Chromatic Derivatives, Chromatic Expansions and Associated Function Spaces

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Details can be found in:

- A. Ignjatovic: Local Approximations Based on Orthogonal Differential Operators, **Journal of Fourier Analysis and Applications**, Vol. 13, Issue 3 (2007).

- A. Ignjatovic: Chromatic derivatives and local approximations, **IEEE Transactions on Signal Processing**, Volume 57, Issue 8 (2009).

- A. Ignjatovic: Chromatic derivatives, chromatic expansions and associated spaces, **East Journal on Approximations**, Volume 15, Number 3 (2009).

How did all of this start: PWM audio amplifier

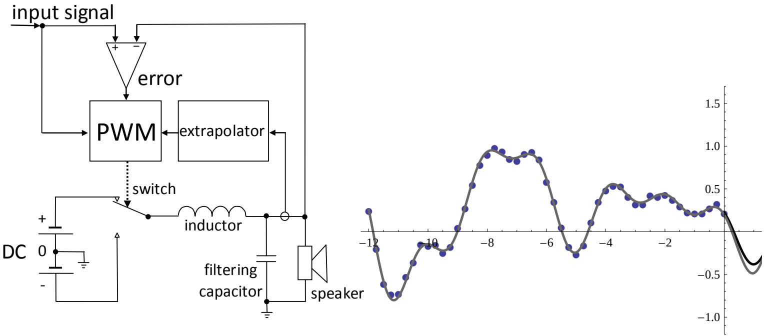


Figure: Left: *PWM amplifier;* Right: *waveform prediction.*

Nyquist's versus Taylor's expansion

- Let $f \in \mathbf{BL}(\pi)$, i.e. $f \in L_2$, $\widehat{f(\omega)}$ supported on $[-\pi, \pi]$; f has two expansions of **complementary nature**:

Nyquist's Expansion:
(Whittaker–Kotelnikov–Nyquist–Shannon)

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \, \frac{\sin \pi(t-n)}{\pi(t-n)}$$

- **global in nature** – requires samples $f(n)$ for all n;

Taylor's Expansion:

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \, \frac{t^n}{n!}$$

- **local in nature** – requires $f^{(n)}(t)$ at a single instant $t = 0$.

- Nyquist expansion: **fundamental** to signal processing;
- Taylor expansion: **very little use** in signal processing**-why?**

Problems with Taylor's expansion of $\mathbf{BL}(\pi)$ signals

1. Numerical evaluation of derivatives of high orders of a sampled signal is **extremely noise sensitive**.

2. – Truncations of Nyquist's expansion of an $f \in \mathbf{BL}(\pi)$ **belong to $\mathbf{BL}(\pi)$ and converge to f uniformly and in L^2**.

   – In comparison, truncations of Taylor's expansion of an $f \in \mathbf{BL}(\pi)$ **do not belong to $\mathbf{BL}(\pi)$, do not converge uniformly but instead accumulate error rapidly and are unbounded and do not converge in L^2.**

3. If A is a filter, then

$$A[f](t) = \sum_{n=-\infty}^{\infty} f(n)\, A\left[\mathrm{sinc}\,\right](t-n), \tag{1}$$

   while, of course, $A[f](t) \neq \sum_{n=-\infty}^{\infty} \frac{f^{(n)}(0)}{n!}\, A\left[t^n\right].$

- **We can fix ALL of these problems!!!**

Numerical differentiation of band limited signals

Let $f \in \mathbf{BL}(\pi)$; then $\dfrac{f^{(n)}(t)}{\pi^n} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} j^n \left(\dfrac{\omega}{\pi}\right)^n \widehat{f(\omega)}e^{j\omega t}d\omega.$

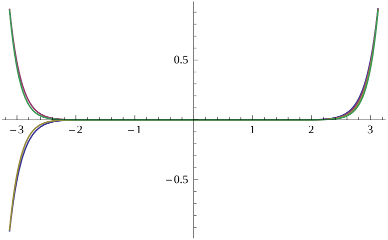


Figure: $(\omega/\pi)^n$ for $n = 15 - 18$

- Derivatives of high order **obliterate the spectrum**.
- Graphs of the transfer functions of the (normalized) derivatives **cluster together and are nearly indistinguishable.**
- **Is there a better base for the vector space of linear differential operators??**

Chromatic derivatives: a better base

- Start with normalized and re-scaled Legendre polynomials:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega)\mathrm{d}\omega = \delta(m-n).$$

- Obtain operator polynomials by replacing ω^k with $j^k\, d^k/dt^k$:

$$\mathcal{K}_t^n = (-j)^n P_n^L\left(j\ \frac{d}{dt}\right)$$

Thus,

$$\begin{aligned}
P_0^L(\omega) &= 1 &\mapsto\quad & \mathcal{K}^0[f](t) = f(t)\\
P_1^L(\omega) &= \frac{\sqrt{3}\,\omega}{\pi} &\mapsto\quad & \mathcal{K}^1[f](t) = \frac{\sqrt{3}\,f'(t)}{\pi}\\
P_2^L(\omega) &= \frac{\sqrt{5}\,(3\,\omega^2-\pi^2)}{2\pi^2} &\mapsto\quad & \mathcal{K}^2[f](t) = \frac{\sqrt{5}\,(3f''(t)+\pi^2 f(t))}{2\pi^2}\\
P_3^L(\omega) &= \frac{\sqrt{7}\,(5\,\omega^3-3\,\omega\pi^2)}{2\pi^3} &\mapsto\quad & \mathcal{K}^3[f](t) = \frac{\sqrt{7}\,(5\,f'''+3\,\pi^2 f'(t))}{2\pi^3}
\end{aligned}$$

Chromatic derivatives: a better base

- Definition of $\mathcal{K}^n$ chosen so that

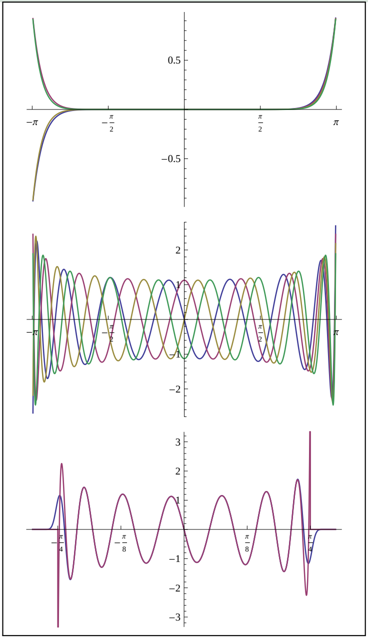
$$\mathcal{K}^n_t[e^{j\,\omega t}] = j^n P^L_n(\omega)\, e^{j\,\omega t}.$$

- Thus, for $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n P^L_n(\omega)\widehat{f(\omega)}\, e^{j\,\omega t} d\omega.$$

- **Why do $\mathcal{K}^n$ form a better base for the vector space of linear differential operators than $\frac{d^n}{dt^n}$???**

Chromatic derivatives: a better base



- Compare the graphs of the transfer functions of $1/\pi^n d^n/dt^n$, i.e., $(\omega/\pi)^n$ (first graph) and of $\mathcal{K}^n$, i.e., $P_n^L(\omega)$ (second graph).

- Transfer functions of $\mathcal{K}^n$ form a sequence of well separated comb filters that preserve spectral features of the signal, thus we call them the **chromatic derivatives**.

- Third graph: transfer function of the ideal filter $\mathcal{K}^{15}$ (red) vs. transfer function of a transversal filter (blue), (128 taps, 2× oversampling).

Fixing Taylor's Expansion: Chromatic Expansion

Proposition: Let $\operatorname{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$ and let $f(t)$ be **any analytic function**. Then,

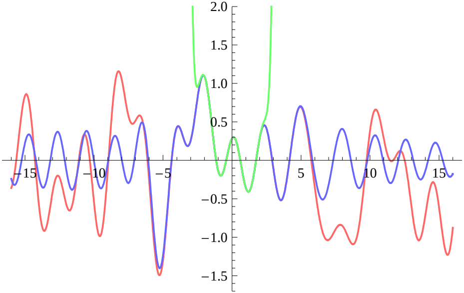
$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[\operatorname{sinc}(t)] \\ &= \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t) \end{aligned}$$

j_n – the spherical Bessel functions

solutions of $x^2 y'' + 2xy' + [x^2 - n(n+1)]y = 0$

- **The truncations of the series belong to BL(π).**
- **If $f \in$ BL(π) the series converges to $f(t)$ both uniformly and in L^2.**

Chromatic approximation versus Taylor's approximation



- **red:** the signal;
  blue: the chromatic approximation of order 15;
  green: Taylor's approximation of order 15.

- $f^{(k)}(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left[\sum_{m=0}^{n} (-1)^m \, \mathcal{K}^m[f](0) \, \mathcal{K}^m[\mathrm{sinc}\,](t)\right]_{t=0}, \quad k \leq n$

- Thus, chromatic approximations are **local**.

Chromatic versus Taylor's expansion of $f \in \mathbf{BL}(\pi)$

$$f(t) = \textstyle\sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, j_n(\pi t) \quad \text{versus} \quad f(t) = \textstyle\sum_{n=0}^{\infty} f^{(n)}(0)\, \frac{t^n}{n!}$$

- Coefficients $\mathcal{K}^n[f](0)$ can be obtained from the samples of the signal in a noise robust manner;
- Expansion functions $\{j_n(\pi t)\}_{n\in\mathbb{N}}$ belong to $\mathbf{BL}(\pi)$;
- For $f \in \mathbf{BL}(\pi)$ convergence is both uniform and in L^2.
- If A is a filter and $f \in \mathbf{BL}(\pi)$,

$$A[f](t) \;=\; \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \mathcal{K}^n[A[\mathrm{sinc}\,]](t).$$

Compare with $A[f](t) = \sum_{n=0}^{\infty} f(n)\; A[\mathrm{sinc}\,](t-n)$.

- **Localized signal processing** from (over-determined) local signal/image representation.

Chromatic expansion vs. Nyquist's expansion

How is Nyquist expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}$$

related to the chromatic expansion

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) \sqrt{2n+1}\, j_n(\pi t)$$

- Transformation $\{f(n)\}_{n\in\mathbb{N}} \Leftrightarrow \{\mathcal{K}^n[f](0)\}_{n\in\mathbb{N}}$ by an orthonormal operator defined by the infinite matrix

$$\left[\sqrt{2k+1}\, j_k(n\pi) \ : \ k \in \mathbb{N}, n \in \mathbb{Z}\right]:$$

$$\begin{aligned} f(n) &= \sum_{k=0}^{\infty} \mathcal{K}^k[f](0) \sqrt{2k+1}\, j_k(n\pi); \\ \mathcal{K}^k[f](0) &= \sum_{n=-\infty}^{\infty} f(n) \sqrt{2k+1}\, j_k(n\pi). \end{aligned}$$

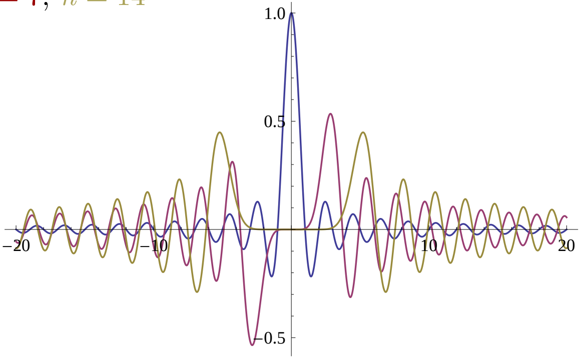
Spherical Bessel Functions

- In practice one CANNOT evaluate $\mathcal{K}^k[f](0)$ using Nyquist rate samples via

$$\mathcal{K}^k[f](0) \approx \sum_{n=-N}^{N} f(n)\, \sqrt{2k+1}\, j_k(n\pi)$$

because $\sqrt{2n+1}\, j_n(\pi t)$ decay very slowly (would need huge N)

$n = 0$, $n = 7$, $n = 14$



Chromatic approximation vs. Nyquist's approximation

- Let $S[f(t)] = f(t+1)$ be the unit shift operator; compare the Nyquist expansion in the form

$$f(t) = \sum_{n=-\infty}^{\infty} f(u+n)\mathrm{sinc}\,(t-(u+n)) = \sum_{n=-\infty}^{\infty} S^n[f](u) S_u^n[\mathrm{sinc}\,(t-u)]$$

with the chromatic expansion

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u)\ \mathcal{K}_u^n[\mathrm{sinc}\,(t-u)];$$

- Corresponding transfer functions $\{\mathrm{e}^{j\,n\,\omega}\}_{n\in\mathbb{N}}$ and $\{j^n P_n(\omega)\}_{n\in\mathbb{N}}$ are two orthonormal bases of **the same space,** $L^2[-\pi,\pi]$.

Local representation of the scalar product in $\mathbf{BL}(\pi)$

Proposition: Assume that $f, g \in \mathbf{BL}(\pi)$; then the sums on the left hand side of the following equations do not depend on the choice of the instant t, and

$$\sum_{n=0}^{\infty} K^n[f](t)^2 = \int_{-\infty}^{\infty} f(t)^2 dt = \|f\|^2$$

$$\sum_{n=0}^{\infty} K^n[f](t)\overline{K^n[g](t)} = \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt = \langle f, g\rangle$$

$$\sum_{n=0}^{\infty} K^n[f](t)K_t^n[g(u-t)] = \int_{-\infty}^{\infty} f(t)g(u-t) dt = (f * g)(u)$$

- These are the **local equivalents** of the usual, "globally defined" norm, scalar product and convolution!
- **"Maximally localized"** signal processing

Channel equalizer

Training: a predetermined pilot signal $p(t)$ is transmitted and in the receiver the output of the equalizer is compared with $p(t)$.

- At an instant k, vector $\mathbf{X}[k] = \langle x[k], \ldots, x[k-n+1] \rangle$ consists of n latest consecutive Nyquist rate samples of the received signal;
- The instantaneous error $e[k]$ of the output of the equalizer at an instant k is equal to $e[k] = \mathbf{W}\mathbf{X}^t[k] - p[k]$;
- The aim: to minimize the square error $e^2[k]$, **averaged either over time or over the ensemble of transmitted signals**.
- $J(\mathbf{W})$: averaged error for the value of coefficients $\mathbf{W}$.
- The optimal choice for $\mathbf{W}$ obtained by the Gradient Descent Method: $\mathbf{W}_{k+1} = \mathbf{W}_k - c \cdot \nabla_{\mathbf{W}} J(\mathbf{W})\big|_{\mathbf{W}=\mathbf{W}_k}$.
- c is a small adaptation constant.

- Averaged error $J(\mathbf{W})$ is unavailable or impractical to get;
- LMS: uses a crude approximation of the true gradient: the gradient of the square of the instantaneous error $e[k]$:

$$\begin{aligned}
\mathbf{G}[k] &= \nabla_{\mathbf{W}} e^2[k] = 2e[k]\nabla_{\mathbf{W}} e[k] \\
&= 2e[k]\nabla_{\mathbf{W}}(p[k] - \mathbf{W}^t\mathbf{X}[k]) \\
&= -2e[k]\nabla_{\mathbf{W}}\mathbf{W}^t\mathbf{X}[k] \\
&= -2e[k]\mathbf{X}[k];
\end{aligned}$$

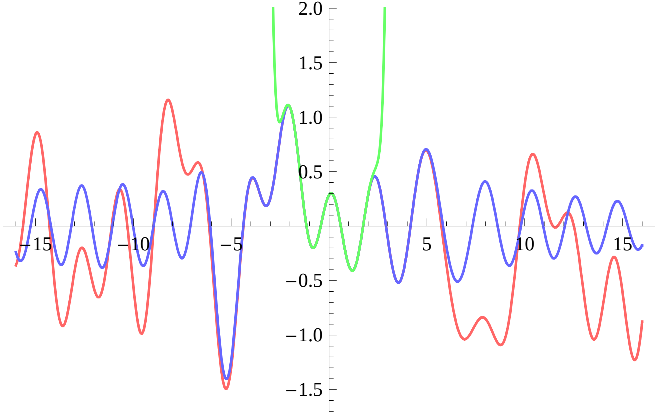
Thus,

- $\mathbf{W}_{k+1} = \mathbf{W}_k - c\mathbf{G}[k] = \mathbf{W}_k + 2c\ e[k]\mathbf{X}[k].$

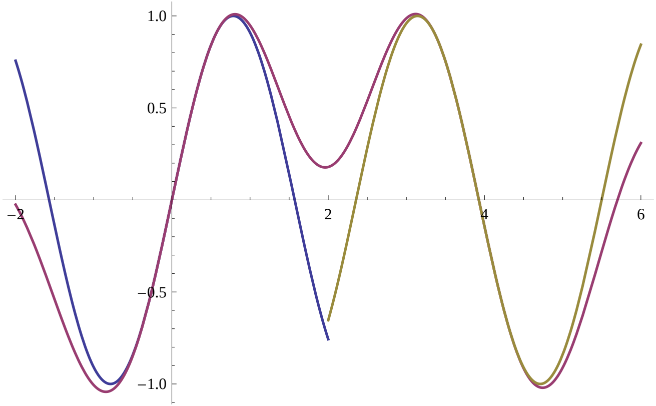
- Since $\mathbf{G}[k]$ is a crude approximation of the true gradient, **convergence of $\mathbf{W}_k$ is slow**.
- This prevents equalizer from tracking rapidly changing channels.
- Idea: replace the error function $e^2[k]$ with a new error function which naturally has averaging properties.
- Minimize

$$\sum_{m=0}^{n}(\mathcal{K}^m[\mathbf{W}^t\mathbf{X}][k]-\mathcal{K}^m[p][k])^2 = \sum_{m=0}^{n}(\mathbf{W}^t\mathcal{K}^m[\mathbf{X}][k]-\mathcal{K}^m[p][k])^2$$

via Gradient Descent. Why should this help??



Minimizing the sum of squares of first n chromatic derivatives of the error has effect of **averaging the error over a continuous interval**, whose duration depends on n.



Pulse shaping via chromatic approximations: 4 (chromatic) derivatives of the approximation (red) agree:

- at $t = 0$ with 4 derivatives of the first symbol (blue) and
- at $t = 4$ with 4 derivatives of the second symbol (yellow).

General families of chromatic derivatives

Question:

What are the families of orthogonal polynomials such that for some associated function $\boldsymbol{m}(t)$ and the corresponding differential operators K^n for all analytic functions $f(t)$

$$f(t) = \sum_{n=0}^{\infty} K^n[f](u) K_u^n[\boldsymbol{m}(t-u)]$$

and when is the convergence uniform?

Examples:
Legendre Polynomials/Spherical Bessel functions

- Let $L_n(\omega)$ be the Legendre polynomials;
- Set $P_n^L(\omega) = \sqrt{2n+1}\, L_n(\omega/\pi)$; then

$$\tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega) P_m^L(\omega) \mathrm{d}\omega = \delta(m-n).$$

- The corresponding recursion coefficients in equation $P_{n+1}^{\mathcal{M}}(\omega) = \frac{1}{\gamma_n}\,\omega\, P_n^{\mathcal{M}}(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\, P_{n-1}^{\mathcal{M}}(\omega)$:

$$\gamma_n = \pi(n+1)/\sqrt{4(n+1)^2-1} \quad \to \pi/2$$

- $\boldsymbol{m}(t) = \operatorname{sinc} t; \qquad \mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{2n+1}\, j_n(\pi t);$
- $f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t)$
- Convergence uniform for functions in $\mathbf{BL}(\pi)$

Examples:
Chebyshev polynomials /Bessel functions

- Let $T_n(\omega)$ be the Chebyshev polynomials of the first kind;
- Set $P_0^T(\omega) = 1$; $P_n^T(\omega) = \sqrt{2}\ T_n(\omega/\pi)$ for $n > 0$; then

$$\int_{-\pi}^{\pi} P_n^T(\omega) P_m^T(\omega) \frac{\mathrm{d}\omega}{\pi^2\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}} = \delta(n-m).$$

- $\gamma_0 = \pi/\sqrt{2}$; $\gamma_n = \pi/2$ for $n > 0$.
- $\boldsymbol{m}(t) = J_0(\pi t)$; $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{2}\, J_n(\pi t)$
- The chromatic expansion is the Neumann series of $f(t)$: $f(t) = f(u) J_0(\pi t) + \sqrt{2}\ \sum_{n=1}^{\infty} \mathcal{K}^n[f](0) J_n(\pi t).$
- Convergence is uniform for functions whose Fourier transform $\widehat{f(\omega)}$ is supported in $[-\pi, \pi]$ and satisfies

$$\int_{-\pi}^{\pi} \sqrt{1-(\omega/\pi)^2}\ |\widehat{f(\omega)}|^2 \mathrm{d}\omega < \infty$$

Examples:
Hermite polynomials/Gaussian monomials

- Let $H_n(\omega)$ be the Hermite polynomials;
- set $P^{\scriptscriptstyle H}_n(\omega) = \sqrt{2^n\, n!}\, H_n(\omega)$; then

$$\int_{-\infty}^{\infty} P^{\scriptscriptstyle H}_n(\omega) P^{\scriptscriptstyle H}_m(\omega)\, \frac{\mathrm{e}^{-\omega^2}}{\sqrt{\pi}}\, \mathrm{d}\omega = \delta(n-m)$$

- $\gamma_n = \sqrt{(n+1)/2}$;
- $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$; $\quad \mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \dfrac{t^n}{\sqrt{2^n\, n!}}\, \mathrm{e}^{-t^2/4}$
- Chromatic expansion converges uniformly for all analytic functions s.t.

$$\int_{-\infty}^{\infty} |\widehat{f(\omega)}|^2\, \mathrm{e}^{\omega^2} \mathrm{d}\omega < \infty$$

thus also for some non-band limited functions.

General families of chromatic derivatives

Let $\mathcal{M} : \mathcal{P}_\omega \to \mathbb{R}$ be a moment functional and $\mu_n = \mathcal{M}(\omega^n)$ be the moment of $\mathcal{M}_\omega$ of order n. We will assume that

- $\mathcal{M}$ is positive definite, i.e., for all n

$$\Delta_n = \begin{vmatrix} \mu_0 & \dots & \mu_n \\ \mu_1 & \dots & \mu_{n+1} \\ \dots & \dots & \dots \\ \mu_{n-1} & \dots & \mu_{2n-1} \\ \mu_n & \dots & \mu_{2n} \end{vmatrix} > 0;$$

- $\mathcal{M}$ is symmetric, i.e. $\mu_{2n+1} = 0$ for all n;
- The moments μ_n satisfy

$$\limsup_{n\to\infty} \left(\frac{\mu_n}{n!}\right)^{1/n} = \mathrm{e} \ \limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} = 0.$$

- For every positive definite moment functional $\mathcal{M}$ there exists a family of orthonormal polynomials $\{P^{\mathcal{M}}_n(\omega)\}_{n\in\mathbb{N}}$

$$\mathcal{M}(P^{\mathcal{M}}_m(\omega)\,P^{\mathcal{M}}_n(\omega)) = \delta(m-n).$$

- A family of polynomials is orthonormal for some symmetric positive definite $\mathcal{M}$ just in case there exists $\gamma_n > 0$ such that

$$P^{\mathcal{M}}_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P^{\mathcal{M}}_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P^{\mathcal{M}}_{n-1}(\omega).$$

- For positive definite symmetric moment functionals $\mathcal{M}$ which satisfy this extra condition we set

$$\mathcal{K}^n = (-j)^n P^{\mathcal{M}}_n\left(j\,\frac{\mathrm{d}}{\mathrm{d}t}\right).$$

then

$$\mathcal{K}^{n+1} = \frac{1}{\gamma_n}\,(\mathrm{d}\circ\mathcal{K}^n) + \frac{\gamma_{n-1}}{\gamma_n}\,\mathcal{K}^{n-1},$$

- $L_2^{\mathcal{M}}$: space of analytic functions s.t. $\sum_{k=0}^{\infty} \mathcal{K}^k[f](0)^2 < \infty$;
- If $f, g \in L_2^{\mathcal{M}}$ then $\sum_{k=0}^{\infty} \mathcal{K}^k[f](t)^2$ and $\sum_{k=0} \mathcal{K}^n[f](t)\mathcal{K}^n[g](t)$ do not depend on t and we can set

$$\|f\|_{\mathcal{M}} = \sum_{k=0}^{\infty} \mathcal{K}^k[f](0)^2 < \infty; \quad \langle f, g\rangle_{\mathcal{M}} = \sum_{k=0}^{\infty} \mathcal{K}^n[f](t)\mathcal{K}^n[g](t)$$

- If we let $\boldsymbol{m}(t) = \sum_{k=0}^{\infty} \mu_{2k} \frac{t^{2k}}{(2k)!}$ then for all $f \in L_2^{\mathcal{M}}$ and all u, t

$$f(t) = \sum_{j=0}^{\infty} (-1)^j \mathcal{K}^j[f](u)\, \mathcal{K}^j[\boldsymbol{m}](t-u).$$

with the series converging uniformly.

- How about local (non-uniform) convergence???

Local convergence of chromatic series

- Bounded families: for some $M \geq 1$ the recursion coefficients γ_n in $P^{\mathcal{M}}_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P^{\mathcal{M}}_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P^{\mathcal{M}}_{n-1}(\omega)$ satisfy:

$$\frac{1}{M} \leq \gamma_n \leq M$$

- Weakly bounded families: for some $M \geq 1$, some integer r and some $0 \leq p < 1$

$$\frac{1}{M} \leq \gamma_n \leq M(n+r)^p$$

$$\frac{\gamma_n}{\gamma_{n+1}} \leq M^2$$

- Bounded families are also weakly bounded with $p = 0$.

- Legendre polynomials: $\gamma_n = \frac{\pi(n+1)}{4(n+1)^2-1}$;
- Chebyshev polynomials: $\gamma_0 = \frac{\pi}{\sqrt{2}}$ and $\gamma_{n+1} = \frac{\pi}{2}$;
- Thus, these two families are bounded ($p = 0$);
- Hermite family: $\gamma_n = \sqrt{(n+1)/2}$;
  thus, the family is only weakly bounded ($p = 1/2$);
- If we want $\boldsymbol{m}(z) = \sum_{k=0}^{\infty} \mu_{2k} t^{2k}/(2k)!$ to be an entire function, then the bound $p < 1$ is sharp.

Lemma: If $\mathcal{M}$ is weakly bounded then there exists $K > 0$, an integer k (which depends on $p < 1$) and a polynomial $P(x)$ such that for every n and every $z \in \mathbb{C}$,

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{|Kz|^n}{n!^{1-p}}\, P(|z|)\, \mathrm{e}^{|Kz|^k}.$$

Theorem: Let $\mathcal{M}$ be weakly bounded and $f(z)$ an entire function. If $\lim_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!^{1-p}}\right|^{1/n} = 0$, then the chromatic expansion of $f(z)$ point-wise converges to $f(z)$ at every $z \in \mathbb{C}$:

$$f(z) = \textstyle\sum_{j=0}^{\infty} (-1)^j \mathcal{K}^j[f](0)\, \mathcal{K}^j[\boldsymbol{m}](z).$$

The convergence is uniform on every disc of finite radius.

Corollary: If $\mathcal{M}$ is bounded then the chromatic expansion of every entire function $f(z)$ point-wise converges to $f(z)$ for all z.

Corollary: Let $P_n^{\mathcal{M}}(\omega)$ be the orthonormal polynomials associated with a weakly bounded moment functional $\mathcal{M}$; then

$$\mathrm{e}^{j\omega t} = \sum_{n=0}^{\infty} j^n P_n^{\mathcal{M}}(\omega)\, \mathcal{K}^n[\boldsymbol{m}](t).$$

is a generalization of the equality for the Chebyshev polynomials /Bessel functions:

$$\mathrm{e}^{j\omega t} = J_0(t) + 2\sum_{n=1}^{\infty} j^n\, T_n(\omega)\, J_n(t)$$

Corollary: The addition formula

$$\boldsymbol{m}(t+u) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[\boldsymbol{m}](u)\mathcal{K}^n[\boldsymbol{m}](t)$$

is a generalization of the addition formula for the Bessel functions

$$J_0(t+u) = J_0(u)J_0(t) + 2\sum_{n=1}^{\infty} (-1)^n J_n(u)J_n(t)$$

Corollary: The equality

$$\boldsymbol{m}(z) + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}} \right) \mathcal{K}^{2n}[\boldsymbol{m}](z) = 1,$$

is a generalization of

$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1.$$

and many more classic formulas for special functions are simply instances of chromatic expansions!

Some non-separable spaces

- Trigonometric functions do not belong to the spaces $L_2^{\mathcal{M}}$:

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[\mathrm{e}^{j\omega t}]|^2 = \sum_{n=0}^{\infty} P_n^{\mathcal{M}}(\omega)^2 \to \infty$$

Definition: Assume $\mathcal{M}$ is weakly bounded. We denote by $\mathcal{C}^{\mathcal{M}}$ the vector space of analytic functions such that the sequence

$$\nu_n^f(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2$$

converges uniformly on every finite interval.

Theorem: Let $f, g \in \mathcal{C}^{\mathcal{M}}$ and

$$\sigma_n^{fg}(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t);$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function.

- Let $\mathcal{C}_0^{\mathcal{M}}$ be the set of all analytic functions such that

$$\lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2 = 0.$$

- We set $\mathcal{C}_2^{\mathcal{M}} = \mathcal{C}^{\mathcal{M}} / \mathcal{C}_0^{\mathcal{M}}$.
- Do the trigonometric functions belong to $\mathcal{C}_2^{\mathcal{M}}$?

$$\frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} |\mathcal{K}^k[\mathrm{e}^{j\omega t}]|^2 = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_n^{\mathcal{M}}(\omega)^2$$

- Chebyshev polynomials: $(p = 0)$ if $0 < \omega < \pi$ then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_n^{T}(\omega)^2 = 1$$

- for all $0 < \sigma, \omega < \pi$, $\sigma \neq \omega$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_k^{T}(\sigma) P_k^{T}(\omega) = 0$$

- Hermite polynomials: $(p = 1/2)$ for all $\omega, \sigma > 0$, $\omega \neq \sigma$,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^{H}(\omega)^2 = \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\omega^2},$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^{H}(\sigma) P_k^{H}(\omega) = 0$$

Thus, in this space every two pure harmonic oscillations with distinct positive frequencies are mutually orthogonal, but their norms (“energy”) depend on their frequency.

Conjecture: Assume that for some $0 \leq p < 1$ the recursion coefficients γ_n satisfy

$$0 < \lim_{n \to \infty} \frac{\gamma_n}{n^p} < \infty.$$

Then for the corresponding family of orthogonal polynomials we have

$$0 < \lim_{n \to \infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_k^{\mathcal{M}}(\omega)^2 < \infty$$

for all ω in the support $sp(a)$ of $a(\omega)$.

Thus, in the corresponding space $\mathcal{C}_2^{\mathcal{M}}$ all pure harmonic oscillations with positive frequencies $\omega \in sp(a)$ have finite positive norm and are mutually orthogonal.

Numerical experiments indicate that this is true...