Chromatic derivatives, chromatic expansions and their applications

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- A. Ignjatovic: *Local Approximations Based on Orthogonal Differential Operators*, **Journal of Fourier Analysis and Applications**, Vol. 13, Issue 3 (2007).
- ——"—— : *Chromatic derivatives and local approximations*, **IEEE Transactions on Signal Processing**, Volume 57, Issue 8 (2009).
- ——"—— : *Chromatic derivatives, chromatic expansions and associated spaces*, **East Journal on Approximations**, Volume 15, Number 3 (2009).
- A. Ignjatovic and Ahmed Zayed: *Multidimensional chromatic derivatives and expansions*, to appear in the **Proceedings of the American Mathematical Society.**

Towards local signal representation

- Let $f \in \mathbf{BL}(\pi)$, i.e., $f \in L^2$ with $\widehat{f(\omega)}$ supported on $[-\pi, \pi]$

Shannon's Expansion:

(Whittaker–Kotelnikov–Nyquist–Shannon)

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \, \frac{\sin \pi(t-n)}{\pi(t-n)}$$

- **global in nature** – requires samples $f(n)$ for all n;
- **fundamental** to signal processing;
- poorly represents local signal behavior

Taylor's Expansion:

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \, \frac{t^n}{n!}$$

- **local in nature** – requires $f^{(n)}(t)$ at a single instant $t = 0$.
- **very little use** in signal processing – **why ?**

Problems with Taylor's expansion of $\mathbf{BL}(\pi)$ signals

1. Numerical evaluation of derivatives of high orders of a (noisy) sampled signal is **unfeasible**.

2. Truncations of Shannon's expansion of an $f \in \mathbf{BL}(\pi)$

   – belong to $\mathbf{BL}(\pi)$

   – converge to f both uniformly and in L^2

   – if A is a filter, then

   $$A[f](t) = \sum_{n=-\infty}^{\infty} f(n)\, A\left[\mathrm{sinc}\,\right](t-n), \qquad (1)$$

   – In comparison, truncations of Taylor's expansion of an $f \in \mathbf{BL}(\pi)$ **have none of these important properties**

Can we fix all of these problems???

Numerical differentiation of band limited signals

Let $f \in \mathbf{BL}(\pi)$; then $\dfrac{f^{(n)}(t)}{\pi^n} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{i}^n \left(\frac{\omega}{\pi}\right)^n \widehat{f(\omega)} e^{\mathrm{i}\,\omega t} d\omega.$

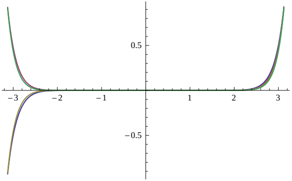


Figure: $(\omega/\pi)^n$ for $n = 15 - 18$

- derivatives of high order **obliterate the spectrum**.
- transfer functions of the (normalized) derivatives **cluster together and are nearly indistinguishable.**
- can we find a better base for the space of linear differential operators? An **orthogonal base**??

Orthogonal base for the space of linear diff. operators

- Start with normalized and re-scaled Legendre polynomials:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P^{L}_{n}(\omega)P^{L}_{m}(\omega)\mathrm{d}\omega = \delta(m-n).$$

- Obtain operator polynomials by replacing ω^k with $\mathrm{i}^k\, d^k/dt^k$:

$$\mathcal{K}^{n}_{t} = (-\mathrm{i})^{n} P^{L}_{n}\left(\mathrm{i}\ \frac{d}{dt}\right)$$

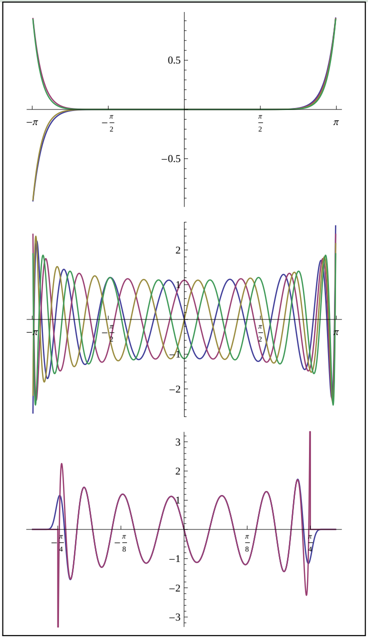
- Definition of $\mathcal{K}^n$ chosen so that

$$\mathcal{K}^{n}_{t}[e^{\mathrm{i}\,\omega t}] = \mathrm{i}^{n} P^{L}_{n}(\omega)\, e^{\mathrm{i}\,\omega t}.$$

- Thus, for $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^{n}[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}^{n} P^{L}_{n}(\omega)\widehat{f(\omega)}\, e^{\mathrm{i}\,\omega t} d\omega.$$

Why are chromatic derivatives a better base?



- Compare the graphs of the transfer functions of $1/\pi^n d^n/dt^n$, i.e., $(\omega/\pi)^n$ (first graph) and of $\mathcal{K}^n$, i.e., $P_n^L(\omega)$ (second graph).

- Transfer functions of $\mathcal{K}^n$ form a sequence of **well separated comb filters** which **preserve spectral features** of the signal, thus we call them the **chromatic derivatives**.

- Third graph: transfer function of the ideal filter $\mathcal{K}^{15}$ (red) vs. transfer function of a transversal filter (blue), (128 taps, 2× oversampling).

Local representation of the scalar product in $\mathbf{BL}(\pi)$

Proposition: Assume that $f, g \in \mathbf{BL}(\pi)$; then the sums on the left hand side of the following equations do not depend on the choice of the instant t, and

$$\sum_{n=0}^{\infty} K^n[f](t)^2 = \int_{-\infty}^{\infty} f(t)^2 dt = \|f\|^2$$

$$\sum_{n=0}^{\infty} K^n[f](t)\overline{K^n[g](t)} = \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt = \langle f, g\rangle$$

$$\sum_{n=0}^{\infty} K^n[f](t)K^n_t[g(u-t)] = \int_{-\infty}^{\infty} f(t)g(u-t) dt = (f * g)(u)$$

- These are the **local equivalents** of the usual, “globally defined” norm, scalar product and convolution!
- **Aim: “maximally localized”** signal processing

Fixing Taylor's Expansion: Chromatic Expansion

Proposition: Let $\mathrm{sinc}\,(t) = \dfrac{\sin(\pi t)}{\pi t}$ and let $f(t)$ be **any analytic function**. Then,

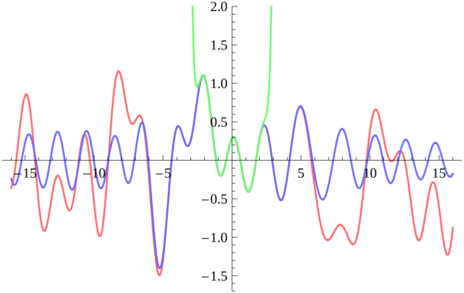
$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[\mathrm{sinc}\,(t)] \\ &= \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t) \end{aligned}$$

j_n – the spherical Bessel functions

solutions of $x^2 y'' + 2xy' + [x^2 - n(n+1)]y = 0$

- **The truncations of the series belong to BL(π).**
- **If $f \in$ BL(π) the series converges to $f(t)$ both uniformly and in L^2.**

Chromatic approximation versus Taylor's approximation



- **red:** the signal;
  blue: the chromatic approximation of order 15;
  green: Taylor's approximation of order 15.

- $f^{(k)}(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\sum_{m=0}^{n}(-1)^m\, \mathcal{K}^m[f](0)\, \mathcal{K}^m[\mathrm{sinc}\,](t)\right]_{t=0}$

- Chromatic approximations are **local approximations**

Chromatic expansion vs. Shannon's expansion

How is Shannon expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \frac{\sin \pi(t-n)}{\pi(t-n)}$$

related to the chromatic expansion

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t)$$

- Transformation $\{f(n)\}_{n\in\mathbb{N}} \Leftrightarrow \{\mathcal{K}^n[f](0)\}_{n\in\mathbb{N}}$ by an unitary operator defined by the infinite matrix

$$\left[\sqrt{2k+1}\, j_k(n\pi) \ : \ k \in \mathbb{N}, n \in \mathbb{Z}\right]:$$

$$\begin{aligned} f(n) &= \sum_{k=0}^{\infty} \mathcal{K}^k[f](0)\, \sqrt{2k+1}\, j_k(n\pi); \\ \mathcal{K}^k[f](0) &= \sum_{n=-\infty}^{\infty} f(n)\, \sqrt{2k+1}\, j_k(n\pi). \end{aligned}$$

- In practice one CANNOT evaluate $\mathcal{K}^k[f](0)$ using Shannon rate samples via

$$\mathcal{K}^k[f](0) \approx \sum_{n=-N}^{N} f(n)\,\sqrt{2k+1}\,j_k(n\pi)$$

because $\sqrt{2n+1}\,j_n(\pi t)$ decay very slowly and we would need huge N.

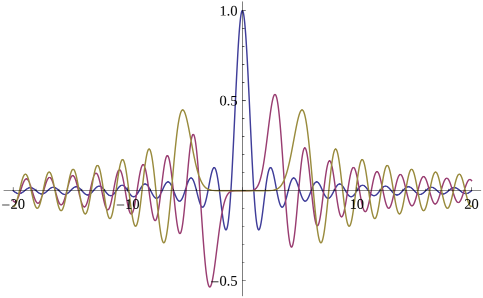
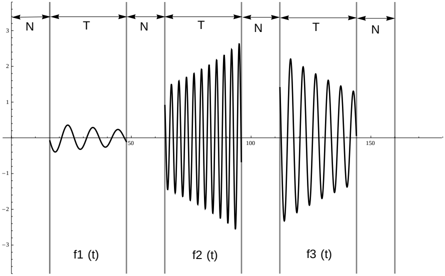


Figure: $n = 0$, $n = 7$, $n = 14$

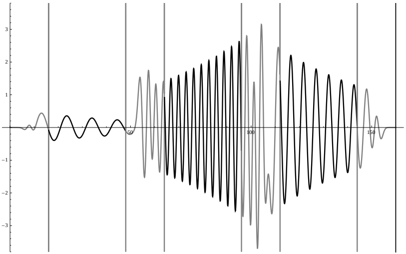
Chromatic derivatives are non-redundant!

- This is a good news because this means that chromatic derivatives are non redundant to the Nyquist rate samples;
- They provide additional “information which:
  - can either be in a more convenient for some signal processing applications,
  - or which can be used **in addition** to the standard Nyquist rate methods making them more powerful!

Application: signal interpolation



Given pieces of band limited signals join them so that the out of band energy is minimal.



We use chromatic expansions to ensure that the resulting signal is N times continuously differentiable Then

$$|\widehat{\phi}(\omega)| \leq \frac{|\widehat{(\phi^{(N)})}(\omega)|}{|\omega|^N} \leq \frac{M}{|\omega|^N}$$

Application: frequency estimation

Idea: *A signal is a sum of at most N shifted and damped sine waves iff it is a solution to a homogeneous linear differential equation with constant coefficients of order at most $2N$.*

A rough sketch of the frequency estimation algorithm:

- Choose the chromatic derivatives which are orthogonal with respect to the power spectrum density of the noise:
  - take polynomials $P_n(\omega)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)S(\omega)d\omega = \delta(m-n)$$

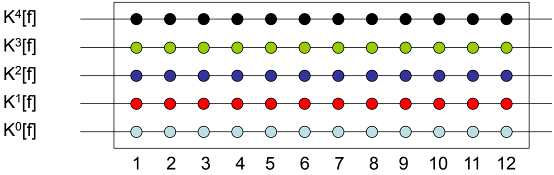
  - Let $\mathcal{K}^n$ be the chromatic derivatives corresponding to the polynomials $P_n(\omega)$, i.e., let

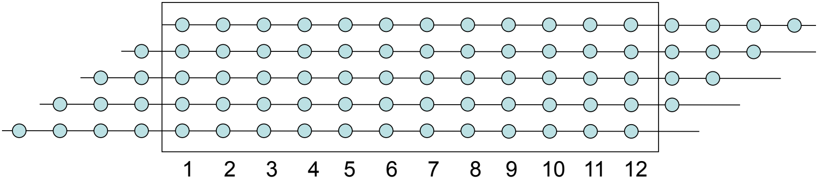
$$\mathcal{K}^n = (-\mathrm{i})^n P_n(-\mathrm{i}\,d/dt).$$

Then, assuming $E[\nu(n)^2] = \rho^2$, we have

$$E\{\mathcal{K}^n[\nu](n)\mathcal{K}^m[\nu](n)\} = \delta(m-n)\rho^2$$

so we can apply the standard *SVD* or *ED* methods.





General families of chromatic derivatives

- Given a family of orthonormal polynomials $P_n(\omega)$ we can always define differential operators

$$\mathcal{K}_t^n = (-\mathrm{i})^n \; P_n^L\left(\mathrm{i}\; \frac{d}{dt}\right)$$

Question:

What are the families of orthogonal polynomials such that for the corresponding differential operators K^n and some associated function $\boldsymbol{m}(t)$ for all analytic functions $f(t)$

$$f(t) = \sum_{n=0}^{\infty} (-1)^n K^n[f](u) K^n[\boldsymbol{m}](t-u)$$

and when is the convergence uniform?

Examples:
Legendre Polynomials/Spherical Bessel functions

- For the (normalized) **Legendre polynomials**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega)P_m^L(\omega)\mathrm{d}\omega = \delta(m-n)$$

and for $\boldsymbol{m}(t) = \frac{\sin(\pi t)}{\pi t}$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2n+1}\, j_n(\pi t)$ and

$$f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\, \sqrt{2n+1}\, j_n(\pi t)$$

holds for all analytic functions;

- The convergence is uniform for functions in $\mathbf{BL}(\pi)$

Examples:
Chebyshev polynomials /Bessel functions

- For the (normalized) **Chebyshev polynomials** of the first kind:

$$\int_{-\pi}^{\pi} \frac{P_n^T(\omega) P_m^T(\omega)}{\pi^2 \sqrt{1 - \left(\frac{\omega}{\pi}\right)^2}} \mathrm{d}\omega = \delta(n-m).$$

for $\boldsymbol{m}(t) = J_0(\pi t)$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \sqrt{2}\, J_n(\pi t)$ and

$$f(t) = f(u) J_0(\pi t) + \sqrt{2} \sum_{n=1}^{\infty} \mathcal{K}^n[f](0) J_n(\pi t)$$

- the Neumann series - converges for all analytic functions;

- Convergence uniform for band limited functions which satisfy

$$\int_{-\pi}^{\pi} \sqrt{1 - (\omega/\pi)^2}\ |\widehat{f}(\omega)|^2 \mathrm{d}\omega < \infty$$

Examples:
Hermite polynomials/Gaussian monomials

- For the (normalized) **Hermite polynomials**

$$\int_{-\infty}^{\infty} P_n^{H}(\omega) P_m^{H}(\omega)\, \frac{\mathrm{e}^{-\omega^2}}{\sqrt{\pi}}\, \mathrm{d}\omega = \delta(n-m)$$

and $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ we have $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \dfrac{t^n}{\sqrt{2^n\, n!}}\, \mathrm{e}^{-t^2/4}$

- chromatic expansion converges for analytic functions s.t.

$$\limsup_{n\to\infty} \frac{|f^{(n)}(z)|^{1/n}}{\sqrt{n}} < \infty$$

- converges uniformly for all analytic functions s.t.

$$\int_{-\infty}^{\infty} |\widehat{f(\omega)}|^2\, \mathrm{e}^{\omega^2} \mathrm{d}\omega < \infty$$

Examples: the hyperbolic family

If $L_n(\omega)$ satisfy

$$\frac{1}{2}\int_{-\infty}^{\infty} L_n(\omega)\; L_m(\omega)\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\mathrm{d}\omega = \delta(m-n)$$

and $\boldsymbol{m}(z) = \mathrm{sech}(z)$ then $\mathcal{K}^n[\boldsymbol{m}](z) = (-1)^n\,\mathrm{sech}(z)\mathrm{tanh}^n(z)$
and

$$f(z) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\,\mathrm{sech}(z)\mathrm{tanh}^n(z)$$

converges uniformly on **the disc** $|z| < \pi/2$ for functions analytic on this disc, whose Fourier transform satisfies

$$\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\cosh(\omega)\mathrm{d}\omega < \infty$$

General families of chromatic derivatives

Definition: A family of polynomials $P_n(\omega)$ which is orthonormal with respect to a non-decreasing bounded **moment distribution function** $a(\omega)$:

$$\int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) \mathrm{d} a(\omega)$$

is **chromatic** if the moments μ_n of $a(\omega)$,

$$\mu_n = \int_{-\infty}^{\infty} \omega^n \mathrm{d}\omega$$

satisfy

$$\rho = \limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} < \infty$$

Lemma: $P_n(\omega)$ are chromatic if and only if for every $0 \leq \alpha < \rho$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|} \mathrm{d} a(\omega) < \infty$$

General families of chromatic derivatives

Theorem: Let $P_n(\omega)$ be a chromatic family of polynomials orthonormal with respect to $a(\omega)$, and let

$$\boldsymbol{m}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega)$$

Then $\boldsymbol{m}(z)$ is analytic on the strip $S_{\rho/2} = \{z : \mathrm{Im(z)} < \rho/2\}$.

Definition: Λ^2 is the space of functions $f(t)$ analytic on $S_{\rho/2}$ such that for the chromatic derivatives $\mathcal{K}^n$ which correspond to $P_n(\omega)$ we have

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty.$$

Definition: $L^2_{a(\omega)}$ is the space of functions $\phi(\omega)$ satisfying

$$\int_{-\infty}^{\infty} |\phi(\omega)|^2 \mathrm{d}a(\omega) < \infty.$$

General families of chromatic derivatives

Theorem: If $P_n(\omega)$ are a chromatic family of polynomials orthonormal with respect to a measure $a(\omega)$, Then they are a complete base of the space $L^2_{a(\omega)}$.

Theorem: A function $f(z)$ is in Λ^2 if and only if there exists a function $\phi_f(\omega)$ such that

$$f(z) = \int_{-\infty}^{\infty} \phi_f(\omega) \mathrm{e}^{\mathrm{i}\,\omega z} \mathrm{d}a(\omega)$$

in which case

$$\phi_f(\omega) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0) P_n(\omega)$$

Corollary: Thus, for all $t \in \mathbb{R}$,

$$\|f\|^2_{\Lambda} = \|\phi_f(\omega)\|^2_{a(\omega)} = \sum_{n=0}^{\infty} |\mathcal{K}^n[\boldsymbol{m}](0)|^2 = \sum_{n=0}^{\infty} |\mathcal{K}^n[\boldsymbol{m}](t)|^2$$

General families of chromatic derivatives

Theorem: If $f(z) \in \Lambda^2$, then

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\mathcal{K}^n[\boldsymbol{m}](t)$$

with the series converging uniformly on strips $S_{\rho/2-\epsilon}$.

How about the **local (non-uniform) convergence** of the chromatic series??

For example, in the case of the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions of the first kind $J_n(\omega)$, we know that the chromatic series is just the Newmann series, and that the above equality holds **for every analytic function** $f(z)$!

Weakly bounded families

Theorem: A family of polynomials is orthonormal with respect to a moment distribution function $a(\omega)$ with all odd moments $\mu_{2n+1} = 0$ if and only if there exist $\gamma_n > 0$ such that

$$P_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}(\omega).$$

Definition: Such family of polynomials $P_n(\omega)$ is:

1. **bounded** if for some M and all n we have $\frac{1}{M} \leq \gamma_n \leq M$.
2. **weakly bounded** if for some $0 \leq p < 1$ we have

$$\frac{1}{M} < \gamma_n < M\,n^p \quad \text{and} \quad \frac{\gamma_n}{\gamma_{n+1}} < M$$

- Bounded families are also weakly bounded with $p = 0$.

Examples:

- **Bounded families** $(p = 0)$:
  - **Legendre** polynomials: $\gamma_n = \frac{\pi(n+1)}{\sqrt{4(n+1)^2-1}} \to \frac{\pi}{2}$
  - **Chebyshev** polynomials: $\gamma_0 = \frac{\pi}{\sqrt{2}}$ and $\gamma_{n+1} = \frac{\pi}{2}$
- **Weakly bounded family** $(p = 1/2)$:
  - **Hermite** polynomials: $\gamma_n = \sqrt{(n+1)/2}$;
- **Non - weakly bounded family** $(p = 1)$:
  - **Hyperbolic** family: $\gamma_n = n + 1$;
- This shows that if we want $\boldsymbol{m}(z)$ to be entire, then the bound $p < 1$ is sharp.

Lemma: Every weakly bounded family of orthonormal polynomials is also chromatic.

Theorem: Let $\{P_n(\omega)\}_{n\in\mathbb{N}}$ be a weakly bounded family and let $f(z)$ be an entire function. If

$$\lim_{n\to\infty}\left|\frac{f^{(n)}(0)}{n!^{1-p}}\right|^{1/n}=0$$

then for every $z\in\mathbb{C}$

$$f(z)=\sum_{j=0}^{\infty}(-1)^j\mathcal{K}^j[f](0)\,\mathcal{K}^j[\boldsymbol{m}](z).$$

The convergence is uniform on every disc of finite radius.

Corollary: If $\mathcal{M}$ is bounded then the chromatic expansion of every entire function $f(z)$ point-wise converges to $f(z)$ for all z.

- It turns out that many of the classical formulas such as

$$\mathrm{e}^{\mathrm{i}\,\omega t} = \textstyle\sum_{n=0}^{\infty} \mathrm{i}^n\, T_n(\omega) J_n(t)$$

$$J_0(t+u) = J_0(u) J_0(t) + 2 \textstyle\sum_{n=1}^{\infty} (-1)^n J_n(u) J_n(t)$$

$$J_0(t)^2 + 2 \textstyle\sum_{k=1}^{\infty} J_n(t)^2 = 1$$

$$J_0(z) + 2 \textstyle\sum_{n=1}^{\infty} J_{2n}(z) = 1$$

are special cases of chromatic expansions valid **for all weakly bounded families** of polynomials and their associated $\boldsymbol{m}(z)$:

$$\mathrm{e}^{\mathrm{i}\,\omega t} = \textstyle\sum_{n=0}^{\infty} \mathrm{i}^n\, P_n(\omega) \mathcal{K}^n[\boldsymbol{m}](t)$$

$$\boldsymbol{m}(t+u) = \textstyle\sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[\boldsymbol{m}](u) \mathcal{K}^n[\boldsymbol{m}](t)$$

$$\textstyle\sum_{k=1}^{\infty} \mathcal{K}^n[\boldsymbol{m}](t)^2 = 1$$

$$\boldsymbol{m}(z) + \textstyle\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}} \right) \mathcal{K}^{2n}[\boldsymbol{m}](z) = 1$$

Trigonometric functions

- Trigonometric functions do not belong to the spaces Λ_2:

$$\|\mathrm{e}^{\mathrm{i}\omega t}\|_\Lambda^2 = \sum_{n=0}^{\infty} |\mathcal{K}^n[\mathrm{e}^{\mathrm{i}\omega t}]|^2 = \sum_{n=0}^{\infty} P_n(\omega)^2 \to \infty$$

Definition: Assume $\mathcal{M}$ is weakly bounded. We denote by $\mathcal{C}$ the vector space of analytic functions such that the sequence

$$\nu_n^f(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2$$

converges uniformly on every finite interval.

Definition: Let $\mathcal{C}_0 \subset \mathcal{C}$ consists of $f(t)$ such that

$$\lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2 = 0.$$

We define $\mathcal{C}_2 = \mathcal{C}/\mathcal{C}_0$.

Theorem: Let $f, g \in \mathcal{C}$ and

$$\sigma_n^{fg}(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t);$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function.

Definition: For $f, g \in \mathcal{C}$ we define

$$\langle f, g \rangle = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)$$

- Do the trigonometric functions belong to $\mathcal{C}_2$?

$$\frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} |\mathcal{K}^k[\mathrm{e}^{\mathrm{i}\omega t}]|^2 = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_n(\omega)^2$$

- Chebyshev polynomials: $(p=0)$ if $0<\omega<\pi$ then

$$\|\mathrm{e}^{\mathrm{i}\omega t}\| = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_n^T(\omega)^2 = 1$$

- for all $0<\sigma, \omega<\pi$, $\sigma \neq \omega$

$$\langle \mathrm{e}^{\mathrm{i}\sigma t}, \mathrm{e}^{\mathrm{i}\omega t} \rangle = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_k^T(\sigma) P_k^T(\omega) = 0$$

- Hermite polynomials: $(p=1/2)$ for all $\omega, \sigma > 0$, $\omega \neq \sigma$,

$$\|\mathrm{e}^{\mathrm{i}\omega t}\| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^H(\omega)^2 = \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\omega^2},$$

$$\langle \mathrm{e}^{\mathrm{i}\sigma t}, \mathrm{e}^{\mathrm{i}\omega t} \rangle = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} P_k^H(\sigma) P_k^H(\omega) = 0$$

Thus, in this space every two pure harmonic oscillations with distinct positive frequencies are mutually orthogonal!

Conjecture: Assume that for some $0 \leq p < 1$ the recursion coefficients γ_n satisfy

$$0 < \lim_{n \to \infty} \frac{\gamma_n}{n^p} < \infty.$$

Then for the corresponding family of orthogonal polynomials we have

$$0 < \lim_{n \to \infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} P_k(\omega)^2 < \infty$$

for all ω in the support $sp(a)$ of $a(\omega)$.

Numerical experiments indicate that this is true...

It turns out that the special case with $p = 0$ is a previously known, still open problem.