Chromatic Derivatives, Chromatic Expansions and Associated Function Spaces

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Abstract:

We present the basic properties of the chromatic derivatives and the chromatic expansions as well as a motivation for introducing these notions. The chromatic derivatives are special, numerically robust linear differential operators; the chromatic expansions are the associated local expansions, which possess the best features of both the Taylor and the Nyquist expansions. This makes them potentially useful in fields involving sampled data, such as signal and image processing.

1. Motivation

The Nyquist–(Whittaker–Kotelnikov–Shannon) expansion $f(t) = \sum_{n=-\infty}^{\infty} f(n) \sin \pi(t-n)/\pi(t-n)$ of a π-band limited signal of finite energy $f(t) \in \mathbf{BL}(\pi)$ is of *global nature*, because it requires samples of the signal at integers of arbitrarily large absolute value. On the other hand, since signals from $\mathbf{BL}(\pi)$ are analytic functions, they can also be represented by the Taylor expansion, $f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\ t^n/n!$. Such expansion is of *local nature*, because the values of the derivatives $f^{(n)}(0)$ are determined by the values of the signal in an arbitrarily small neighborhood of zero.

While the Nyquist expansion has a central role in digital signal processing, the Taylor expansion is of very limited use there, for several reasons.

(1) Numerical evaluation of higher order derivatives of a signal from its samples is very noise sensitive; in general, one is cautioned against numerical differentiation of signals given by empirical samples.

(2) The Taylor expansion of a signal $f \in \mathbf{BL}(\pi)$ converges non-uniformly; its truncations are unbounded and have rapid error accumulation.

(3) The Nyquist expansion of a signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$ and thus the action of a filter A on any $f \in \mathbf{BL}(\pi)$ can be expressed using the samples of f and the impulse response $A[\mathrm{sinc}\,]$ of A, i.e.,

$$A[f](t) = \sum_{n=-\infty}^{\infty} f(n)\ A\,[\mathrm{sinc}\,]\,(t-n). \qquad (1)$$

In contrast, the polynomials obtained by truncating the Taylor series do not belong to $\mathbf{BL}(\pi)$ and nothing similar to (1) holds for the Taylor expansion.

The chromatic derivatives and the chromatic expansions and approximations were introduced to obtain local signal representations which do not suffer from these problems.

2. Chromatic Derivatives

To explain our notions, we first consider normalized and rescaled Legendre polynomials $P_n^L(\omega)$ which satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega)\ P_m^L(\omega)\mathrm{d}\omega = \delta(m-n),$$

and then define operator polynomials

$$\mathcal{K}_t^n = \frac{1}{\mathrm{i}^n} P_n^L\left(\mathrm{i}\ \frac{d}{dt}\right). \qquad (2)$$

It is easy to verify that for $f \in \mathbf{BL}(\pi)$ and its Fourier transform $\widehat{f(\omega)}$ we have

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}^n P_n^L(\omega)\widehat{f(\omega)}\ \mathrm{e}^{\mathrm{i}\,\omega t} d\omega.$$

Figure 1 compares the plots of $P_n^L(\omega)$ and ω^n/π^n for $n = 15$ to $n = 18$, which are the transfer functions (save a factor of i^n) of the operators $\mathcal{K}^n$ and of the (normalized) derivatives $1/\pi^n\ \mathrm{d}^n/\mathrm{d}t^n$, respectively. While the transfer functions of the normalized "standard derivatives" $1/\pi^n\ \mathrm{d}^n/\mathrm{d}t^n$ obliterate the spectrum of the signal, leaving only its edges which in practice contain mostly noise, the transfer functions of operators $\mathcal{K}^n$ form a family of well separated, interleaved and increasingly refined comb filters. Due to their spectrum preserving property, we call the operators $\mathcal{K}^n$ *the chromatic derivatives* associated with the Legendre polynomials. Both analytic estimates and empirical tests have shown that the chromatic derivatives

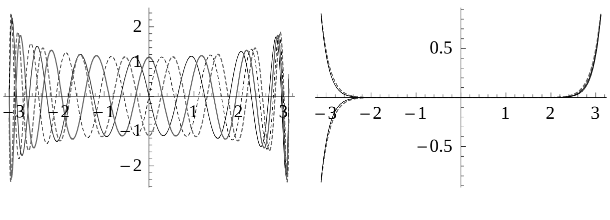

Figure 1: Graphs of $P_n^L(\omega)$ (left) and ω^n/π^n (right), for $n = 15 - 18$.

can be accurately and robustly evaluated from samples of the signal taken at a small multiple (2 to 4) of the usual Nyquist rate, thus solving problem (1) associated with numerical evaluation of the standard derivatives, mentioned above. Chromatic expansions, on the other hand, were introduced to solve problems (2) and (3).

3. Chromatic Approximations

Proposition 1 *Let $\mathcal{K}^n$ be the chromatic derivatives associated with the Legendre polynomials, and let $f(t)$ be any analytic function; then for all t,*

$$f(t) = \sum_{n=0}^{\infty}(-1)^n\ K^n[f](u)\ K^n[\mathrm{sinc}](t-u). \quad (3)$$

If $f \in \mathbf{BL}(\pi)$ the series converges uniformly and in L_2.

The series in (3) is denoted by $\mathrm{CE}[f,u](t)$ and is called *the chromatic expansion* of $f(t)$ associated with the Legendre polynomials; a truncation of this series up to first $n+1$ terms is denoted by $\mathrm{CA}[f,n,u](t)$ and is called a *chromatic approximation* of $f(t)$. Just like a Taylor approximation, a chromatic approximation is also a local approximation: its coefficients are the values of differential operators $\mathcal{K}^m[f](u)$ at a single instant u, and for all $k \leq n$, $f^{(k)}(u) = \mathrm{d}^k/\mathrm{d}t^k\ \mathrm{CA}[f,n,u](t)\big|_{t=u}$.
Figure 2 compares the behavior of the chromatic approximation (black) of a signal $f \in \mathbf{BL}(\pi)$ (gray) with the behavior of the Taylor approximation of $f(t)$ (dashed). Both approximations are of order sixteen. The plot reveals that, when approximating a signal $f \in \mathbf{BL}(\pi)$, a chromatic approximation has a much gentler error accumulation when moving away from the point of expansion than the Taylor approximation of the same order.
Functions $\mathcal{K}^n[\mathrm{sinc}\,](t)$ appearing in the chromatic expansion associated with the Legendre polynomials are given by $\mathcal{K}^n[\mathrm{sinc}\,](t) = (-1)^n\sqrt{2n+1}\ j_n(\pi t)$, where j_n is the spherical Bessel function of the first kind of order n. Thus, unlike the monomials that appear in the Taylor formula, functions $\mathcal{K}^n[\mathrm{sinc}\,](t)$ belong to $\mathbf{BL}(\pi)$ and satisfy $|\mathcal{K}^n[\mathrm{sinc}\,](t)| \leq 1$ for all $t \in \mathbb{R}$. Consequently, the chromatic approximations are bounded on $\mathbb{R}$ and belong to $\mathbf{BL}(\pi)$. Also, as Proposition 1 asserts, the chromatic approximation of a signal $f \in \mathbf{BL}(\pi)$ converges in $\mathbf{BL}(\pi)$. Thus, if A is a filter, then A commutes with the differential

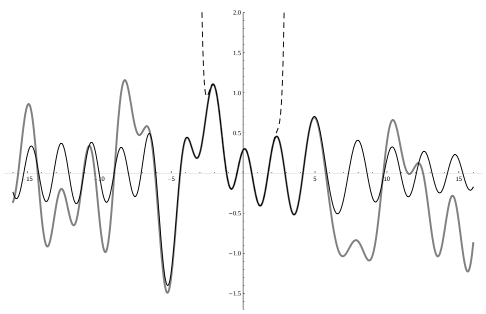

Figure 2: Chromatic approximation (black) and Taylor's approximation (dashed) of a signal from $\mathbf{BL}(\pi)$ (gray).

operators $\mathcal{K}^n$ and for every $f \in \mathbf{BL}(\pi)$,

$$A[f](t) = \sum_{n=0}^{\infty}(-1)^n\ \mathcal{K}^n[f](0)\ \mathcal{K}^n[A[\,\mathrm{sinc}\,]](t).$$

This shows that, while local, the chromatic expansion possesses the features that make the Nyquist expansion useful in signal processing. This, together with numerical robustness of the chromatic derivatives, makes chromatic approximations applicable in fields involving empirically sampled data, such as signal and image processing.
The next proposition demonstrates another remarkable feature of the chromatic derivatives which is relevant to signal processing.

Proposition 2 *Let $\mathcal{K}^n$ be the chromatic derivatives associated with the (re-scaled and normalized) Legendre polynomials, and $f, g \in \mathbf{BL}(\pi)$. Then*

$$\begin{aligned}
\sum_{n=0}^{\infty} K^n[f](t)^2 &= \int_{-\infty}^{\infty} f(x)^2 dx;\\
\sum_{n=0}^{\infty} K^n[f](t)K^n[g](t) &= \int_{-\infty}^{\infty} f(x)g(x)dx;\\
\sum_{n=0}^{\infty} K^n[f](t)K^n_t[g(u-t)] &= \int_{-\infty}^{\infty} f(x)g(u-x)dx.
\end{aligned}$$

Thus, the sums on the left hand side of the above equations do not depend on the choice of the instant t.

Note that the above equations provide local representations of the usual norm, the scalar product and the convolution, respectively, which are defined in L_2 globally, as improper integrals.
Given the above properties of the Legendre polynomials, it is natural to ask if other families of orthonormal polynomials have similar properties. This question was answered in [1].

4. General Chromatic Derivatives

Let $\mathcal{M} : \mathcal{P}_\omega \rightarrow \mathbb{R}$ be a linear functional on the vector space $\mathcal{P}_\omega$ of real polynomials in the variable ω. Such $\mathcal{M}$ is called a moment functional and $\mu_n = \mathcal{M}(\omega^n)$ is the moment of $\mathcal{M}$ of order n.

Definition 1 *A moment functionals $\mathcal{M}$ is chromatic if it satisfies the following conditions (condition (iii) is not essential, but simplifies the technicalities):*
(i) $\mathcal{M}$ is positive definite;
(ii) $\limsup_{n\to\infty} \mu_n^{1/n}/n < \infty$;
(iii) $\mathcal{M}$ is symmetric, i.e., $\mu_{2n+1} = 0$ for all n.

For functionals $\mathcal{M}$ which satisfy conditions (i) and (iii) there exists a family of real polynomials $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$, such that $P_n^{\mathcal{M}}(\omega)$ contains only powers of ω of the same parity as n and which are orthonormal with respect to $\mathcal{M}$; i.e., for all m, n,

$$\mathcal{M}(P_m^{\mathcal{M}}(\omega)\,P_n^{\mathcal{M}}(\omega)) = \delta(m-n).$$

The family $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ corresponds to a symmetric positive definite moment functional just in case there exists a sequence of positive reals $\{\gamma_n\}_{n\in\mathbb{N}}$ such that

$$P_{n+1}^{\mathcal{M}}(\omega) = \tfrac{1}{\gamma_n}\,\omega\,P_n^{\mathcal{M}}(\omega) - \tfrac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}^{\mathcal{M}}(\omega). \qquad (4)$$

For every positive definite moment functional there exists a non-decreasing bounded function $a(\omega)$, called an m–distribution function, such that for the associated Stieltjes integral we have

$$\textstyle\int_{-\infty}^{\infty} \omega^n \,\mathrm{d}a(\omega) = \mu_n, \qquad (5)$$

$$\textstyle\int_{-\infty}^{\infty} P_n^{\mathcal{M}}(\omega)\,P_m^{\mathcal{M}}(\omega)\,\mathrm{d}a(\omega) = \delta(m-n). \qquad (6)$$

If $\mathcal{M}$ is chromatic, then condition (3) implies that $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ is a complete system in $L^2_{a(\omega)}$.

Let $\varphi \in L^2_{a(\omega)}$; we can define a corresponding function $f_\varphi : \mathbb{R} \to \mathbb{C}$ by

$$f_\varphi(t) = \textstyle\int_{-\infty}^{\infty} \varphi(\omega) e^{\mathrm{i}\omega t}\,\mathrm{d}a(\omega), \qquad (7)$$

and one can show that (7) can be differentiated under the integral sign any number of times. Setting

$$\mathcal{K}^n = \tfrac{1}{\mathrm{i}^n} P_n^{\mathcal{M}}(\omega)\left(\mathrm{i}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\right)$$

we get that for all t

$$\mathcal{K}^n[f_\varphi](t) = \textstyle\int_{-\infty}^{\infty} \mathrm{i}^n\, P_n^{\mathcal{M}}(\omega)\,\varphi(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}a(\omega), \qquad (8)$$

i.e., $\langle \varphi(\omega)\mathrm{e}^{\mathrm{i}\omega t}, P_n^{\mathcal{M}}(\omega)\rangle_{a(\omega)} = (-\mathrm{i})^n \mathcal{K}^n[f_\omega](t)$. By Parseval's Theorem, for every $t \in \mathbb{R}$,

$$\textstyle\sum_{n=0}^{\infty} |\mathcal{K}^n[f_\varphi](t)|^2 = \|\varphi(\omega)\mathrm{e}^{\mathrm{i}\omega t}\|^2_{a(\omega)} = \|\varphi(\omega)\|^2_{a(\omega)}.$$

Thus, if $\varphi \in L^2_{a(\omega)}$, then the sum $\sum_{n=0}^{\infty} |\mathcal{K}^n[f_\varphi](t)|^2$ converges to a constant function on $\mathbb{R}$. In particular, if we let

$$\boldsymbol{m}(t) = \textstyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}a(\omega), \qquad (9)$$

then (5) implies $\boldsymbol{m}^{(k)}(0) = \mathrm{i}^k \mu_k$. It can be shown that condition (iii) of Definition 1 implies that $\boldsymbol{m}(t)$ is analytic on $\mathbb{R}$ (moreover, it is analytic on a strip in $\mathbb{C}$). Let

$$\mathrm{CA}^{\mathcal{M}}[f,n,u](t) = \textstyle\sum_{k=0}^{n} (-1)^k \mathcal{K}^k[f](u)\mathcal{K}^k[\boldsymbol{m}](t-u);$$

then it can be shown that

$$|f_\varphi(t) - \mathrm{CA}^{\mathcal{M}}[f,n,u](t)| < \textstyle\sum_{k=n+1}^{\infty} \left|\mathcal{K}^k[f_\varphi](u)\right|^2.$$

Thus, $f_\varphi(t) = \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f_\varphi](u)\,\mathcal{K}^k[\boldsymbol{m}](t-u)$, and the convergence is uniform on $\mathbb{R}$.

Definition 2 $L_2^{\mathcal{M}}$ *denotes the space of functions analytic on* $\mathbb{R}$ *which satisfy* $\sum_{k=0}^{\infty} \mathcal{K}^k[f](0)^2 < \infty$.

Let $f(t) \in L_2^{\mathcal{M}}$; then

$$\varphi_f(\omega) = \textstyle\sum_{k=0}^{\infty} (-\mathrm{i})^k \mathcal{K}^k[f](0) P_k^{\mathcal{M}}(\omega)$$

belongs to $L^2_{a(\omega)}$ and for all t,

$$f(t) = \textstyle\int_{-\infty}^{\infty} \varphi_f(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}a(\omega).$$

On the space $L_2^{\mathcal{M}}$ one can now introduce locally defined norm, inner product and convolution using equations from Proposition 2, and for every fixed u, the chromatic expansion of an $f \in L_2^{\mathcal{M}}$ is just the Fourier series of f in the orthonormal and complete base $\{\mathcal{K}^n[\boldsymbol{m}](t-u) : n \in \mathbb{N}\}$.

5. Examples

Example 1. (Legendre polynomials/Spherical Bessel functions) Let $L_n(\omega)$ be the Legendre polynomials; if we set $P_n^L(\omega) = \sqrt{2n+1}\,L_n(\omega/\pi)$, then

$$\textstyle\int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) \frac{\mathrm{d}\omega}{2\pi} = \delta(m-n).$$

The corresponding recursion coefficients in equation (4) are given by the formula $\gamma_n = \pi(n+1)/\sqrt{4(n+1)^2 - 1}$. In this case $\boldsymbol{m}(t) = \operatorname{sinc} t$, and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{(2n+1)}\; j_n(\pi t)$, where $j_n(x)$ is the spherical Bessel function of the first kind of order n. The corresponding space $L_2^{\mathcal{M}}$ consists of all analytic functions which belong to L_2 and have a Fourier Transform supported in $[-\pi,\pi]$.

Example 2. (Chebyshev polynomials of the first kind/Bessel functions) Let $P_n^T(\omega)$ be the family of orthonormal polynomials obtained by normalizing and rescaling the Chebyshev polynomials of the first kind, $T_n(\omega)$, by setting $P_0^T(\omega) = 1$ and $P_n^T(\omega) = \sqrt{2}\,T_n(\omega/\pi)$ for $n > 0$. In this case

$$\textstyle\int_{-\pi}^{\pi} P_n^T(\omega) P_m^T(\omega)\; \frac{\mathrm{d}\omega}{\pi^2\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}} = \delta(n-m).$$

The corresponding function (9) is $\boldsymbol{m}(t) = J_0(\pi t)$ and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2}\,J_n(\pi t)$, where $J_n(t)$ is the Bessel function of the first kind of order n. In the recurrence relation (4) the coefficients are given by $\gamma_0 = \pi/\sqrt{2}$ and $\gamma_n = \pi/2$ for $n > 0$. The corresponding space $L_2^{\mathcal{M}}$ consists of analytic functions whose Fourier transform $\widehat{f(\omega)}$ is supported in $(-\pi,\pi)$ and satisfies $\int_{-\pi}^{\pi} \sqrt{1-(\omega/\pi)^2}\; |\widehat{f(\omega)}|^2 \mathrm{d}\omega < \infty$. The chromatic expansion of a function $f(t)$ is the Neumann series of $f(t)$,

$$f(t) = f(0)J_0(\pi t) + \sqrt{2}\;\textstyle\sum_{n=1}^{\infty} \mathcal{K}^n[f](0) J_n(\pi t).$$

Thus, the chromatic expansions corresponding to various families of orthogonal polynomials can be seen as generalizations of the Neumann series, while the families of corresponding functions $\{\mathcal{K}^n[\boldsymbol{m}](t)\}_{n\in\mathbb{N}}$ can be seen as generalizations (and a uniform representation) of some familiar families of special functions.

Example 3. (Hermite polynomials/Gaussian monomial functions) Let $H_n(\omega)$ be the Hermite polynomials; then the polynomials given by $P_n^H(\omega) = (2^n\, n!)^{-1/2}\, H_n(\omega)$ satisfy

$$\textstyle\int_{-\infty}^{\infty} P_n^H(\omega) P_m^H(\omega)\; \frac{\mathrm{e}^{-\omega^2}}{\sqrt{\pi}}\; \mathrm{d}\omega = \delta(n-m).$$

The corresponding function defined by (9) is $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\, t^n \mathrm{e}^{-t^2/4}/\sqrt{2^n\, n!}$. The corresponding recursion coefficients are given by $\gamma_n = \sqrt{(n+1)/2}$. The corresponding space $L_2^{\mathcal{M}}$ consists of analytic functions whose Fourier transform $\widehat{f(\omega)}$ satisfies $\int_{-\infty}^{\infty} |\widehat{f(\omega)}|^2\; \mathrm{e}^{\omega^2} \mathrm{d}\omega < \infty$. The chromatic expansion of $f(t)$ is just the Taylor expansion of $f(t)\,\mathrm{e}^{t^2/4}$, multiplied by $\mathrm{e}^{-t^2/4}$.

6. Weakly Bounded Moment Functionals

To study pointwise convergence of chromatic expansions we somewhat restrict the class of moment functionals we consider.

Definition 3 *Let $\mathcal{M}$ be a symmetric positive definite moment functional and let $\gamma_n > 0$ be such that* (4) *holds.*

(i) $\mathcal{M}$ is weakly bounded if there exist some $M \geq 1$, some $0 \leq p < 1$ and some integer r, such that for all $n \geq 0$, $1/M \leq \gamma_n \leq M(n+r)^p$ and $\gamma_n/\gamma_{n+1} \leq M^2$.

(ii) $\mathcal{M}$ is bounded if there exists some $M \geq 1$ such that $1/M \leq \gamma_n \leq M$ for all $n \geq 0$.

Thus, every bounded functional $\mathcal{M}$ is also weakly bounded with $p = 0$. Functionals in our Example 1 and Example 2 are bounded. For bounded moment functionals $\mathcal{M}$ the corresponding m-distribution $a(\omega)$ has a finite support and consequently $\boldsymbol{m}(t)$ is a band-limited signal. However, $\boldsymbol{m}(t)$ can be of infinite energy (i.e., not in L_2) as is the case in our Example 2. Moment functional in Example 3 is weakly bounded but not bounded ($p = 1/2$). We note that all important examples of classical orthogonal polynomials which correspond to weakly bounded moment functionals in fact satisfy a stronger condition $0 < \lim_{n\to\infty} \gamma_n/n^p < \infty$ for some $0 \leq p < 1$.

Lemma 3 *If $\mathcal{M}$ is a weakly bounded moment functional, then $\lim_{k\to\infty} (\mu_k/k!)^{1/k} = 0$. Thus, $\mathcal{M}$ is chromatic; moreover, $\boldsymbol{m}(z) = \sum_{n=0}^{\infty} \mathrm{i}^n \mu_n z^n/n!$ is an entire function on $\mathbb{C}$.*

Lemma 4 *Let $\mathcal{M}$ be weakly bounded and $p < 1$ as in Definition 3(i); then there exists $K > 0$ such that for every integer $k \geq 1/(1-p)$, every $n \in \mathbb{N}$ and every $z \in \mathbb{C}$,*

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < (K|z|)^n \mathrm{e}^{(K|z|+1)^k}/n!^{1-p}.$$

This Lemma is used to prove the following Proposition.

Proposition 5 *Let $\mathcal{M}$ be as in Lemma 4, $f(z)$ an entire function and $u \in \mathbb{C}$. If $\lim_{n\to\infty} |f^{(n)}(u)/n!^{1-p}|^{1/n} = 0$, then the chromatic expansion of $f(z)$ centered at u converges everywhere to $f(z)$, and the convergence is uniform on every disc of finite radius.*

Thus, if $\mathcal{M}$ is bounded ($p = 0$) and f is an entire function, then the chromatic expansion $CE[f,u](t)$ converges to $f(t)$ for all t.
Proposition 5 yields generalizations of many well known equalities for the Bessel functions of the first kind. For example, for every weakly bounded moment functional $\mathcal{M}$,

$$\mathrm{e}^{\mathrm{i}\omega t} = \sum_{n=0}^{\infty} \mathrm{i}^n P_n^{\mathcal{M}}(\omega)\, \mathcal{K}^n[\boldsymbol{m}](t);$$
$$\boldsymbol{m}(z) + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\right) \mathcal{K}^{2n}[\boldsymbol{m}](z) = 1,$$
$$\boldsymbol{m}(t+u) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[\boldsymbol{m}](u) \mathcal{K}^n[\boldsymbol{m}](t),$$

which generalize the following well known equalities:

$$\mathrm{e}^{\mathrm{i}\,\omega t} = J_0(t) + 2\sum_{n=1}^{\infty} \mathrm{i}^n T_n(\omega) J_n(t);$$
$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1;$$
$$J_0(t+u) = J_0(u)J_0(t) + 2\sum_{n=1}^{\infty} (-1)^n J_n(u) J_n(t).$$

7. Non-Separable Inner Product Spaces

Periodic functions do not belong to $L_2^{\mathcal{M}}$; for example, it is easy to see that $\sum_{k=0}^{\infty} |\mathcal{K}^k[\mathrm{e}^{\mathrm{i}\,\omega t}]|^2$ diverges. We now consider some inner product spaces in which pure harmonic oscillations have finite positive norms ([2]).

Definition 4 *Assume again that $\mathcal{M}$ is weakly bounded and let p be as in Definition 3. We denote by $\mathcal{C}^{\mathcal{M}}$ the vector space of analytic functions such that the sequence*

$$\nu_n^f(t) = 1/(n+1)^{1-p} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2$$

converges uniformly on every finite interval.

Proposition 6 *Let $f, g \in \mathcal{C}^{\mathcal{M}}$ and*

$$\sigma_n^{fg}(t) = 1/(n+1)^{1-p} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t);$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function. In particular, $\nu_n^f(t)$ is constant.

Corollary 7 *Let $\mathcal{C}_0^{\mathcal{M}}$ be the vector space consisting of analytic functions $f(t)$ such that $\lim_{n\to\infty} \nu_n^f(t) = 0$; then in the quotient space $\mathcal{C}_2^{\mathcal{M}} = \mathcal{C}^{\mathcal{M}}/\mathcal{C}_0^{\mathcal{M}}$ the limit $\lim_{n\to\infty} \sigma_n^{fg}(t)$ is independent of t and defines a scalar product on $\mathcal{C}_2^{\mathcal{M}}$.*

Proposition 8 *Let $\mathcal{M}$ correspond to Chebyshev polynomials as in our Example 2; then functions $f_\omega(t) = \sqrt{2}\,\sin\omega t$ and $g_\omega(t) = \sqrt{2}\,\cos\omega t$ for all $0 < \omega < \pi$ form an uncountable orthonormal system of vectors in $\mathcal{C}_2^{\mathcal{M}}$.*

Proposition 9 *Let $\mathcal{M}$ correspond to Hermite polynomials as in our Example 3; then for all $\omega > 0$ functions $f_\omega(t) = \sin\omega t$ and $g_\omega(t) = \cos\omega t$ form an uncountable orthogonal system of vectors in $\mathcal{C}_2^{\mathcal{M}}$, and $\|f_\omega\|^{\mathcal{M}} = \|g_\omega\|^{\mathcal{M}} = \mathrm{e}^{\omega^2/2}/\sqrt[4]{2\pi}$.*

Conjecture 1 *Assume that for some $0 \leq p < 1$ the recursion coefficients γ_n in* (4) *are such that γ_n/n^p converges to a finite positive limit. Then, for the corresponding family of orthogonal polynomials we have*

$$0 < \lim_{n\to\infty} 1/(n+1)^{1-p} \sum_{k=0}^{n} P_k^{\mathcal{M}}(\omega)^2 < \infty$$

for all ω in the support $sp(a)$ of the corresponding m-distribution function $a(\omega)$. Thus, in the corresponding space $\mathcal{C}_2^{\mathcal{M}}$ all pure harmonic oscillations with positive frequencies $\omega \in sp(a)$ have finite positive norm and are mutually orthogonal.

Most detailed presentation of the theory of chromatic derivatives can be found in [3]; preprints of unpublished manuscripts are available at `http://www.cse.unsw.edu.au/~ignjat/diff`.

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