|}(\pi t)\right)$$

Proof. A consequence of the previous two propositions. □

3.5 Chromatic Derivatives Of "Real Degrees"

We define the following operator acting on π-band limited functions (signals):

$$\mathbf{K}[\delta, f, t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)\, \cos\left(\delta\left(\pi - \arccos\left(\frac{\omega}{\pi}\right)\right)\right) e^{-\frac{i\delta\pi}{2}} e^{i\omega t}\,\mathrm{d}\omega$$

For positive integer values of δ, it is easy to see, using basic properties of Chebyshev polynomials, for example the equality $T_n(cos(\theta)) = cos(n\theta)$, that $\mathbf{K}[n, f, t]$ defined as above, coincides with the usual definition of chromatic derivative of order n, associated with Chebyshev polynomials. More over, using the equalities for Bessel functions of the first kind,

$$\int_0^{\infty} J_\delta(\alpha\, x) \sin(\beta\, x)\,\mathrm{d}x = \frac{\sin\left(\delta\, \arcsin\left(\frac{\beta}{\alpha}\right)\right)}{\sqrt{\alpha^2 - \beta^2}}$$

and

$$\int_0^{\infty} J_\delta(\alpha\, x) \cos(\beta\, x)\,\mathrm{d}x = \frac{\cos\left(\delta\, \arcsin\left(\frac{\beta}{\alpha}\right)\right)}{\sqrt{\alpha^2 - \beta^2}}$$

which hold for $\beta < \alpha$, one can easily obtain that

$$\int_{-\infty}^{\infty} J_\delta(\pi\, t)\, e^{-i\,\omega\, t}\,\mathrm{d}t = \frac{2}{\pi\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}} e^{\frac{i\,\delta\,\pi}{2}} \cos\left(\delta\left(\pi - \arccos\left(\frac{\omega}{\pi}\right)\right)\right)$$

Thus, $\mathbf{K}[\delta, J_0(\pi t), t] = e^{-i\delta\pi} J_\delta(\pi t)$, extending $\mathbf{K}[n, J_0(\pi t), t] = (-1)^n J_n(\pi t)$; also, for every analytic function f, we have $\mathbf{K}[-n, f, t] = (-1)^n \mathbf{K}[n, f, t]$.

It is now natural to associate with $f(t)$ the "chromatic differential transform" $\widetilde{f}(\delta, \omega)$ defined by

$$\widetilde{f}(\delta, \omega) = \widehat{f}(\omega)\, \cos\left(\delta\left(\pi - \arccos\left(\frac{\omega}{\pi}\right)\right)\right) e^{-\frac{i\,\delta\,\pi}{2}}$$

Then clearly

$$\mathbf{K}[n, f, t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{f}(n, \omega) e^{i\,\omega\, t}\,\mathrm{d}\omega$$

Problem: Study the properties of the transform $\widetilde{f}(\delta, \omega)$, and its relationship with $f(t)$ and $\widehat{f}(\omega)$.

4 The Space $\mathfrak{J}_2[-\pi,\pi]$

We first formulate the main relationship between Bessel functions of the first kind and chromatic derivatives associated with Chebyshev polynomials.

Proposition 5 *Chromatic derivatives associated with Chebyshev polynomials are orthogonal with respect to the pair $(J_0(\pi t), 0)$, and thus also with respect to every pair $(J_0(\pi(t-t_0)), t_0)$.*

Proof. Since

$$\mathbf{K}[m, J_n(\pi t)] = (-1)^m \frac{J_{m+n}(\pi t) + (-1)^{\min(m,n)} J_{|m-n|}(\pi t)}{2}$$

we get that

$$\mathbf{K}[m, \mathbf{K}[n, J_0(\pi t)]](0) = (-1)^n \mathbf{K}[m, J_n(\pi t)](0) = \tfrac{(-1)^{m+n}}{2}(J_{m+n}(0) + (-1)^{\min(m,n)} J_{|m-n|}(0)) \quad (15)$$

Since for all $k \neq 0$, $J_k(0) = 0$ we get that for $m \neq n$ $\mathbf{K}[m, \mathbf{K}[n, J_0(\pi t)]](0) = 0$. For $m = n \neq 0$ we get that $\mathbf{K}[m, \mathbf{K}[n, J_0(\pi t)]](0) = \frac{(-1)^n J_0(0)}{2} = (-1)^n \frac{1}{2}$. Thus, $\mathbf{K}[m, \mathbf{K}[n, J_0(\pi t)]](0) = (-1)^n c_n \delta_{m,n}$ with $c_0 = 1$ and $c_n = \frac{1}{2}$ for $n \neq 0$. □

We can now investigate for what functions f

$$A_P[f, M, t] = \sum_{n=0}^{M} \mathbf{K}[n, f](0) \frac{(-1)^n}{c_n} \mathbf{K}[n, J_0(\pi t)](t) = f(0) J_0(\pi t) + 2 \sum_{n=1}^{M} \mathbf{K}[n, f](0) J_n(\pi t)$$

is an approximation of $f(t)$, and the nature of this approximation. By translation for t_0 we get a more general form:

$$A_P[f, M, t_0, t] = f(t_0) J_0(\pi(t-t_0)) + 2 \sum_{n=1}^{M} \mathbf{K}[n, f](t_0) J_k(\pi(t-t_0)) \quad (16)$$

Using the facts that $\mathbf{K}[n, J_0(\pi t), t] = (-1)^n J_n(\pi t)$, $J_{-n}(t) = (-1)^n J_n(t)$ and $\mathbf{K}[-n, f, t] = (-1)^n \mathbf{K}[n, f, t]$, we get that

$$A_P[f, M, t_0, t] = \sum_{n=-M}^{M} \mathbf{K}[n, f](t_0) J_n(\pi(t-t_0))$$

Definition 2 *We denote by $\mathfrak{J}_2[-\pi,\pi]$ the collection of all band limited functions $f(t)$ such that the Fourier transform $\widehat{f}(\omega)$ of $f(t)$ satisfies that $\widehat{f}(\omega)\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}$ is continuous on the interval $[-\pi,\pi]$, and such that*

$$\int_{-\pi}^{\pi} \left|\widehat{f}(\omega)\right|^2 \sqrt{1-\left(\tfrac{\omega}{\pi}\right)^2} d\omega < \infty$$

Thus, all functions from $\mathfrak{J}_2[-\pi,\pi]$ have finite norm with respect to the scalar product defined as

$$\langle f, g \rangle_{\overline{CH}} = \frac{1}{4} \int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{g}(\omega)} \sqrt{1-\left(\frac{\omega}{\pi}\right)^2} d\omega.$$

It is easy to verify that the above formula is a correct definition of a scalar product.

Proposition 6 *Translated Bessel functions of the first kind $J_n(\pi(t-t_0))$ are orthogonal with respect to the scalar product $\langle f, g\rangle_{\overline{CH}}$. The sequence of functions $J_0(\pi(t-t_0)), \sqrt{2}J_1(\pi(t-t_0)), \ldots, \sqrt{2}J_n(\pi(t-t_0)), \ldots$ is a set of orthonormal vectors with respect to the above scalar product.*

Proof. Follows from the fact that Fourier transform of $J_k(\pi(t-t_0))$ is equal to $\frac{2}{\pi}\frac{(-i)^k T_k(\frac{\omega}{\pi}) e^{-i\omega t_0}}{\sqrt{1-(\frac{\omega}{\pi})^2}}$, $-\pi < \omega < \pi$ and the fact that

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi} \frac{T_k\left(\frac{\omega}{\pi}\right) T_m\left(\frac{\omega}{\pi}\right)}{\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}} d\omega = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m = 0 \\ \frac{1}{2} & \text{if } k = m \neq 0 \end{cases}$$

Consequently, the above integral is equal to 0 if $k \neq m$; if $k = m$ the above integral is real and equal to 1 if $k = m = 0$, and is equal to 1/2 if $k = m \neq 0$. We set $C_n = \langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}}$. Thus, we get $C_0 = 1$ and $C_n = 1/2$ if $n \neq 0$. From propositions 5 and 6 $C_k = c_k$ for all k. □

If we introduce scalar product

$$\langle f, g\rangle_{CH} = \frac{1}{4}\int_{-\pi}^{\pi} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}}\, d\omega$$

then polynomials $\frac{2}{\pi\sqrt{C_n}} i^n T_n\left(\frac{\omega}{\pi}\right)$ are orthonormal with respect to the above scalar product.

Proposition 7

$$\langle f, J_n(\pi(t-t_0))\rangle_{\overline{CH}} = \mathbf{K}[n, f, t_0]$$

and the sum

$$\sum_{n=0}^{M} \frac{\mathbf{K}[n,f](t_0) J_n(\pi(t-t_0))}{\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}}} =$$

$$\sum_{n=0}^{M} \frac{\langle f, J_n(\pi(t-t_0))\rangle_{\overline{CH}}}{\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}}} J_n(\pi(t-t_0))$$

converges to $f(t)$ in the sense of the norm $\|\cdot\|_{\overline{CH}}$.

For a proof see [3]. By normalizing we get that

$$\sum_{n=0}^{M} \frac{\mathbf{K}[n,f](t_0)}{(\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}})^{1/2}} \frac{J_n(\pi(t-t_0))}{(\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}})^{1/2}}$$

converges to $f(t)$ in the sense of the norm $\|\cdot\|_{\overline{CH}}$, and since the base vectors

$$\frac{J_n(\pi(t-t_0))}{(\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}})^{1/2}}$$

are orthonormal, we can apply Parseval's equality:

$$(\|f\|_{\overline{CH}})^2 = \frac{1}{4}\int_{-\pi}^{\pi} \left|\widehat{f(\omega)}\right|^2 \sqrt{1-\left(\tfrac{\omega}{\pi}\right)^2} d\omega$$
$$= \sum_{n=0}^{\infty}\left(\frac{\mathbf{K}[n,f](t_0)}{(\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}})^{1/2}}\right)^2$$
$$= \mathbf{K}[0,f](t_0)^2 + \sum_{n=1}^{\infty}\left(\frac{\mathbf{K}[n,f](t_0)}{(1/2)^{1/2}}\right)^2$$
$$= (\mathbf{K}[0,f](t_0))^2 + 2\sum_{n=1}^{\infty}(\mathbf{K}[n,f](t_0))^2 = (\|f\|_{\Psi})^2$$

Thus, for $B(t-t_0) = J_0(\pi(t-t_0))$ we have $C_k = c_k$ and

$$\langle f(t), J_n(\pi(t-t_0))\rangle_{\overline{CH}} = \mathbf{K}[n, f, t_0] = \langle f(t), B_n(t-t_0)\rangle_{\Psi}$$

as well as

$$\|f\|_{\overline{CH}} = \tfrac{1}{4}\int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2\sqrt{1-\left(\tfrac{\omega}{\pi}\right)^2}\, d\omega = \|f\|_{\Psi} = (\mathbf{K}[0,f](t_0))^2 + 2\textstyle\sum_{n=1}^{\infty}(\mathbf{K}[n,f](t_0))^2$$

This shows that *the norm and the projection defined purely by orthogonal differential operators have integral representations.* In general, integral operators tend to be numerically stable. We will show that one can obtain approximations of chromatic derivatives by using a stable integration over a finite interval of time.

5 Approximations of functions from $\mathfrak{J}_2[-\pi,\pi]$

We have shown that

$$Ap[f,M,t_0,t] = \sum_{n=0}^{M} \frac{\mathbf{K}[n,f](t_0) J_n(\pi(t-t_0))}{\langle J_n(\pi(t-t_0)), J_n(\pi(t-t_0))\rangle_{\overline{CH}}}$$

converges to f in the sense of the norms $\|\cdot\|_\Psi$ and $\|\cdot\|_{\overline{CH}}$. We now want to investigate *pointwise* convergence of the approximation $Ap[f,M,t_0,t]$ to the value of $f(t)$.

Let

$$Ap(f,M,t_0,t) = \mathbf{K}[0,f](t_0) J_0(\pi t) + 2\sum_{k=1}^{M} \mathbf{K}[k,f](t_0) J_k(\pi t)$$

our aim is to show that this approximation formula provides a good approximation of the values of $f(t_0+t)$ for all sufficiently small t, and that the chromatic derivatives of this approximation formula are good approximations of the chromatic derivatives of $f(t_0+t)$.

Theorem 2 *Let $f(t) \in \mathfrak{J}_2[-\pi,\pi]$, i.e., such that $\widehat{f(\omega)}\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}$ is continuous on $[-\pi,\pi]$, and*

$$\|f\|_{\overline{CH}} = \frac{1}{4}\int_{-\pi}^{\pi} \left|\widehat{f(\omega)}\right|^2 \sqrt{1-\left(\tfrac{\omega}{\pi}\right)^2}\, d\omega < \infty$$

Then

$$\begin{aligned}
&|f(t) - Ap(f,M,t_0,t)| \\
&\quad < \frac{\|f\|_{\overline{CH}}}{2}\sqrt{1 - J_0(\pi t)^2 - 2\sum_{k=1}^{M} J_k(\pi t)^2} \\
&\quad = \frac{1}{2}\sqrt{\mathbf{K}[0,f](t_0)^2 + 2\sum_{n=1}^{\infty} \mathbf{K}[n,f](t_0)^2} \\
&\quad \sqrt{1 - J_0(\pi t)^2 - 2\sum_{k=1}^{M} J_k(\pi t)^2}
\end{aligned}$$

Proof. Using an argument essentially based on Parseval's formula; detailed and somewhat involved proof can be found in [3]. □

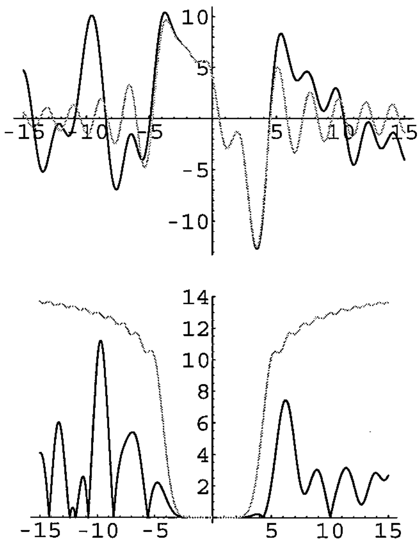

Figure 4: Top: gray - the approximation, black - the signal; bottom: black - actual error of approximation; gray - theoretical bound

The local nature of chromatic approximation can be seen from the graph representing a band limited function $f(t)$ and its chromatic approximation, as well as the error of the same approximation (second graph), obtained by subtracting the function from its chromatic approximation and taking the absolute value of the difference; the theoretical bound from the approximation theorem, of the form $C\sqrt{1-\left(J_0(\pi t)^2 + 2\sum_{k=1}^{M} J_k(\pi t)^2\right)}$ is also shown, majorizing the actual error.

We now look at the basic properties of the chromatic approximations.

Proposition 8 *Let $\gamma_0, ..., \gamma_M$ be a sequence of real numbers; then*

$$\beta(t) = \gamma_0 J_0(\pi(t-t_0)) + 2\sum_{k=1}^{M} \gamma_k J_k(\pi(t-t_0))$$

belongs to $\mathfrak{J}_2[-\pi,\pi]$ and for $m \leqslant M$, $\mathbf{K}[m,\beta(t)](t_0) = \gamma_m$.

This theorem also shows that, since $J_0 \in \mathfrak{J}_2[-\pi,\pi]$, the set $\mathfrak{I}_2[-\pi,\pi]$ properly extends the set of absolutely integrable band limited functions.

One can easily bound the values of chromatic approximations. Their asymptotic behavior towards infinity is determined by the asymptotic behavior of the Bessel functions of the first kind.

Proposition 9

$$\begin{aligned}
&Ap(f,M,t_0,t) = \\
&\mathbf{K}[0,f](t_0)J_0(\pi(t-t_0)) \\
&+2\sum_{k=1}^{M}\mathbf{K}[k,f](t_0)J_k(\pi(t-t_0)) \\
&\leq \sqrt{\mathbf{K}[0,f](t_0)^2+2\sum_{k=1}^{M}(\mathbf{K}[k,f](t_0))^2} \\
&\sqrt{J_0(\pi(t-t_0))^2+2\sum_{k=1}^{M}J_k(\pi(t-t_0))^2} \\
&\leq \|f\|_{\overline{CH}}\sqrt{J_0(\pi(t-t_0))^2+2\sum_{k=1}^{M}J_k(\pi(t-t_0))^2}
\end{aligned}$$

6 Smoothness Of Chromatic Approximation

We now show that the chromatic derivative of order m of the chromatic approximation of order M of a band limited function f is essentially approximation of the chromatic derivative of the function f of order $M-m$. This corresponds to the fact that the standard derivatives of order m of Taylor's formula of order M are Taylor's approximations of order $M-m$ of the derivative of f of order $M-m$.

Lemma 2

$$\begin{aligned}
&\mathbf{K}[m, Ap(f,M,t_0)] \\
&\quad = Ap(\mathbf{K}[m,f], M-m, t_0) + \\
&\quad \sum_{k=M-m+1}^{M+m}\mathbf{K}[|m-k|,f](t_0)(-1)^{Min[m,k]}J_k(\pi t)
\end{aligned}$$

Proof. Via a tedious and straight forward calculations. For details see [3]. □

Definition 3 *An approximation with the error bound of the form*

$$C\sqrt{1-\left(J_0(\pi t)^2+2\sum_{k=1}^{M}J_k(\pi t)^2\right)}$$

for some positive constant C we call an M-approximation.

Proposition 10 *The m^{th} chromatic derivative of an M-approximation of $f(t)$ is an $(M-m)$-approximation of the k^{th} chromatic derivative of $f(t)$.*

for a proof see [3]

6.1 Piece-wise Approximations Of Band-Limited Signals

The last theorem shows that, if for the sequence $\{t_j\}_{0\leq j\leq K}$ of numbers the sequence of chromatic approximations $Ap_k(M,f,t_j,t)$ of a band-limited function f are such that the first k chromatic derivatives of these approximations are sufficiently accurate in the interval $[t_j-H/2, t_j+H/2]$ and if $t_{j+1}-t_j=H$, then the function obtained by concatenating the restrictions of these approximations to the intervals $[t_j-H/2,t_j+H/2]$ will have k essentially continuous chromatic derivatives. By this we mean that the difference between derivatives as determined by the adjacent pieces will be smaller than certain small number ϵ.

At the end points $t_0-H/2$ and $t_K+H/2$ of the sequence of the chromatic approximations let us extend the approximations with band-limited functions of finite energy, denoted by $Ap_{-1}(t_0-H/2,t)$ and $Ap_{K+1}(t_K+H/2,t)$, such that the first k chromatic derivatives of $Ap_{-1}(t_0-H/2,t)$ agree with the first k chromatic derivatives of $Ap_0(M,f,t_0,t)$ at $t=t_0-H/2$, and the first k chromatic derivatives of $Ap_{K+1}(t_0,t)$ agree with the first k chromatic derivatives of $Ap_{K+1}(M,f,t_K,t)$ at $t_K+H/2$. From the definition of the Fourier integral, one can easily see that this implies that there exists a constant C such that for all $0\leq j\leq k$ the norm of the Fourier transform $\widehat{\mathbf{K}[j,F]}(\omega)$ of $\mathbf{K}[j,F(t)]$ is bounded by C, i.e., such that $\widehat{\mathbf{K}[j,F]}(\omega)<C$. However, if $\widehat{F}(\omega)$ is the Fourier transform of F, then

$$\widehat{K}[j,F(\omega)] = i^j T_j\left(\tfrac{\omega}{\pi}\right)\widehat{F}(\omega)$$

which implies that for all $0 \leq j \leq k$

$$|\widehat{F}(\omega)| \leq \frac{C}{|T_j\left(\frac{\omega}{\pi}\right)|}$$

Thus, $|\widehat{F}(\omega)|$ decays faster than $\frac{C}{|T_k(\frac{\omega}{\pi})|}$.

Consequently, the functions obtained by concatenating pieces of chromatic approximations even of purely π-band-limited signals do not have a "brick wall" cut off frequency, but rather a fast decay past the band limit of the signal.

However, even "standard" band limited functions are truncated to a finite interval, which, by the Uncertainty Principle, necessarily produces out of band noise.

Chromatic approximation can be seen as a concatenation of "transients" which avails us with a better model for signals in practice. For example, the transient appearing at the beginning of each note played by a musical instrument has a clear beginning in time, and thus sound might be better represented by "patches" of band limited signals than by a single band limited signal.

Thus, the significance of the above lies in the fact that any band-limited signal can be replaced by a "Bessel-spline signal", i.e., by sequence of the chromatic approximations of the original signal.

Here is a theorem proved in Papoulis' paper "Generalized Sampling Expansion"[2]. For any band limited signal which is in L_2 (i.e. which has finite energy ($\int_{-\infty}^{\infty} f(t)^2 dt < \infty$), or it satisfies some other conditions, for example that it is absolutely integrable ($\int_{-\infty}^{\infty} |f(t)| dt < \infty$), and for an interval T equal to the total length of m Nyquist rate sampling intervals, there are functions $y_0(t), \ldots, y_{m-1}(t)$ such that

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{m-1} f^{(k)}(nT) y_k(t - nT)$$

Thus, if we know the values of the function $f(t)$ and $m-1$ of its derivatives at every m Nyquist rate sampling points, then there is a perfect reconstruction based on these samples and interpolation functions $y_0(t), \ldots, y_{m-1}(t)$.

If we restrict ourselves to signals with finite energy, then the same holds for chromatic derivatives generated by any family of orthogonal polynomials on $[-\pi, pi]$, because such chromatic derivatives are linear combinations of the standard derivatives. (Chromatic derivatives are linear differential operators with constant and real coefficients). To see this, if $K_k[f](t)$ denotes the k^{th} chromatic derivative, then for some constants $c[p,q]$ we have:

$$K_k[f](t) = \sum_{p=0}^{k} c[k,p] f^{(p)}(t)$$

Solving this system for the standard derivatives $f^{(k)}(t)$, we get that for some constants $d[p,q]$ the following holds

$$f^{(k)}(t) = \sum_{q=0}^{k} d[k,q] K_q[f](t)$$

By replacing this in the expansion for $f(t)$ we get:

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{m-1} \left(\sum_{q=0}^{k} d[k,q] K_q[f](nT) \right) y_k(t-nT)$$

By multiplying with $y_k(t-nT)$ and by regrouping, we get that for some constants $e[k,p]$

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{m-1} K_k[f](nT) \sum_{p=k}^{m-1} e[k,p] y_p(t-nT)$$

Setting

$$\overline{y}_k(t-nT) = \sum_{p=k}^{m-1} e[k,p] y_p(t-nT)$$

We get that

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{m-1} K_k[f](nT) \overline{y}_k(t-nT)$$

Thus, the interpolation functions for chromatic derivatives are linear combinations of the interpolation functions for the standard derivatives.

In the case of chromatic derivatives associated with the Legendre polynomials, we have perfect reconstruction for the entire family for which the chromatic expansion converges.

In the case of chromatic derivatives associated with Chebyshev polynomials the above argument shows that perfect reconstruction holds only for signals of finite energy. At the moment we do not know if this is true for the entire family which includes functions with infinite energy, like Bessel's functions.

Notice the different nature of piece-wise approximation ("super-splines") and the perfect reconstruction approximation. The piece-wise approximation has no critical sampling rate and the information generally has certain level of redundancy since k derivatives are generally spaced less than k Nyquist rate points apart. It is better localized since in general fewer sampling points provide accurate reconstruction between the sampling points.

7 Other Families Of Chromatic Derivatives

We now consider chromatic derivatives associated with an arbitrary family of polynomials of appropriate type. Let $f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega$ be a representation of a signal f by its Fourier transformation. Then an application of a linear differential operator $D[n,f] = \sum_{m=0}^{n} \frac{q_m}{\pi^m}(\partial/\partial t)^m$ of order n to the input signal produces a multiplicative factor in the Fourier transform of the signal which is a polynomial $P_n(i\omega/\pi)$:

$$D[n,f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n\left(\frac{i\omega}{\pi}\right)\widehat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

We want to consider what families of differential operators are chromatic differentiation families, i.e. what families preserve information about the spectrum of the input signal.

We say that system of polynomials P_n *feasibly separates points* on $[-\pi,\pi]$ if, for every two numbers x and y such that $x \neq y$, $x, y \in [-\pi,\pi]$, there is a polynomial $P_n\left(\frac{\omega}{\pi}\right)$ in the system of arbitrary large degree, such that, for $M = \max\{P_n\left(\frac{\omega}{\pi}\right), -\pi \leq \omega \leq \pi\}$, it holds that

$$\left|P_n\left(\frac{x}{\pi}\right) - P_n\left(\frac{y}{\pi}\right)\right| \geq \frac{M}{k}$$

for a reasonably small positive number k. Thus, it is not the case that all polynomials in the system with high index attain small values for all points within an interval inside the interval $[-\pi,\pi]$.

A family of differential operator is a chromatic differential family if the corresponding family of polynomial multiplicative factors $P_n\left(\frac{i\omega}{\pi}\right)$ feasibly separates points. This insures that the family preserves the spectral features of the signal since at no interval within the frequency bounds all of the polynomials have excessively small values.

The main example of polynomials which feasibly separate points are orthogonal polynomials.

Thus, we choose a continuous, non negative, even function $w(u)$, (*i.e.*, such that $w(u) = w(-u)$) which has finitely many zeros in $[-\pi,\pi]$, and such that $\int_{-\pi}^{\pi} w(u)P\left(\frac{u}{\pi}\right)\,\mathrm{d}u$ converges for every polynomial. Then we consider the scalar product defined by

$$< f, g >= \int_{-\pi}^{\pi} w(\omega)^{-1}\widehat{f}(\omega)\overline{\widehat{g}(\omega)}\,\mathrm{d}\omega$$

on the vector space of all functions $f(\omega)$ such that $\int_{-\pi}^{\pi} w(\omega)^{-1}|\widehat{f}(\omega)|^2\,\mathrm{d}\omega$ exists and $w(\omega)^{-1}\widehat{f}(\omega)$ is continuous for all $\omega \in [-\pi,\pi]$, where $\widehat{f}(\omega)$ is the Fourier transform of $f(t)$.

Linear independence of polynomials $1, x, x^2, \ldots, x^n, \ldots$ and the Gram-Schmidt procedure guarantee the existence of a unique sequence of polynomials, $SP = \{P_i\}_{i\in\mathbb{N}}$ such that $\int_{-\pi}^{\pi} w(\omega)P_i[\omega/\pi]P_j[\omega/\pi]\,\mathrm{d}\omega = 0$ for all $i \neq j$ and such that $\int_{-\pi}^{\pi} w(\omega)P_i[\omega/\pi]^2\,\mathrm{d}\omega = h_i \neq 0$ for each $i \geq 0$.

As it is well known, any family of orthogonal polynomials of the form $P_n(u) = k_n u^n + k'_n u^{n-1} + \cdots$ satisfies the recursion of the formula obtained as follows: Let $b_n = \frac{k_{n+1}}{k_n}$, $a_n = b_n\left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n}\right)$ and $c_n = \frac{k_{n+1}k_{n-1}h_n}{k_n^2 h_{n-1}}$. Then $P_{n+1}(u) = (a_n + ub_n)P_n(u) - c_n P_{n-1}(u)$, with h_n from the orthogonality condition.

We call a system of polynomials on $[-\pi,\pi]$ *regular* if

i. It contains exactly one polynomial for of each degree n (thus also a constant function);

ii. Polynomials $P_{2k}\left(\frac{i\omega}{\pi}\right)$ which belongs to a regular family and are of even index contain only even powers of $i\omega$, while $P_{2k+1}\left(\frac{i\omega}{\pi}\right)$ which belongs to a regular family and are of odd index contain only odd powers of $i\omega$. Thus, polynomials containing only even powers are real while the polynomials containing only odd powers are of the form $iQ(\omega)$ where Q is a real polynomial in ω.

The importance of a regular systems comes from the fact that the Fourier transform of a real function has even real part and odd imaginary part.

In case of orthogonal polynomials, if $a_n = 0$, *i.e.*, if $\frac{k'_{n+1}}{k_{n+1}} = \frac{k'_n}{k_n}$ for all $n > 0$ and in addition also $k'_n = 0$, these polynomials form a regular system, satisfying $P_0(u) = k_0, P_1(u) = k_1 u$ and $P_{n+1}(u) = b_n u P_n(u) - c_n P_{n-1}(u)$.

For every regular set of orthogonal polynomials SP we associate a class SD of linear differential operators defined by the following recursion scheme. Let $P_0(t) = k_0$ and $P_1(t) = k_1 u$. We set:

$$\begin{aligned}\Delta_0[f] &= k_0 f(t);\\ \Delta_1[f] &= \frac{k_1}{\pi}\frac{\mathrm{d}f(t)}{\mathrm{d}t};\\ \Delta_{n+1}[f] &= \frac{b_n}{\pi}\frac{\mathrm{d}(\Delta_n[f])(t)}{\mathrm{d}t} + c_n\Delta_{n-1}[f]\end{aligned}$$

If for a family of differential operators SD the corresponding family of polynomial multiplicative factors $P_n\left(\frac{i\omega}{\pi}\right)$ is a regular family of orthogonal polynomials, then SD is a chromatic differentiation family. Chebyshev and Legendre polynomials are typical examples of such polynomials. Legendre polynomials have an important feature that the weight in the corresponding scalar product is equal to one, $w(u) = 1$. Thus, signals with finite norm with respect to this scalar product are exactly the signals with finite L_2 norm (i.e., with finite energy).

On the other hand it is clear that the family $1, \frac{\omega}{\pi}, \left(\frac{\omega}{\pi}\right)^2, \ldots, \left(\frac{\omega}{\pi}\right)^n, \ldots$ does NOT feasibly separate points.

Let $B_0(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\omega)e^{i\omega t}\,\mathrm{d}\omega$; then we call $B_0(t)$ the fundamental basis function of order 0 associated with a *regular family of polynomials* SP, orthogonal with respect to the scalar product $< f, g >= \int_{-\pi}^{\pi} w(u)f(u)\overline{g(u)}\,\mathrm{d}u$ Let also $B_n(t) = (-1)^n\Delta_n[B_0]$. Then $B_n(t)$ is called the fundamental basis function of order n.

Any band-limited function $f(t)$ such that

$$\int_{-\pi}^{\pi} w(\omega)^{-1}|\widehat{f}(\omega)|^2\,\mathrm{d}\omega$$

converges and $w(\omega)^{-1}\widehat{f}(\omega)$ is continuous on $[-\pi, \pi]$, can be approximated by a formula of the form

$$App[M, f, t_0, t] = \sum_{i=0}^{M} A[i]\Delta[i, f](t_0)B_i(t)$$

with properties similar to the properties of the expansion based on chromatic derivatives associated with Chebyshev polynomials and with Bessel functions of the first kind as interpolating functions.

8 Linear Operators And Chromatic Approximations

We now consider the action of a continuous linear shift-invariant operator. The Fourier transform of $L[f]$ is obtained from the Fourier transform of f by multiplying it with the symbol of L. Thus, if L is acting on a function whose Fourier transform has support $[-\pi, \pi]$ and if $\widehat{L}(\omega)$ is the symbol of L then

$$L[f] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\widehat{L}(\omega)e^{i\omega t}d\omega$$

Action of a linear operator can be localized, if $L[Ap(f, M, t_0)](t_0 + t)$ is a good approximation of $L[f](t_0 + t)$ for all t which are sufficiently small for given M and L. Localizing the action of a linear operator is similar in nature to using a FIR algorithm, which ignores samples sufficiently distant from the central point. By linearity of L, to determine $L[Ap(f, M, t_0)](t_0 + t)$ it is enough to find $L[J_k(\pi t)]$ for $k \leq M$. We notice that $L[J_k(\pi t)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2(-i)^k}{\pi} \frac{\widehat{L}(\omega)T_k(\frac{\omega}{\pi})}{\sqrt{1-(\frac{\omega}{\pi})^2}} e^{i\omega t}d\omega$. The following calculation shows that this is in $\mathfrak{I}_2[-\pi, \pi]$ whenever $L(\omega)$ is continuous on the closed interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \left(\frac{|L(\omega)T_k(\frac{\omega}{\pi})|}{\sqrt{1-(\frac{\omega}{\pi})^2}}\right)^2 \sqrt{1-\left(\frac{\omega}{\pi}\right)^2}d\omega \leq$$
$$\|L(\omega)\|_\infty^2 \int_{-\pi}^{\pi} \frac{|T_k(\frac{\omega}{\pi})|^2}{\sqrt{1-(\frac{\omega}{\pi})^2}}d\omega$$

This implies that $L[J_k(\pi t)] \in \mathfrak{I}_2[-\pi, \pi]$ and thus, it can be expanded as

$$L[J_k(\pi t)] = L[J_k(\pi t)](0)J_0(\pi t)$$
$$+2\sum_{m=1}^{N} K[m, L[J_k(\pi t)]](0)J_k(\pi t)$$

To determine $L[J_k(\pi t)]$ we notice that $L[J_k(\pi t)] = L[(-1)^k\mathbf{K}[k, J_0(\pi t)]] = (-1)^k\mathbf{K}[k, L[J_0(\pi t)]]$. Thus it is enough to determine $L[J_0(\pi t)]$. This is a representation of a linear operator L similar to the impulse response representation in standard signal processing methods. However,

$$l(t) = L[J_0(\pi t)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2\widehat{L(\omega)}}{\pi\sqrt{1-(\frac{\omega}{\pi})^2}} e^{i\omega t}d\omega$$

and

$$l(t) = \mathbf{K}[0, l(t)](0)J_0(\pi t)$$
$$+2\sum_{m=1}^{\infty} \mathbf{K}[m, l(t)](0)J_m(\pi t)$$

Since

$$\mathbf{K}[m, l(t)](0) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \frac{\widehat{L(\omega)}i^m T_m(\frac{\omega}{\pi})}{\sqrt{1-(\frac{\omega}{\pi})^2}} d\omega$$

we get that the coefficients of the expansion of $l(t)$ into series of Bessel functions can be obtained from the coefficients of the expansion of $\widehat{L}(\omega)$ into Chebyshev polynomials of the form $i^k T_k\left(\frac{\omega}{\pi}\right)$ on the interval$[-\pi, \pi]$.

9 Numerical Methods: Basic Least Square Fit Approximation

The fact that the accuracy of chromatic approximation decreases through a "gentle drift" from the true value allows a simple least square fit to be used for evaluation of the chromatic derivatives with a remarkable accuracy.

9.1 *Monad*: The Basic Interpolation

To obtain such least square fits, one considers the following sum:

$$S = v\left(A(0)^2 + 2\sum_{k=1}^{N} A(k)^2\right) + u \sum_{i=-o_s M}^{o_s M} W\left(\frac{i}{o_s}\right)^2$$
$$\left(J_0\left(\frac{\pi i}{o_s}\right) A(0) + 2\sum_{k=1}^{N} J_k\left(\frac{\pi i}{o_s}\right) A(j) - f\left(\frac{i}{o_s}\right)\right)$$

Here v and u are constants and o_s is the oversampling factor. $W(t)$ is the windowing function obtained from the error estimate:

$$W(t) = \frac{a+1}{a^2}\left(\sqrt{1 - J_0(\pi t)^2 - 2\sum_{k=1}^{N} J_k(\pi t)^2 + \frac{1}{a}}\right)^{-1} - \frac{1}{a+1}$$

and it is represented on the graph above, for degree of approximation $N = 12$ and the "error constant" $a = 16$. The first part of the sum limits the "weighted energy" of the approximation. The quotient u/v corresponds to the signal to noise ratio and it provides noise robustness and numerical stability of the method.

The sum

$$\sum_{i=-M}^{M}\left(J_0\left(\frac{\pi i}{o_s}\right) A(0) + 2\sum_{k=1}^{N} J_k\left(\frac{\pi i}{o_s}\right) A(k)\right)$$

is an interpolation evaluated at o_s times oversampled points in the interval $[-M, M]$ consisting of

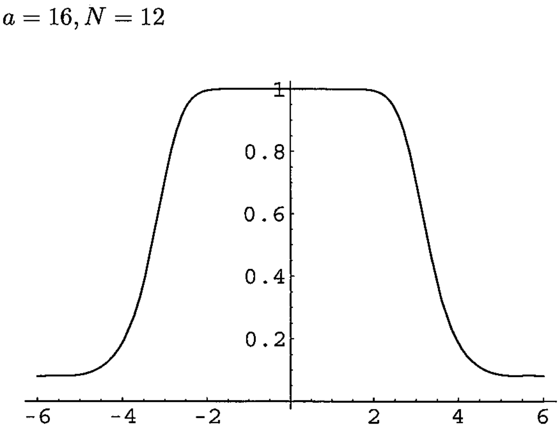

Figure 5:

$2M$ Nyquist rate unit intervals. Thus,

$$\sum_{i=-M}^{M} W\left(\frac{i}{o_s}\right)^2 \left(J_0\left(\frac{\pi i}{o_s}\right) A(0) + 2\sum_{k=1}^{N} J_k\left(\frac{\pi i}{o_s}\right) A(k) - f\left(\frac{i}{o_s}\right)\right)^2$$

corresponds to the sum of squares of the differences of the interpolated values and (o_s times over-) sampled values of the signal $f(t)$, windowed by $W(t)$, corresponding to the error of the interpolation, thus, "equalizing" the differences between the sampled and approximated value. In this way, allowing correspondingly larger errors away from the center in the same way how the accuracy of interpolation decreases, the least squares produces the best approximations $A(j)$ of the values of the corresponding chromatic derivatives $\mathbf{K}[j, f](0)$.

To determine the span $[-M, M]$, we consider the window function represented on the above graph. A good candidate for the domain over which the least squares should be taken is the first local minima of the windowing function. It is easy to see that this local minima corresponds to the first zero of the expression

$$1 - J_0(\pi t)^2 - 2\sum_{i=0}^{N} J_i(\pi t)^2$$

Differentiation gives a "telescopic" sum as follows:

$$\left(1 - J_0(\pi t)^2 - 2\sum_{i=0}^{N} J_i(\pi t)^2\right)' =$$

$$-2(J_0(\pi t)J_0(\pi t)' + 2\sum_{i=0}^{N} J_i(\pi t)J_i(\pi t)') =$$

$$-2(-J_0(\pi t)J_1(\pi t)$$

$$+\sum_{i=0}^{N} J_i(\pi t)(J_{i-1}(\pi t) - J_{i+1}(\pi t))) =$$

$$2J_N(\pi t)J_{N+1}(\pi t)$$

Thus, the proper value of M for the span of the least squares fit is a number close to the first positive zero of the function $J_N(\pi t)$.

The total sum is then minimized using the standard numerical techniques, *i.e.,* by differentiating the sum S with respect to the variables $A(i)$ and setting all such partial derivatives equal to 0. This produces a system of linear equations which is then solved for the variables $A(i)$.

9.2 Approximations by a *Complex*

If we increase the degree of the basic approximation for the least squares fit, the fact that Bessel functions are *not* orthogonal over any finite interval starts causing significant errors of the approximate values of higher order derivatives. Thus, in order to extend the interval of approximation without such "aliasing" of higher order derivatives, we employ a construction which to a certain degree resembles the idea of polynomial splines.

A **complex** is a sequence of basic approximations as described above, parameterized (simultaneously) by a set of sequences of variables, one sequence for each basic approximation as employed in the basic least squares approximation. To the usual conditions associated with every basic approximation (every "monad") one adds conditions ensuring that the (adjacent) basic approximations "agree" about the values of the chromatic derivatives evaluated at the same point. We first explain this on the example of a complex with Bessel functions as interpolants. Using the properties of the chromatic derivatives associated with Chebyshev polynomials, one can introduce the following three basic approximations for the n^{th} derivative of a band-limited function, associated with each basic approximation of a 3-complex, i.e. a complex involving three basic approximations, and a slightly modified windowing function $WN[m,t]$.

$$AP1[n,s] = X_1[n]J_0\left(\pi\left(\frac{s}{z}\right)\right) + \sum_{i=1}^{N-n} \left(X_1[n+i] + (-1)^{\min[i,n]}X_1[|n-i|]\right) J_i\left(\pi\left(\frac{s}{z}\right)\right)$$

$$AP2[n,s] = X_2[n]J_0\left(\pi\left(\frac{s}{z}\right)\right) + \sum_{i=1}^{N-n} \left(X_2[n+i] + (-1)^{\min[i,n]}X_2[|n-i|]\right) J_i\left(\pi\left(\frac{s}{z}\right)\right)$$

$$AP3[n,s] = X_3[n]J_0\left(\pi\left(\frac{s}{z}\right)\right) + \sum_{i=1}^{N-n} \left(X_3[n+i] + (-1)^{\min[i,n]}X_3[|n-i|]\right) J_i\left(\pi\left(\frac{s}{z}\right)\right)$$

$$WN(n,t) = \frac{a+1}{a^2} \left(\sqrt{1 - J_0(\pi t)^2 - 2\sum_{k=1}^{N-n} J_k(\pi t)^2} + \frac{1}{a}\right)^{-1} - \frac{1}{a+1}$$

Thus, the parameter n in $WN[n,t]$ reduces the degree of approximation. This is necessary because if the signal is represented by a chromatic approximation of order N, then its n^{th} derivative is approximated by $AP[n,t]$ which is essentially an approximation of order $N-n$, as shown.

The following three sums are introduced, each corresponding to the appropriate basic approximation, with o_s equal to the oversampling factor. The interval $[-M,M]$ is chosen on the basis of the corresponding window $WN[0,t]$ so that it contains the first point ρ at which the first derivative of the function $WN[0,t]$ is equal to 0 and thus where $WN[0,t]$ is no longer decreasing in a monotone way for $0 < t < \rho$.

$$S1 = \sum_{i=-o_sM}^{o_sM} WN\left(0, \frac{i}{o_s}\right)^2 \left(-f\left(\frac{i}{o_s}\right) + J_0\left(\pi\frac{i}{o_s}\right) X_1[0] + 2\sum_{j=1}^{N} J_j\left(\pi\frac{i}{o_s}\right) X_1[j]\right)^2$$

$$S2 = \sum_{i=-o_sM}^{o_sM} WN\left(0, \frac{i}{o_s}\right)^2 \left(-f\left(\frac{i}{o_s}-1\right) + J_0\left(\pi\frac{i}{o_s}\right) X_2[0] + 2\sum_{j=1}^{N} J_j\left(\pi\frac{i}{o_s}\right) X_2[j]\right)^2$$

$$S3 = \sum_{i=-o_sM}^{o_sM} WN\left(0, \frac{i}{o_s}\right)^2 \left(-f\left(\frac{i}{o_s}+1\right) + J_0\left(\pi\frac{i}{o_s}\right) X_3[0] + 2\sum_{j=1}^{N} J_j\left(\pi\frac{i}{o_s}\right) X_3[j]\right)^2$$

Notice that these approximations are centered at -1,0 and 1. This is manifested in the shifted values $f(\frac{i}{o_s}+1)$, $f(\frac{i}{o_s})$, $f(\frac{i}{o_s}-1)$ of the signal f.

The sums $E1, E2, E3$ defined as

$$\begin{aligned} E1 &= X_1[0]^2 + 2\sum_{s=1}^{N} X_1[s]^2, \\ E2 &= X_2[0]^2 + 2\sum_{s=1}^{N} X_2[s]^2, \\ E3 &= X_3[0]^2 + 2\sum_{s=1}^{N} X_3[s]^2, \end{aligned}$$

play the same role as in the basic "monad" approximation.

Finally, $D1$ and $D2$ defined below correspond to an approximation of the RMS value of the sum of differences of chromatic derivatives calculated using each of the three interpolations. Thus, $D1$ and $D2$ will ensure that the three basic approximations mutually "agree" about the values of the chromatic derivatives within the error given by the appropriate windowing function $WN[k,t]$.

$$D1 = \sum_{j=0}^{o_s}\sum_{k=0}^{N} WN\left(k, \frac{j}{o_s}\right) WN\left(k, 1-\frac{j}{o_s}\right) \left(AP1\left(k, -\frac{j}{o_s}\right) - AP2\left(k, 1-\frac{j}{o_s}\right)\right)^2$$

$$D2 = \sum_{j=0}^{o_s}\sum_{k=0}^{N} WN\left(k, \frac{j}{o_s}\right) WN\left(k, 1-\frac{j}{o_s}\right) \left(AP2\left(k, -\frac{j}{o_s}\right) - AP3\left(k, 1-\frac{j}{o_s}\right)\right)^2$$

The above sum can be extended to include the "agreement" of the first and the third approximations, if the degree of approximation is sufficiently high.

The weighted sum $S = u(S1+S2+S3) + v(E1+E2+E3) + w(D1+D2)$ is now minimized using the standard procedure.

Thus, the complex generalizes the idea of splines, by replacing the "agreement" of the derivatives of the individual (polynomial) pieces at the end points only, with a more "distributed" agreement of the (chromatic) derivatives of the pieces within certain windows which depend on the order of the derivatives. Clearly, such construction is possible only if the interpolating functions allow differentiation of higher order. One uses complex instead of a monad in the presence of significant noise, since a complex is more noise robust than a simple monad.

10 Chromatic Derivatives and the Spectrum

Assume that $f(t)$ is a π band limited signal of finite energy whose Fourier transform is $\widehat{f}(\omega)$, i.e.,

$$f[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega) e^{i\omega t}\, d\omega$$

Then $\widehat{f}(\omega)$ is a continuous complex function and its real part is an even function while its imaginary part is an odd function. Thus, one can look for a linear combination of Chebyshev polynomials such that $\sum_{k=0}^{M} \gamma[k] i^k T_k\left(\frac{\omega}{\pi}\right) \approx \widehat{f}(\omega)$ in the interval $[-\pi, \pi]$.

For such a combination we let:

$$\begin{aligned} f_M(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=0}^{M}\gamma[k]i^kT_k\left(\frac{\omega}{\pi}\right)e^{i\omega t}\,\mathrm{d}\omega \\ &= \sum_{k=0}^{M}\frac{\gamma[k]}{2\pi}\int_{-\pi}^{\pi}i^kT_k\left(\frac{\omega}{\pi}\right)e^{i\omega t}\,\mathrm{d}\omega \\ &= \sum_{k=0}^{M}\gamma[k]\mathbf{K}[k,\mathrm{sinc}[t]] \\ &= \sum_{k=0}^{M}\gamma[k]S_k(t) \end{aligned}$$

for

$$S_k(t) = \mathbf{K}[k,\mathrm{sinc}[t]]$$

This implies that an approximation of the spectrum of the signal by Chebyshev polynomials of the form $\sum_{k=0}^{M}\gamma[k]i^kT_k\left(\frac{\omega}{\pi}\right) \approx H_f[\omega]$ produces an approximation of the signal in the time domain which is a linear combination of the functions $S_k(t)$ with the same coefficients $\gamma[k]$.

We also note that chromatic derivatives $S_j(t)$ of the sinc function form a *dual base* with Bessel functions.This allows to determine the coefficients $\gamma[k]$ as convolutions.

However, one can obtain such functions using also a least square fit in several ways.

First, we note that

$$f_M(t) = \sum_{k=0}^{M}\frac{2}{\pi}\int_{-\pi}^{\pi}i^k\gamma[k]T_k\left(\frac{\omega}{\pi}\right)e^{i\omega t}\,\mathrm{d}\omega$$

Implies

$$\begin{aligned} &\mathbf{K}[k,f_M[t]] = \\ &\sum_{m=0}^{M}\frac{2}{\pi}\int_{-\pi}^{\pi}i^{m+k}\gamma[m]T_m\left(\frac{\omega}{\pi}\right)T_k\left(\frac{\omega}{\pi}\right)e^{i\omega t}\,\mathrm{d}\omega = \\ &\sum_{m=0}^{M}\gamma[m]\mathbf{K}[k,\mathbf{K}[m,\mathrm{sinc}[t]]] = \\ &\sum_{m=0}^{M}\frac{\gamma[m]}{2}(\mathbf{K}[m+k,\mathrm{sinc}[t]] \\ &+(-1)^{\min[m,k]}\mathbf{K}[|m-k|,\mathrm{sinc}[t]]\Big) = \\ &\sum_{m=0}^{M}\frac{\gamma[m]}{2}\left(S_{m+k}[t]+(-1)^{\min[m,k]}S_{|m-k|}[t]\right) \end{aligned}$$

Thus, if not only the signal is given by its Nyquist rate values, but also N of its derivatives, the coefficients $\gamma[m]$ must provide not only the best least square fit for f(t) but they must also satisfy the requirement that

$$\mathbf{K}[k,f_M[t]] = \sum_{p=0}^{M}\frac{\gamma[m]}{2}\left(S_{m+n}[t]+(-1)^{\min[m,n]}S_{|m-n|}[t]\right)$$

is the best least square fit of the values $\mathbf{K}[k,f]$. Thus, the values of chromatic derivatives produce a conditions for $\gamma[j]$'s which reduces the number of Nyquist rate points needed for a faithful representation of the Fourier transform of $f(t)$. The $\gamma[m]$'s can be obtained, for example by determining values for $\gamma[m]$, $0 \leq m \leq M$ which minimize the following expression:

$$\sum_{i=-P}^{P}\left(\left(f[i]-\sum_{j=0}^{M}\gamma[j]S_j[i]\right)^2 + \sum_{m=1}^{N}\left(\mathbf{K}[m,f](i)-\sum_{j=0}^{M}\frac{\gamma[j]}{2}\left(S_{m+n}[i]+(-1)^{\min[m,n]}S_{|m-n|}[i]\right)\right)^2\right)$$

if the Nyquist rate values of $f[t]$ and its chromatic derivatives $\mathbf{K}[m,f,t]$ for $m \leq N$ are given for integers i such that $|i| \leq P$, i.e if $\mathbf{K}[m,f](i)$ are known for $0 \leq m \leq Nand - P \leq i \leq P$.

10.1 Representation of Signals using chromatic derivatives at several points

Chromatic expansion offers a compact representation of the oversampled data. Assume that, using a data acquisition unit which operates on the oversampled scale, the values of k chromatic derivatives are obtained every m Nyquist rate intervals, for some $m < k$. In this case one can obtain an interpolation of the input waveform over a period of time containing M Nyquist intervals by considering an approximation of sufficiently high order ($N >> M$). For this purpose, we consider an approximation of the form:

$$Ap[t] = X[0]J_0(\pi t) + 2\sum_{s=1}^{N} X[s]J_s(\pi t)$$

with $X[s]$ as variables, and find the values of $X[s]$ which minimize the value of the sum

$$\mu(X[0]^2 + 2\sum_{s=1}^{N} X[s]^2) + \sum_{s=0}^{M}\sum_{q=0}^{k-1} W[N-q, t_s]^2 (K[q, Ap](t_s) - D_q^s)^2$$

As before $X[0]^2 + 2\sum_{s=1}^{N} X[s]^2$ is the energy of the approximation. Values $\{D_q^s\}_{q<k,s\leq M}$ are the values of k consecutive chromatic derivatives of the input signal at s moments in time which are m Nyquist rate intervals apart. $\{K[q, Ap](t_s)\}_{q<k,s\leq M}$ are the values of the corresponding chromatic derivatives of the approximation $Ap[t]$ at these moments in time. The moment $t = 0$ corresponds to the central point of the interval determined by the sequence $\{t_s\}_{q<k,s\leq M}$. μ is a small constant controlling the quotient between the energy of the approximation and the error of the fit, thus adapting the algorithm to the signal to noise ratio and the quantization error. In practice six chromatic derivatives (k=0-5) per moment in time with moments in time four Nyquist rate intervals apart provides an excellent approximation of the analog waveform.

11 conclusion

The above methods represent the core algorithms of the *Chromatic Signal Processing method.* This method offers a compact representation of an analog waveform via broad band chromatic differentiation filters, which allow design of fast converging adaptive filters, prediction filters, novel modulation schemas, pulse-width-modulators etc. Chromatic Signal Processing method is being developed and implemented by Kromos Technology Inc., a subsidiary of Comstellar Technologies Inc.

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