Periodica Mathematica Hungarica Vol. 33 (1), (1996), pp. 45–50

A COMMENT ON THE JOINT EMBEDDING PROPERTY*

ALEKSANDER IGNJATOVIĆ (Pittsburgh)
MILAN Z. GRULOVIĆ (Novi Sad)

Abstract

The paper presents an alternative proof of the known result that no recursively enumerable number theory has the joint embedding property.

1.

By a *theory* (of the language L) we mean a consistent, deductively closed set of sentences (of L). The models will be denoted by $\mathbf{A}, \mathbf{B}, \ldots$, while their domains are, respectively $A, B, \ldots$. If $\mathbf{A}$ is a model of the language L then $L(A)$ is a simple expansion of L, obtained by adding to L the new set of constants which is in one to one correspondence with the domain A. It became customary that for an element of A and its corresponding constant one uses the same notation (whence, instead of c_a we will rather write a); it is always clear from the context what is meant by it. For a model $\mathbf{A}$, $Diag(\mathbf{A})$ is its *positive diagram* — the set of all basic sentences of the language $L(A)$ which hold in $\mathbf{A}$. For a formula ϕ, $fv(\phi)$ is the set of its free variables. When we write $\phi(v_{i_1}, \ldots, v_{i_m})$ it just means that $fv(\phi)$ is a subset (possibly empty) of $\{v_{i_1}, \ldots, v_{i_m}\}$. For a set of formulas Γ, $fv(\Gamma) = \cup\{fv(\phi) \mid \phi \in \Gamma\}$ and Γ is a type iff $fv(\Gamma)$ is finite. A type is *open* (Σ_n, Π_n) iff each of its formulas is open – equivalent to a formula without quantifiers (Σ_n, Π_n formula). A type Γ is a *type of a theory* T iff some model of T realizes Γ. For a theory T, T^f is its (Robinson) forcing companion.

DEFINITION 1.1. A theory T has the *joint embedding property* (JEP) iff any two models can be embedded into a third one.

Mathematics subject classification numbers, 1991. Primary 03C62, 03C52; Secondary 03C25.

Key words and phrases. Joint embedding property, amalgamation property, Peano arithmetic, number theory.

*Presented at the Conference on Ordered structures and Universal Algebra — in Honor of László Fuchs, Budapest, 1994.

0031-5303/96/$5.00
© *Akadémiai Kiadó, Budapest*
Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

A quintuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f, g)$ is an *amalgamation base* of a theory T iff $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are models of T and f and g are embeddings of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$, respectively. The amalgamation base $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f, g)$ can be amalgamated in T iff there is a model $\mathbf{D}$ of T and embeddings f_1 and g_1 of $\mathbf{B}$ and $\mathbf{C}$ into D such that $f_1 \circ f = g_1 \circ g$. A model $\mathbf{A}$ of the theory is an *amalgamation core* iff each amalgamation base $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f, g)$ (of T) can be amalgamated in T (in a wider interpretation of an amalgamation core one requires only that $\mathbf{A}$ is just a submodel of the theory T, thus a model of $T \cap \Pi_1 = \{\varphi \in T \mid \varphi \text{ is an } \Pi_1\text{-sentence}\}$).

A theory T has the *amalgamation property* (AP) iff each of its models is an amalgamation core.

A theory T has a *model-consistent completion* iff there is a complete theory T^* which is a mutually model-consistent with T (i.e. $T \cap \Pi_1 = T^* \cap \Pi_1$ holds).

A lot of equivalents to the joint embedding property are known, both of semantic and syntactical nature. We will mention some of them.

THEOREM 1.2. [16] *For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:*

(1) *Theory T has the joint embedding property.*

(2) *If $\mathbf{A}$ and $\mathbf{B}$ are models of T and $a_1, \ldots, a_n$ are some elements from A, then there is a model $\mathbf{C}$ of T, which is an expansion of $\mathbf{B}$ such that for some of its elements $c_1, \ldots, c_n$, $(\mathbf{A}, a_1, \ldots, a_n) \equiv_0 (\mathbf{C}, c_1, \ldots, c_n)$ holds (i.e. the submodels of $\mathbf{A}$ and $\mathbf{C}$ generated by the constant of the language L and the new constants are isomorphic).*

(3) *If ϕ and ψ are universal sentences of the language L such that $T \vdash \phi \vee \psi$ then either $T \vdash \phi$ or $T \vdash \psi$.*

(4) *If ϕ and ψ are existential sentences such that both $T \cup \{\phi\}$ and $T \cup \{\psi\}$ are consistent sets of sentences then $T \cup \{\phi, \psi\}$ is consistent as well.*

(5) *If $\mathbf{A}$ and $\mathbf{B}$ are models of T then $T \cup Diag(\mathbf{A}) \cup Diag(\mathbf{B})$ is a consistent set of sentences.*

(6) *If Γ_1 and Γ_2 are open types of the theory T such that $fv(\Gamma_1) \cap fv(\Gamma_2) = \emptyset$, then $\Gamma_1 \cup \Gamma_2$ is also an open type of T.*

(7) *The same as in the previous condition but with Γ_1, Γ_2 being existential.*

(8) *If $\mathbf{A}_\alpha$, $\alpha < \lambda$ are models of T then all of them can be embedded in some model of T.*

(9) *If $\Sigma = \{\phi_\alpha \mid \alpha < \lambda\}$ is the set of existential sentences, each of which is consistent with T, then $T \cup \Sigma$ is consistent.*

(10) *There is a model $\mathbf{A}$ of T such that any model of T can be embedded into some ultrapower of $\mathbf{A}$.*

(11) *Theory T has the model-consistent completion.*

REMARK. [16] contains a detailed proof of a statement analogous to the previous theorem (stated in terms of amalgamation cores). Since a model $\mathbf{A}$ of a theory T is an amalgamation core iff the theory $T \cup Diag(\mathbf{A})$ has the joint embedding property, the theorem is an easy consequence of the cited statement.

2.

From now on our language is the standard language of Peano arithmetic. The next recursive functions and recursive or recursively enumerable properties are described in many textbooks on mathematical logic. However, the notations can vary, therefore we need to fix our notation. The only purpose of this section is to exclude any misunderstandings.

A code number of a term t or a formula ϕ will be denoted, respectively, by $\lceil t \rceil$, $\lceil \phi \rceil$. The variables are $v_1, v_2, \ldots$, but also $x, y, \ldots$; the numerals are: $s_0 \equiv 0$, $s_{n+1} = s(s_n)$. Tm is the recursive property such that (for a natural number x) $Tm(x)$ holds iff x is a *term*, i.e. the code number of some term. Similarly, Fml is the recursive property such that $Fml(x)$ holds iff x is a *formula*, that is the code number of some formula. x is an *expression* is a shorter way to say that x is either a *term* or a *formula*. Vbl is the binary recursive relation such that $Vbl(x, y)$ holds iff $y > 0$ and either x is a *term* containing the y^{th} variable v_y or a *formula* containing v_y free. Num is the recursive function defined by $Num(x) = \lceil s_x \rceil$, Neg is the recursive function for which $Neg(\lceil \phi \rceil) = \lceil \neg \phi \rceil$ holds. Sub is a (total) ternary recursive function, such that if x is an *expression* and if $y > 0$ then $Sub(x, y, z)$ is the *expression* obtained from it by substituting the numeral s_z for the variable v_y, otherwise $Sub(x, y, z) = x$. Finally, Prf is the binary recursive relation such that $Prf(m, n)$ holds iff m is the code number of the proof of the formula with code number n.

Some folklore theorems are assumed to be known.

3.

In the next propositions we will be using the equivalence of the conditions (1) and (4) in theorem 1.2. Namely, it will be shown that no recursively enumerable theory T has the joint embedding property by presenting two existential sentences, both consistent with T but whose conjunction is not consistent with T.

LEMMA 3.1. *Let $\varphi(v_1, v_2)$ and $\psi(v_1, v_2)$ be any two formulas of the language of Peano arithmetic (PA). Then there exist sentences ϕ, θ such that:*

$$PA \vdash \phi \Leftrightarrow \varphi(\lceil \phi \rceil, \lceil \theta \rceil), \quad PA \vdash \theta \Leftrightarrow \psi(\lceil \phi \rceil, \lceil \theta \rceil).$$

PROOF. Let

$$\varphi_1(x, y) \doteq \varphi\Big(Sub\,(\, Sub(x, 1, x), 2, y), Sub\,(\, Sub(y, 2, y), 1, x)\Big),$$

$$\psi_1(x, y) \doteq \psi\Big(Sub\,(\, Sub(x, 1, x), 2, y), Sub\,(\, Sub(y, 2, y), 1, x)\Big),$$

and $m = \lceil \varphi_1(v_1, v_2) \rceil$, $n = \lceil \psi_1(v_1, v_2) \rceil$, while $\phi \doteq \varphi_1(m, n)$, $\theta \doteq \psi_1(m, n)$. Then, obviously, $PA \vdash \phi \Leftrightarrow \varphi(\lceil \phi \rceil, \lceil \theta \rceil)$. For, just by the definition of the given recursive

relation Sub, we have for a model $\mathbf{M}$ of Peano arithmetic:

$$\mathbf{M} \models \phi$$

iff

$$\mathbf{M} \models \varphi\Big(Sub\ \big(Sub(m,1,m),2,n\big), Sub\ \big(Sub(n,2,n),1,m\big)\Big)$$

iff

$$\mathbf{M} \models \varphi\Big(Sub\ \big(Sub(\lceil\varphi_1(v_1,v_2)\rceil,1,m),2,n\big), Sub\ \big(Sub(\lceil\psi(v_1,v_2)\rceil,2,n),1,m\big)\Big)$$

iff

$$\mathbf{M} \models \varphi\big(Sub(\varphi_1(m,v_2)\rceil,2,n), Sub(\lceil\psi_1(v_1,n)\rceil,1,m)\big)$$

iff

$$\mathbf{M} \models \varphi(\lceil\varphi_1(m,n)\rceil,\lceil\psi_1(m,n)\rceil)$$

iff

$$\mathbf{M} \models \varphi(\lceil\phi\rceil,\lceil\theta\rceil).$$

Analogously, $PA \vdash \theta \Leftrightarrow \psi(\lceil\phi\rceil,\lceil\theta\rceil)$. ■

THEOREM 3.2. [5] *PA does note have the joint embedding property.*

PROOF. Let

$$\varphi(x,y) \equiv \varphi \doteq \exists z\Big(Prf\ (z, Neg(x)) \wedge (\forall w < z)\neg\, Prf\ (w, Neg(y))\Big)$$

$$\psi(x,y) \equiv \psi \doteq \exists z\Big(Prf\ (z, Neg(y)) \wedge (\forall w < z)\neg\, Prf\ (w, Neg(x))\Big)$$

By the previous lemma there exist sentences ϕ and θ such that

(1) $$PA \vdash \phi \Leftrightarrow \exists z\big(Prf(z,\lceil\neg\phi\rceil) \wedge \forall w < z \neg\, Prf(w,\lceil\neg\theta\rceil)\big),$$

(2) $$PA \vdash \theta \Leftrightarrow \exists z\big(Prf(z,\lceil\neg\theta\rceil) \wedge \forall w < z \neg\, Prf(w,\lceil\neg\phi\rceil)\big),$$

First, we show that both $PA \cup \{\phi\}$ and $PA \cup \{\theta\}$ are consistent. Let us suppose that $PA \vdash \neg\phi$ and let n be the least code of the proof for it. Hence, $PA \vdash Prf(n,\lceil\neg\phi\rceil)$ which, together with $PA \nvdash \phi$, implies $PA \nvdash (\forall w < n)\neg\, Prf(w,\lceil\neg\theta\rceil)$. Being quantifier free (bounded quantifiers "are not counted") and unprovable in PA, the sentence $(\forall w < n)\neg\, Prf(w,\lceil\neg\theta\rceil)$ does not hold in $\mathbf{N}$ ($\mathbf{N}$ is, of course, the standard model of Peano aritmetic). Therefore, for some $k < n$, $\mathbf{N} \models Prf(k,\lceil\neg\theta\rceil)$ and, consequently, $PA \vdash \neg\theta$. But now, by (2), $PA \nvdash (\forall w < k)\neg\, Prf(w,\lceil\neg\phi\rceil)$, and so, for some $m < k$, $\mathbf{N} \models Prf(m,\lceil\neg\phi\rceil)$, contradicting the fact that n is the least code number of the proof for $\neg\phi$. By symmetry, $PA \nvdash \neg\theta$.

On the other hand, $PA \cup \{\phi,\theta\}$ is not consistent. For if $\mathbf{M}$ is a model of $PA \cup \big\{\exists z\big(Prf(z,\lceil\neg\phi\rceil) \wedge (\forall w < z)\neg\, Prf(w,\lceil\neg\theta\rceil)\big)\big\}$, then any code number of the proof for $\neg\theta$ is greater than the least code number of the proof for $\neg\phi$, thus θ does not hold in $\mathbf{M}$.

Finally, let us note that by Matijasevič theorem (*a relation is recursively enumerable iff it is diophantine*) the sentences ϕ and θ are equivalent in Peano

arithmetic to some existential sentences ([1], [8]). Now "(1) $\Leftrightarrow$ (2)" of theorem 1.2 completes the proof. ■

A *number theory* is any theory containing the Π_2-part of Peano arithmetic ([2], [7]). From the fact that for the proof of Matijasevič theorem just this segment of Peano arithmetic is sufficient (see, for instance, [12]), it follows directly:

COROLLARY 3.3. *No recursively enumerable number theory has the joint embedding property.*

COROLLARY 3.4. *No recursively enumerable number theory has the amalgamation property.*

COROLLARY 3.5. *The forcing companion of a recursively enumerable number theory is not complete.*

PROOF. Generally it holds for any theory T: T^f is complete iff T has the joint embedding property ([14]) — one more condition equivalent to JEP. ■

NOTE. Much more on this subject can be found in [7], [10], [11], [12].

REFERENCES

[1] GAIFMAN, H., A Note on Models and Submodels of Arithmetic, *Lecture Notes in Mathematics* **255**, Springer-Verlag, 1972, 128–144.

[2] GOLDREI, D.C., MACINTYRE, A. and SIMMONS, H., The Forcing Companions of Number Theories, *Israel Journal of Mathematics* **14** (1973), 317–337.

[3] GRÄTZER, G., *General Lattice Theory*, Akademie-Verlag, Berlin, 1978.

[4] GRULOVIĆ, M.Z. On n-Finite Forcing Companions, *Review of Research, Faculty of Science, University of Novi Sad, Mathematics Series* **14**(2), (1984), 211–222.

[5] IGNJATOVIĆ, A. *Modeli aritmetike*, Magistarski rad (Master thesis), Beograd, 1984.

[6] KISS, E. W., MÁRKI, L., PRÖHLE P. and THOLEN, W., Categorical Algebraic Properties. A Compendium on Amalgamation, Congruence Extension, Epimorphisms, Residual Smallness and Injectivity. *Studia Sci. Math. Hungar.* **18**, 1983, 79–141.

[7] MACINTYRE, A., SIMMONS, H., Algebraic Properties of Number Theories, *Israel Journal of Mathematics* **22(1)** (1975), 7–27.

[8] MATIJASEVIČ, Y., Enumerable sets are diophantine, *Sov. Math. Dokl.*, 1970, 354–357.

[9] MENDELSON, E., *Introduction to Mathematical Logic*, D. Van Nostrand Company, 1979.

[10] OTERO, M., A Note on Joint Embedding Property in Fragments of Arithmetic, *Bull. London Math. Soc.* **24** (1992), 417–423.

[11] OTERO, M., The joint embedding property in normal open induction, *Annals of Pure and Applied Logic* **60** (1993), 275–290.

[12] OTERO, M., The Amalgamation Property in Normal Open Inductions, *Notre Dame Journal of Formal Logic*, Volume **34**, Number **1**, Winter 1993.

[13] PARIS, J.B. and KIRBY, L.A., Σ_n-collection schemas in arithmetic, *Logic Colloquium '77* (ed. A.J. Macintyre et al), Norht-Holland, Amsterdam, 1978, 199–209.

[14] ROBINSON, A., Forcing in Model Theory, *Proceedings of Colloquium of Model Theory, Rome,* 1969, 69–82.

[15] SHOENFIELD, J.R. *Mathematical Logic*, Addison-Wesley Publishing Company, 1967.

[16] ZEITLER, H., *Modeltheoretic Investigations on the Amalgamation Property*, Ph. D. Thesis, Heidelberg, 1976.

CARNEGIE MELLON
DEPARTMENT OF PHILOSOPHY
PITTSBURGH
USA
E-MAIL: aizc@andrew.cmu.edu

INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
21000 NOVI SAD
YUGOSLAVIA
E-MAIL: grulovic@unsim.ns.ac.yu