Chromatic derivatives and local approximations

Aleksandar Ignjatović

***Abstract*—We present a detailed motivation for the notions of chromatic derivatives and chromatic expansions. Chromatic derivatives are special, numerically robust linear differential operators; chromatic expansions are the associated local expansions, which possess the best features of both the Taylor and the Nyquist expansions. We give a simplified treatment of some of the basic properties of chromatic derivatives and chromatic expansions which are relevant for applications. We also consider some signal spaces with a scalar product defined by a Cesàro–type sum of products of chromatic derivatives, as well as an approximation of such a scalar product which is relevant for signal processing. We also introduce a new kind of local approximations based on trigonometric functions.**

***Index Terms*—chromatic derivatives, chromatic expansions, local signal representation, generalized Fourier series, orthogonal systems**

I. Introduction

The special case of the chromatic derivatives presented in Example 1 below was introduced in [1]; chromatic expansions were subsequently introduced in [2]. The theory emerged in the course of the author's design of a pulse-width modulation switching power amplifier. Subsequently, the author applied similar reasoning to propose a communication channel equalizer and a modulation method also based on chromatic derivatives and chromatic expansions. The theory was generalized and extended to various systems corresponding to several classical families of orthogonal polynomials by the research team of the author's startup, *Kromos Technology Inc*. [3], [4]. The results were initially published as technical reports on the company web site.[1] Kromos' team designed and implemented a channel equalizer [5] and a digital transceiver (unpublished) based on chromatic expansions. It also developed a novel image compression method motivated by chromatic expansions [6], [7]. In [8] chromatic expansions were related to the work of Papoulis [9] and Vaidyanathan [10]. In [11] and [12] the theory was cast in the framework commonly used in signal processing. Chromatic expansions were also studied in [3], [13] and [14]. Pointwise convergence of chromatic expansions was studied in [15]. A generalization of chromatic derivatives, with the prolate spheroidal wave functions replacing orthogonal polynomials, was introduced in [16]; the theory was also extended to prolate spheroidal wavelet series that combine chromatic series with sampling series.

The present paper is structured as follows. To give an appropriate motivation, section I-A presents detailed heuristics underpinning the notions of chromatic derivatives and chromatic expansions. Section II presents a simplified treatment of some of the material from [15] which is relevant for applications. The presentation in this paper centers around the Fourier transform, and such an approach should be more familiar to engineers than the one taken in [15]. In section III we discuss some non-separable inner product spaces; in one example of such spaces, associated with the Hermite polynomials, all pairs of sine waves with different positive frequencies are orthogonal. We also discuss an approximation of the norm in such spaces, which can have practical applications, such as estimation of local energy of a signal. Finally, in section IV we introduce a new type of local approximation using the trigonometric functions.

A. Motivation

Let $\mathbf{BL}(\sigma)$ denote the space of σ-band limited signals of finite energy, i.e., the space of continuous L^2 functions whose Fourier transform is supported within $[-\sigma,\sigma]$. The expansion of a signal $f \in \mathbf{BL}(\pi)$ given by the Whittaker–Kotelnikov–Nyquist–Shannon Sampling Theorem (for brevity the Nyquist Theorem),

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,(t-n), \qquad (1)$$

is global in nature, because it requires samples of the signal at integers of arbitrarily large absolute value.[2] As is well known, the truncations of such expansion are not satisfactory local approximations of a signal, because, for a fixed t, the values of $\mathrm{sinc}\,(t-n)$ decay slowly as $|n|$ grows.

On the other hand, signals in $\mathbf{BL}(\pi)$ are analytic functions and their representation by the Taylor formula,

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{t^n}{n!},$$

is local in nature, because the values of the derivatives $f^{(n)}(0)$ are determined by the values of the signal in an arbitrarily small neighborhood of zero.

In sharp contrast with the Nyquist expansion, which has a central role in digital signal processing, the Taylor expansion has found very limited use there, due to several problems associated with its application to sampled signals.

Aleksandar Ignjatović is with the School of Computer Science and Engineering, University of New South Wales, and with the National ICT Australia (NICTA), Sydney, Australia; E-mail: ignjat@cse.unsw.edu.au; http://www.cse.unsw.edu.au/~ignjat

[1] These technical reports, as well as some additional unpublished manuscripts and preprints, can be found at author's web site `http://www.cse.unsw.edu.au/~ignjat/diff/`

[2] Here $\mathrm{sinc}\,t$ denotes the (normalized) cardinal sine function, $\mathrm{sinc}\,t = \sin(\pi t)/\pi t$.

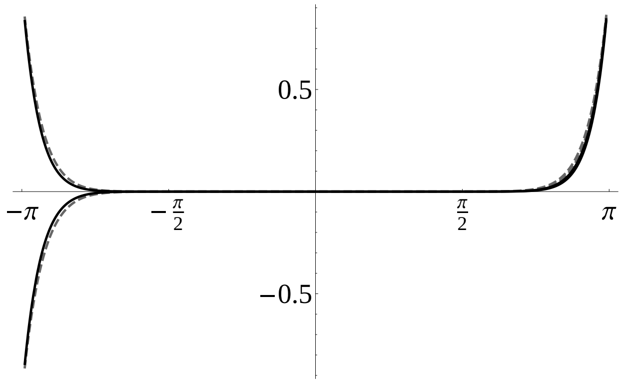


Fig. 1. Graphs of $(\omega/\pi)^n$ for $n = 15$ to $n = 18$.

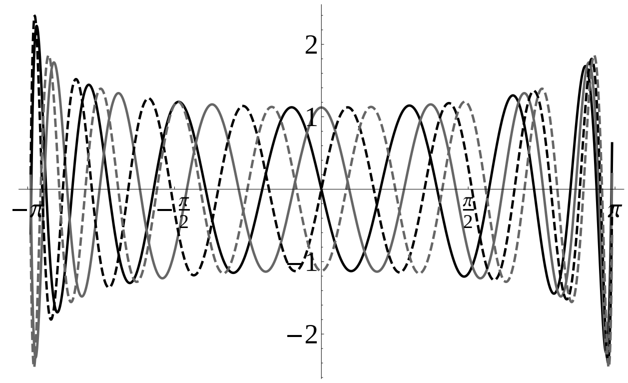


Fig. 2. Graphs of $P^L_n(\omega)$ for $n = 15$ to $n = 18$.

1) Numerical evaluation of higher order derivatives of a signal from its samples is very noise sensitive. In general, one is cautioned against numerical differentiation:

   "…numerical differentiation should be avoided whenever possible, particularly when the data are empirical and subject to appreciable errors of observation"[17].

2) The Taylor expansion of a signal $f \in \mathbf{BL}(\pi)$ converges non-uniformly on $\mathbb{R}$; its truncations have rapid error accumulation and are unbounded.
3) The Nyquist expansion of a signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$ and thus the action of a filter (a continuous linear shift invariant operator) on any $f \in \mathbf{BL}(\pi)$ can be expressed using samples of f and the *impulse response* $A[\mathrm{sinc}]$ of A, i.e.,

$$A[f](t) = \sum_{n=-\infty}^{\infty} f(n)\; A[\mathrm{sinc}](t-n). \tag{2}$$

   In contrast, the polynomials obtained by truncating the Taylor series do not belong to $\mathbf{BL}(\pi)$ and nothing similar to (2) is true of the Taylor expansion.

The chromatic derivatives and the chromatic approximations were introduced to obtain local approximations of band-limited signals which do not suffer from any of the above problems. To motivate these notions, we first examine the problem of numerical differentiation of band limited signals.

Assume that $f \in \mathbf{BL}(\pi)$ and let $\widehat{f(\omega)}$ be its Fourier transform; then $f^{(n)}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(j\,\omega)^n \widehat{f(\omega)} \mathrm{e}^{j\,\omega t} d\omega$. Figure 1 shows, for $n = 15$ to $n = 18$, the plots of $(\omega/\pi)^n$, which are the transfer functions (the symbols) of the normalized derivatives $1/\pi^n\ \mathrm{d}^n/\mathrm{d}t^n$ (modulo a factor of j^n). These plots reveal why numerical evaluation of higher order derivatives from signal samples makes no practical sense. Multiplication of the Fourier transform of a signal by the transfer function of a derivative of higher order essentially obliterates the spectrum of the signal, leaving only its edges, which in practice contain mostly noise. Note also that the graphs of the transfer functions of the normalized derivatives of high orders and the same parity cluster so tightly together that they are essentially indistinguishable; see Figure 1.

This, however, *does not* imply that numerical evaluation of all differential operators of high order is unfeasible, but rather, that from a numerical perspective, the set of the derivatives $\{f, f', f'', \ldots\}$ is a very poor base of the vector space of linear differential operators with real coefficients. It is thus natural to look for a different base for this vector space which does not have such shortcomings.

To obtain such a base we normalize and rescale the Legendre polynomials so that the resulting polynomials $P^L_n(\omega)$ satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P^L_n(\omega)\; P^L_m(\omega)\mathrm{d}\omega = \delta(m-n),$$

and then define operator polynomials[3]

$$\mathcal{K}^n_t = (-j)^n P^L_n\left(j\ \frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{3}$$

Since polynomials $P^L_n(\omega)$ contain only powers of the same parity as n, operators $\mathcal{K}^n$ have real coefficients, and it is easy to verify that

$$\mathcal{K}^n_t[\mathrm{e}^{j\,\omega t}] = j^n P^L_n(\omega)\,\mathrm{e}^{j\,\omega t}. \tag{4}$$

Consequently, for $f \in \mathbf{BL}(\pi)$,

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n P^L_n(\omega)\widehat{f(\omega)}\,\mathrm{e}^{j\,\omega t}\mathrm{d}\omega.$$

Figure 2 shows the plots of $P^L_n(\omega)$, for $n = 15$ to $n = 18$, which are the transfer functions (again save a factor of j^n) of the operators $\mathcal{K}^n$. Unlike the transfer functions of the (normalized) derivatives $1/\pi^n\ \mathrm{d}^n/\mathrm{d}t^n$, the transfer functions of the chromatic derivatives $\mathcal{K}^n$ form a family of well separated, interleaved and increasingly refined comb filters. Instead of obliterating, such operators encode the spectral features of the signal. For this reason, we call operators $\mathcal{K}^n$ *the chromatic derivatives associated with the Legendre polynomials*.

Chromatic derivatives can be accurately and robustly evaluated from samples of the signal taken at a moderate multiple (two to four) of the usual Nyquist rate. Figure 3 shows the plots of the transfer function of a transversal filter $A[f](t) = \sum_{i=-128}^{128} c_i\, f(t-i/4)$ which approximates the chromatic

[3]Thus, $\mathcal{K}^n_t$ is obtained from $P^L_n(\omega)$ by replacing ω^k with $j^k\ \mathrm{d}^k/\mathrm{d}t^k$ for $k \leq n$. If $\mathcal{K}^n_t$ is applied to a function of a single variable, we drop index t in $\mathcal{K}^n_t$.

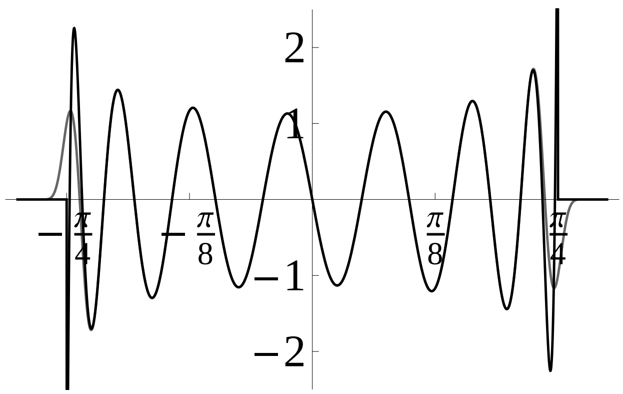


Fig. 3. The transfer functions of the operator $\mathcal{K}^{15}$ (black) and of its transversal filter approximation (gray).

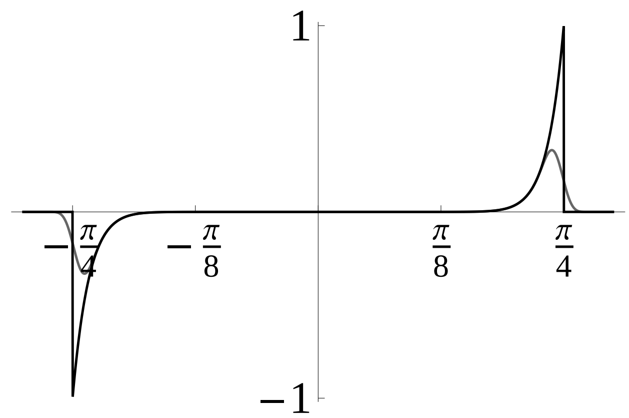


Fig. 4. The transfer functions of $(4/\pi)^{15}\ \mathrm{d}^{15}/\mathrm{d}t^{15}$ (black) and of its transversal filter approximation (gray).

derivative $\mathcal{K}^{15}$ (gray), and the transfer function of $\mathcal{K}^{15}$ (black). The filter was designed using the Remez exchange method [18], and has 257 taps, spaced four taps per Nyquist rate interval. Thus, the transfer function of the corresponding ideal filter $\mathcal{K}^{15}$ is $P^{L}_{15}(4\omega)$ for $|\omega| \leq \pi/4$, and zero outside this interval. The pass-band of the filter is 90% of the interval $[-\pi/4, \pi/4]$, and the transition region extends 10% of the bandwidth $\pi/4$ on each side of the boundaries $-\pi/4$ and $\pi/4$. The coefficients c_i of the filter satisfy $|c_i| < 0.1$; outside the transition region the error of approximation is less than 10^{-4}. Implementations of filters for operators $\mathcal{K}^n$ of orders $0 \leq n \leq 24$ have been tested in practice and proved to be both accurate and noise robust, as expected from the above considerations.

For comparison, Figure 4 shows the transfer function of a transversal filter approximating the (normalized) "standard" derivative $(4/\pi)^{15}\ \mathrm{d}^{15}/\mathrm{d}t^{15}$ restricted to the same bandwidth $[-\pi/4, \pi/4]$ (gray) and the transfer function of this ideal filter (black). It is clear from Figure 4 (right) that such a filter is of no practical use.

Thus, numerical evaluation of the chromatic derivatives associated with the Legendre polynomials does not suffer Problem 1 mentioned above, which precludes numerical evaluation of the "standard" derivatives of higher orders. On the other hand, the chromatic expansions, defined in Proposition 1.1 below, were conceived as a solution to Problems 2 and 3 above, associated with the use of the Taylor expansion.

Proposition 1.1: Let $\mathcal{K}^n$ be the chromatic derivatives associated with the Legendre polynomials, and let $f(t)$ be any function analytic on $\mathbb{R}$; then for all $t \in \mathbb{R}$,

$$f(t) = \sum_{n=0}^{\infty} K^n[f](u)\ K^n_u[\mathrm{sinc}\,(t-u)] \tag{5}$$

$$= \sum_{n=0}^{\infty} (-1)^n\ K^n[f](u)\ K^n[\mathrm{sinc}\,](t-u). \tag{6}$$

If $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on $\mathbb{R}$ and in the space $\mathbf{BL}(\pi)$.

The series in (5) is called *the chromatic expansion of f associated with the Legendre polynomials*; a truncation of this series is called a *chromatic approximation* of f. Just like a Taylor approximation, a chromatic approximation is also a local approximation: its coefficients are the values of differential operators $\mathcal{K}^m[f](u)$ at a single instant u, and for all $k \leq n$,

$$f^{(k)}(u) = \frac{\mathrm{d}^k}{\mathrm{d}t^n}\left[\sum_{m=0}^{n}(-1)^m\ \mathcal{K}^m[f](u)\mathcal{K}^m[\mathrm{sinc}\,](t-u)\right]_{t=u}.$$

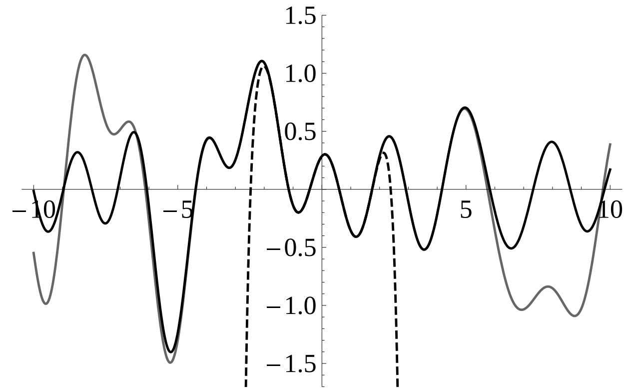


Fig. 5. A signal $f \in \mathbf{BL}(\pi)$ (gray) and its chromatic and Taylor approximations (black, dashed).

Figure 5 compares the behavior of the chromatic approximation (black) of a signal $f \in \mathbf{BL}(\pi)$ (gray) with the behavior of the Taylor approximation of $f(t)$ (dashed). Both approximations are of order sixteen, and the signal $f(t)$ is defined using the Nyquist expansion, with randomly generated samples $\{f(n)\ :\ |f(n)| < 1,\ -32 \leq n \leq 32\}$.

By Proposition 1.1, the chromatic expansion of such a signal converges uniformly on $\mathbb{R}$, and the plot reveals that, when approximating a signal $f \in \mathbf{BL}(\pi)$, a chromatic approximation has a much gentler error accumulation when moving away from the point of expansion than the Taylor approximation of the same order.

Functions $\mathcal{K}^n[\mathrm{sinc}](t)$ in the chromatic expansion associated with the Legendre polynomials are given by $\mathcal{K}^n[\mathrm{sinc}](t) = (-1)^n\sqrt{2n+1}\ \ j_n(\pi t)$, where j_n is the spherical Bessel function of the first kind of order n. Unlike the monomials that appear in the Taylor formula, functions $\mathcal{K}^n[\mathrm{sinc}](t)$ belong to $\mathbf{BL}(\pi)$ and satisfy $|\mathcal{K}^n[\mathrm{sinc}](t)| \leq 1$ for all $t \in \mathbb{R}$. Consequently, the chromatic approximations are bounded on $\mathbb{R}$ and belong to $\mathbf{BL}(\pi)$.

Since by Proposition 1.1 the chromatic approximation of a signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$, if A is a filter, then A commutes with the differential operators $\mathcal{K}^n$ and thus for every $f \in \mathbf{BL}(\pi)$,

$$A[f](t) = \sum_{n=0}^{\infty}(-1)^n\ \mathcal{K}^n[f](0)\ \mathcal{K}^n[A[\,\mathrm{sinc}]](t); \qquad (7)$$

compare (7) with (2).

The above considerations show that, while local just like the Taylor expansion, the chromatic expansion associated with the Legendre polynomials possesses the features that make the Nyquist expansion so useful in signal processing. This, together with numerical robustness of chromatic derivatives, makes chromatic approximations applicable in fields involving empirically sampled data, such as digital signal and image processing.

Proposition 1.2 below demonstrates another remarkable property of chromatic derivatives which is relevant to signal processing.

Proposition 1.2: Let $\mathcal{K}^n$ be the chromatic derivatives associated with the (rescaled and normalized) Legendre polynomials, and $f, g \in \mathbf{BL}(\pi)$. Then for all $t \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} K^n[f](t)^2 \ = \ \int_{-\infty}^{\infty} f(x)^2 dx; \qquad (8)$$

$$\sum_{n=0}^{\infty} K^n[f](t) K^n[g](t) \ = \ \int_{-\infty}^{\infty} f(x) g(x) dx; \qquad (9)$$

$$\sum_{n=0}^{\infty} K^n[f](t) K^n_t[g(u-t)] \ = \ \int_{-\infty}^{\infty} f(x) g(u-x) dx. (10)$$

Thus, the sums on the left hand side of the above equations do not depend on the choice of the instant t. Moreover, if a function f is analytic on $\mathbb{R}$ and $\sum_{n=0}^{\infty} K^n[f](0)^2 < \infty$, then such f must belong to $\mathbf{BL}(\pi)$.

Note that (8), (9) and (10) provide local representations of the usual norm, the scalar product and the convolution, respectively, which are defined in $\mathbf{BL}(\pi)$ globally, as improper integrals.

Given these properties of the Legendre polynomials, it is natural to ask the following question.

Question 1: What are the families of orthonormal polynomials for which there exists an associated function $\boldsymbol{m}(t)$ such that for every analytic function $f(t)$ the equality

$$f(t) = \sum_{n=0}^{\infty}(-1)^n K^n[f](u)\ K^n[\mathbf{m}](t-u) \qquad (11)$$

holds for all $t \in \mathbb{R}$? Given such a family, for which functions is the convergence uniform?

This question was answered in [15]; in this paper we present a simplification of these results.

We wish to make two observations which point to a close relationship between the Nyquist expansion and the chromatic expansion associated with the Legendre polynomials, and ask a related question.

Firstly, by (6),

$$f(n) = \sum_{k=0}^{\infty}\ \mathcal{K}^k[f](0)\ (-1)^k \mathcal{K}^k[\mathrm{sinc}](n). \qquad (12)$$

Since the series (1) converges uniformly, and since $\mathcal{K}^n[\mathrm{sinc}](t)$ is an even function for even n and an odd function for odd n,

$$\mathcal{K}^k[f](0) = \sum_{n=-\infty}^{\infty}\ f(n)\ (-1)^k K^k[\mathrm{sinc}](n). \qquad (13)$$

Equations (12) and (13) show that the coefficients of the Nyquist expansion of a signal — the samples $f(n)$, and the coefficients of the chromatic expansion of the signal — the simultaneous samples of the chromatic derivatives $\mathcal{K}^n[f](0)$, are related by an orthonormal operator defined by the infinite matrix

$$\left[(-1)^k \mathcal{K}^k\,[\mathrm{sinc}\,]\,(n)\ \ :\ \ k \in \mathbb{N}, n \in \mathbb{Z}\right].$$

Secondly, let $\mathcal{S}_u[f(u)] = f(u+1)$ be the unit shift operator in the variable u (f might have other parameters). The Nyquist expansion for the set of sampling points $\{u+n\ \ :\ \ n \in \mathbb{Z}\}$ can be written in a form entirely analogous to the chromatic expansion, using operator polynomials $\mathcal{S}_u^n = \mathcal{S}_u \circ \ldots \circ \mathcal{S}_u$, as

$$f(t) \ = \ \sum_{n=0}^{\infty} f(u+n)\,\mathrm{sinc}\,(t-(u+n)) \qquad (14)$$

$$\ = \ \sum_{n=0}^{\infty} \mathcal{S}_u^n[f](u)\ \mathcal{S}_u^n[\mathrm{sinc}\,(t-u)]; \qquad (15)$$

compare now (15) with (5). Note that the family of operator polynomials $\{\mathcal{S}_u^n\}_{n\in\mathbb{Z}}$ is also an orthonormal system, in the sense that their corresponding transfer functions $\{\mathrm{e}^{j\,n\,\omega}\}_{n\in\mathbb{Z}}$ form an orthonormal system in $L^2[-\pi,\pi]$. Moreover, the transfer functions of the families of operators $\{\mathcal{K}^n\}_{n\in\mathbb{N}}$ and $\{\mathcal{S}^n\}_{n\in\mathbb{Z}}$, where $\mathcal{K}^n$ are the chromatic derivatives associated with the Legendre polynomials, are orthogonal on $[-\pi,\pi]$ with respect to *the same*, constant weight $\mathrm{w}(\omega) = 1/(2\pi)$.

Finally, having in mind the form of expansions (5) and (15), one can ask a more general (and somewhat vague) question.

Question 2: What are the operators A for which there exists a family of operator polynomials $\{P_n(A)\}$, orthogonal under a suitably defined notion of orthogonality, such that for an associated function $\mathbf{m}_A(t)$,

$$f(t) = \sum_n P_n(A)[f](u)\ P_n(A)[\mathbf{m}_A(t-u)]$$

for all functions from a corresponding (and significant) class $\mathcal{C}_A$?

II. General families of chromatic derivatives

We do not present here the theory in its full generality, but rather, in a form sufficient for signal processing applications; see [19] for a more general treatment. The results were originally proven in [15]; here we present significantly simplified proofs based on the properties of the Fourier transform.

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of positive reals, $\gamma_{-1} = 1$, and let the family of polynomials $\{P_n(\omega)\}_{n\in\mathbb{N}}$ be defined by recursion so that $P_{-1}(\omega) = 0$, $P_0(\omega) = 1$, and for all $n \in \mathbb{N}$,

$$P_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\,P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}(\omega). \tag{16}$$

Then there exists a linear mapping $\mathcal{M}$ from the vector space $\mathcal{P}_\omega$ of real polynomials in the variable ω into $\mathbb{R}$, such that $\mathcal{M}(P_m(\omega)\,P_n(\omega)) = \delta(m-n)$ for all m, n, [20]. Such linear mapping $\mathcal{M}$ is a *positive definite moment functional* and the value $\mu_n = \mathcal{M}(\omega^n)$ is called the *moment of order* n. It can be shown that (16) implies that $\mu_{2k+1} = 0$ for all k; such moment functionals are called *symmetric*. Since $P_0(\omega) = 1$, we have $\mu_0 = \mathcal{M}(P_0(\omega)) = \mathcal{M}(P_0(\omega)^2) = 1$. The family $\{P_n(\omega)\}_{n\in\mathbb{N}}$ is the family of *orthonormal polynomials* that correspond to the moment functional $\mathcal{M}$. Note that (16) also implies that $P_n(\omega)$ has only powers of ω of the same parity as n. Thus, if we define linear differential operators by the operator polynomials

$$\mathcal{K}^n = (-j)^n P_n\left(j\,\frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{17}$$

then these operators have real coefficients and they can be shown to satisfy the three term recurrence relation

$$\mathcal{K}^{n+1} = \frac{1}{\gamma_n}\,(\mathrm{d} \circ \mathcal{K}^n) + \frac{\gamma_{n-1}}{\gamma_n}\,\mathcal{K}^{n-1}, \tag{18}$$

with the same coefficients γ_n as in (16). Using (16) and (18), it is easy to verify that $\mathcal{K}^n$ and $P_n(\omega)$ satisfy

$$\mathcal{K}^n_t[\mathrm{e}^{j\,\omega t}] = j^n P_n(\omega)\,\mathrm{e}^{j\,\omega t}. \tag{19}$$

Let f and g be two infinitely differentiable functions; the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{m=0}^{n}\mathcal{K}^m[f]\mathcal{K}^m[g]\right] = \gamma_n\left(\mathcal{K}^{n+1}[f]\mathcal{K}^n[g] + \mathcal{K}^n[f]\mathcal{K}^{n+1}[g]\right) \tag{20}$$

corresponds to the Christoffel–Darboux identity for orthogonal polynomials,

$$\begin{aligned}(\omega - \sigma)&\sum_{k=0}^{n} P_k(\omega)P_k(\sigma)\\ &= \gamma_n(P_{n+1}(\omega)P_n(\sigma) - P_{n+1}(\sigma)P_n(\omega)).\end{aligned} \tag{21}$$

Setting $\sigma = -\omega$ in (21) we get that for $\omega \neq 0$,

$$\sum_{k=0}^{n} P_{2k+1}(\omega)^2 - \sum_{k=0}^{n} P_{2k}(\omega)^2 = \frac{\gamma_{2n+1}}{\omega}\,P_{2n+2}(\omega)\,P_{2n+1}(\omega), \tag{22}$$

while letting $\sigma \to \omega$ in (21) we get

$$\sum_{k=0}^{n} P_k(\omega)^2 = \gamma_n\,\left(P_{n+1}(\omega)' P_n(\omega) - P_{n+1}(\omega)P_n(\omega)'\right). \tag{23}$$

For every symmetric positive definite moment functional there exists a non-decreasing bounded function $a(\omega)$, called an m–distribution function, such that for the associated Stieltjes integral we have

$$\int_{-\infty}^{\infty} \omega^n\,\mathrm{d}a(\omega) = \mu_n \tag{24}$$

and

$$\int_{-\infty}^{\infty} P_n(\omega)\,P_m(\omega)\,\mathrm{d}a(\omega) = \delta(m-n) \tag{25}$$

for all m and n, [21].

Definition 2.1: (1) $L^2_{a(\omega)}$ is the Hilbert space consisting of functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that the associated Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} |\varphi(\omega)|^2\,\mathrm{d}a(\omega)$ is finite, with the scalar product defined by $\langle\alpha, \beta\rangle_{a(\omega)} = \int_{-\infty}^{\infty} \alpha(\omega)\,\overline{\beta(\omega)}\,\mathrm{d}a(\omega)$. The corresponding norm is denoted by $\|\varphi\|_{a(\omega)} = (\int_{-\infty}^{\infty} |\varphi(\omega)|^2\,\mathrm{d}a(\omega))^{1/2} < \infty$.
(2) $\mathcal{F}L^2_{a(\omega)}$ is the subspace of $L^2_{a(\omega)}$ consisting of all functions $\varphi \in L^2_{a(\omega)}$ such that the function $f(t)$ defined by (28) is real valued.
(3) $L_2^{\mathcal{M}}$ is the space of real analytic functions such that $\sum_{n=0}^{\infty} \mathcal{K}^n[f](0)^2$ converges.

If a symmetric positive definite functional $\mathcal{M}$ satisfies

$$\lim_{n\to\infty}\left(\frac{\mu_n}{n!}\right)^{1/n} = \mathrm{e}\,\lim_{n\to\infty}\frac{\mu_n^{1/n}}{n} = 0, \tag{26}$$

then by a theorem of Riesz, the family of polynomials $\{P_n(\omega)\}_{n\in\mathbb{N}}$ is a complete system in $L^2_{a(\omega)}$; see, for example, Theorem 5.1 in §II.5 of [21], where it is shown that for the completeness of $\{P_n(\omega)\}_{n\in\mathbb{N}}$, existence of a finite $\liminf_{n\to\infty}\left(\frac{\mu_n}{n!}\right)^{1/n}$ suffices. We need at least the condition $\liminf_{n\to\infty}\left(\frac{\mu_n}{n!}\right)^{1/n} < \infty$ to ensure that the function $\boldsymbol{m}(t)$ defined by (34) below is analytic in a neighborhood of zero; see [19]. The stronger assumption (26), which we make for the rest of this paper, amounts to requiring $\boldsymbol{m}(t)$ to be analytic on $\mathbb{R}$. Since in signal processing we usually deal with signals which correspond to real analytic functions, assumption (26) is not significantly restrictive.

Let $\varphi \in L^2_{a(\omega)}$; since

$$\begin{aligned}\int_{-\infty}^{\infty} &|(j\,\omega)^n\varphi(\omega)\,\mathrm{e}^{j\,\omega t}|\mathrm{d}a(\omega)\\ &\leq \left(\int_{-\infty}^{\infty}\omega^{2n}\mathrm{d}a(\omega)\int_{-\infty}^{\infty}|\varphi(\omega)|^2\mathrm{d}a(\omega)\right)^{1/2}\\ &= \mu_{2n}^{1/2}\,\|\varphi\|_{a(\omega)}^{1/2} < \infty,\end{aligned} \tag{27}$$

we can define a corresponding function $f : \mathbb{R} \to \mathbb{C}$ by

$$f(t) = \int_{-\infty}^{\infty}\varphi(\omega)e^{j\omega t}\,\mathrm{d}a(\omega), \tag{28}$$

and we can differentiate (28) under the integral sign any number of times. By (26) and (27),

$$\left|\frac{f^{(n)}(t)}{n!}\right|^{1/n} \leq \frac{\mathrm{e}\;\mu_{2n}^{1/(2n)}\,\|\varphi\|_{a(\omega)}^{1/(2n)}}{n}. \tag{29}$$

Equation (29) can be used to estimate the Lagrange form of the remainder term of the Taylor approximations. Since the right hand side converges to zero and does not depend on t, $f(t)$ given by (28) is analytic on $\mathbb{R}$ and (19) implies

$$\mathcal{K}^n[f](t) = \int_{-\infty}^{\infty} j^n\,P_n(\omega)\,\varphi(\omega)\,\mathrm{e}^{j\omega t}\mathrm{d}a(\omega). \tag{30}$$

Since the family $\{P_n(\omega)\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $L^2_{a(\omega)}$, for every $\varphi \in L^2_{a(\omega)}$ and for the corresponding $f(t)$ such that (28) holds, (30) implies that in $L^2_{a(\omega)}$,

$$\varphi(\omega) = \sum_{n=0}^{\infty} (-j)^n \mathcal{K}^n[f](0)\, P_n(\omega). \tag{31}$$

Thus, for every $f(t)$ there can be at most one function $\varphi(\omega)$ in $L^2_{a(\omega)}$ such that (28) holds; we call such $\varphi(\omega)$ the $\mathcal{M}$–*Fourier–Stieltjes transform* of $f(t)$, and write $\varphi = \mathcal{F}^{\mathcal{M}}[f]$. More generally, since for every $\varphi(\omega) \in L^2_{a(\omega)}$ and every $t \in \mathbb{R}$ also $\varphi(\omega)\,\mathrm{e}^{j\omega t} \in L^2_{a(\omega)}$, (30) implies that for every fixed t, the following holds in $L^2_{a(\omega)}$:

$$\varphi(\omega)\,\mathrm{e}^{j\omega t} = \sum_{n=0}^{\infty} (-j)^n \mathcal{K}^n[f](t)\, P_n(\omega). \tag{32}$$

Thus, by the Parseval equality,

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[f](t)|^2 = \left\|\varphi(\omega)\mathrm{e}^{j\omega t}\right\|^2_{a(\omega)} = \|\varphi(\omega)\|^2_{a(\omega)}. \tag{33}$$

Note that this implies that $\sum_{n=0}^{\infty} |\mathcal{K}^n[f](t)|^2$ does not depend on t.

On the other hand, assume that $f(t)$ is an analytic function such that $\sum_{n=0}^{\infty} \mathcal{K}^n[f](0)^2$ converges. Then the series in (31) converges in $L^2_{a(\omega)}$ to some $\varphi_f(\omega)$; let the corresponding analytic function defined by (28) be $f_{\varphi_f}(t)$. Then (30) implies that $\mathcal{K}^n[f_{\varphi_f}](0) = \mathcal{K}^n[f](0)$; thus $f_{\varphi_f} = f$, and so $f(t)$ satisfies (30). By (33) the sum $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ is independent on t and we get the following proposition.

Proposition 2.2: Let $\mathcal{M}$ be a symmetric positive definite moment functional such that (26) holds, and let $f(t)$ be any real analytic function. If $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ converges for some t, then it converges everywhere to a constant function, and its $\mathcal{M}$–Fourier–Stieltjes transform $\mathcal{F}^{\mathcal{M}}[f]$ belongs to $\mathcal{F}L^2_{a(\omega)}$.

We can now introduce a scalar product, the associated norm and a convolution on $L_2^{\mathcal{M}}$ by the following sums that are independent of t; see [15], [19] for more details.

$$\begin{aligned}
\|f\|^2_{\mathcal{M}} &= \sum_{n=0}^{\infty} K^n[f](t)^2 = \|\mathcal{F}^{\mathcal{M}}[f]\|^2_{a(\omega)}; \\
\langle f, g\rangle_{\mathcal{M}} &= \sum_{n=0}^{\infty} K^n[f](t) K^n[g](t) \\
&= \langle \mathcal{F}^{\mathcal{M}}[f](\omega), \mathcal{F}^{\mathcal{M}}[g](\omega)\rangle_{a(\omega)}; \\
(f *_{\mathcal{M}} g)(u) &= \sum_{n=0}^{\infty} K^n[f](t) K^n_t[g(u-t)] \\
&= \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[f](\omega)\mathcal{F}^{\mathcal{M}}[g](\omega)\mathrm{e}^{j\omega t}\mathrm{d}a(\omega).
\end{aligned}$$

If we define

$$\boldsymbol{m}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{j\omega t}\mathrm{d}a(\omega) \tag{34}$$

then (30) implies that for every t

$$\begin{aligned}
\langle \mathcal{K}^m[\boldsymbol{m}], \mathcal{K}^n[\boldsymbol{m}]\rangle_{\mathcal{M}} &= \langle P_m(\omega), P_n(\omega)\rangle_{a(\omega)} \\
&= \delta(m-n),
\end{aligned}$$

i.e., $\{\mathcal{K}^n[\boldsymbol{m}]\}_{n\in\mathbb{N}}$ is an orthonormal system in $L_2^{\mathcal{M}}$, and in particular

$$\|\boldsymbol{m}\|^2_{\mathcal{M}} = \sum_{n=0}^{\infty} \mathcal{K}^n[\boldsymbol{m}](t)^2 = 1. \tag{35}$$

Also,

$$\boldsymbol{m}^{(n)}(0) = j^n \int_{-\infty}^{\infty} \omega^n \mathrm{d}a(\omega) = j^n \mu_n.$$

Since $\mathcal{M}$ is symmetric, $\boldsymbol{m}(t)$ is real valued. By (29), $\boldsymbol{m}(t)$ is analytic on $\mathbb{R}$ and for all t,

$$\boldsymbol{m}(t) = \sum_{n=0}^{\infty} (-1)^n \mu_{2n} t^{2n}. \tag{36}$$

Let $f \in C^{\infty}$; the formal series

$$\mathrm{CE}^{\mathcal{M}}[f,u](t) = \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](u)\; \mathcal{K}^k[\boldsymbol{m}](t-u) \tag{37}$$

is called the *chromatic expansion* of f associated with $\mathcal{M}$, and

$$\mathrm{CA}^{\mathcal{M}}[f,n,u](t) = \sum_{k=0}^{n} (-1)^k \mathcal{K}^k[f](u) \mathcal{K}^k[\boldsymbol{m}](t-u) \tag{38}$$

is the *chromatic approximation* of f of order n, centered at u.

Assume that $f(t) \in L_2^{\mathcal{M}}$; since (35) implies

$$\begin{aligned}
&\sum_{k=m}^{\infty} (-1)^k \mathcal{K}^k[f](0) \mathcal{K}^k[\boldsymbol{m}](t) \\
&\qquad \le \sum_{k=m}^{\infty} \mathcal{K}^k[f](0)^2 \sum_{k=m}^{\infty} \mathcal{K}^k[\boldsymbol{m}](t)^2 \\
&\qquad \le \sum_{k=m}^{\infty} \mathcal{K}^k[f](0)^2,
\end{aligned} \tag{39}$$

the series $\mathrm{CE}^{\mathcal{M}}[f,0]$ converges uniformly to an analytic function. Since for all m

$$\begin{aligned}
&\mathcal{K}^m \left[\sum_{k=0}^{\infty} \mathcal{K}^k[f](0) \mathcal{K}^k[\boldsymbol{m}](t)\right]\Bigg|_{t=0} \\
&\qquad = \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](0) (\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](0) \\
&\qquad = \mathcal{K}^m[f](0),
\end{aligned}$$

and since both $f(t)$ and $\mathrm{CE}^{\mathcal{M}}[f,u](t)$ are analytic, $f(t) = \mathrm{CE}^{\mathcal{M}}[f,u](t)$ for all t. Thus, also

$$f(t+u) = \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](u)\; \mathcal{K}^k[\boldsymbol{m}](t). \tag{40}$$

Note that (39) implies that $\mathrm{CE}^{\mathcal{M}}[f,u](t)$ converges to $f(t)$ also in $L_2^{\mathcal{M}}$. Thus, functions $\{\mathcal{K}^n[\boldsymbol{m}](t)\}$ form a complete orthonormal system in $L_2^{\mathcal{M}}$, and for a fixed u, the chromatic expansion (40) is the Fourier expansion of $f(t+u)$ in this base.

If A is a continuous, time invariant linear operator on $L_2^{\mathcal{M}}$, then A commutes with differential operators, and for all $f \in$

$L_2^{\mathcal{M}}$,

$$\begin{aligned} A[f](t) &= \sum_{n=0}^{\infty}(-1)^n\mathcal{K}^n[f](0)\,\mathcal{K}^n[A[\boldsymbol{m}]](t)\\ &= \sum_{n=0}^{\infty}(-1)^n\mathcal{K}^n[A[\boldsymbol{m}]](0)\,\mathcal{K}^n[f](t). \end{aligned}$$

Thus, $A[\boldsymbol{m}]$ plays the role of the *impulse response* of a filter.

If $a(\omega)$ is absolutely continuous, then there exists a non-negative weight function $\mathrm{w}(\omega)$ such that almost everywhere $a'(\omega) = \mathrm{w}(\omega)$. Thus, in this case, if $f(t)$ has the usual Fourier transform $\widehat{f(\omega)}$ such that $f(t) = \int_{-\infty}^{\infty}\widehat{f(\omega)}\,\mathrm{e}^{j\,\omega t}\mathrm{d}t$, then $\widehat{f(\omega)} = 2\pi\,\mathrm{w}(\omega)\,\mathcal{F}^{\mathcal{M}}[f](\omega)$, and $\mathcal{F}^{\mathcal{M}}[f](\omega) \in L^2_{a(\omega)}$ just in case $\widehat{f(\omega)}\mathrm{w}(\omega)^{-1} \in L^2$.

A. Examples

Example 1. (Legendre polynomials/Spherical Bessel functions) Let $L_n(\omega)$ be the Legendre polynomials; if we set $P_n^L(\omega) = \sqrt{2n+1}\,L_n(\omega/\pi)$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^L(\omega)P_m^L(\omega)\mathrm{d}\omega = \delta(m-n).$$

The corresponding recursion coefficients in equation (16) are given by the formula $\gamma_n = \pi(n+1)/\sqrt{4(n+1)^2-1}$. In this case $\boldsymbol{m}(t) = \operatorname{sinc} t$, and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{(2n+1)}\;j_n(\pi t)$, where $j_n(x)$ is the spherical Bessel function of the first kind of order n.[4] The corresponding equality (38) holds point-wise for every analytic function [15]. If also $f \in L_2$, then the convergence is both uniform and in L_2.

Example 2. (Chebyshev polynomials of the first kind/Bessel functions) Let $P_n^T(\omega)$ be the family of orthonormal polynomials obtained by normalizing and rescaling the Chebyshev polynomials of the first kind, $T_n(\omega)$, by setting $P_0^T(\omega) = 1$ and $P_n^T(\omega) = \sqrt{2}\,T_n(\omega/\pi)$ for $n > 0$. In this case

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\frac{P_n^T(\omega)P_m^T(\omega)}{\sqrt{1-\left(\frac{\omega}{\pi}\right)^2}}\,\mathrm{d}\omega = \delta(n-m).$$

The corresponding function (34) is $\boldsymbol{m}(t) = J_0(\pi t)$, and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2}\,J_n(\pi t)$, where $J_n(t)$ is the Bessel function of the first kind of order n. In the recurrence relation (18) the coefficients are given by $\gamma_0 = \pi/\sqrt{2}$ and $\gamma_n = \pi/2$ for $n > 0$. The corresponding chromatic expansion is the Neumann series (see [22]):

$$f(t) = f(u)J_0(\pi(t-u)) + \sqrt{2}\,\sum_{n=1}^{\infty}\mathcal{K}^n[f](u)J_n(\pi(t-u)),$$

which converges point-wise for every analytic function. If $f(t)$ has a Fourier transform $\widehat{f(\omega)}$ such that $\int_{-\pi}^{\pi}\sqrt{1-(\omega/\pi)^2}\;|\widehat{f(\omega)}|^2\mathrm{d}\omega < \infty$, then the convergence is also uniform. Thus, the chromatic expansions corresponding to various families of orthogonal polynomials can be seen as generalizations of the Neumann series, while the families of corresponding functions $\{\mathcal{K}^n[\boldsymbol{m}](t)\}_{n\in\mathbb{N}}$ can be seen as generalizations of familiar special functions.

[4]Thus, $j_n(t) = J_{n+1/2}(t)/\sqrt{2\,t}$, where $J_n(x)$ is the Bessel function of the first kind of order n.

Example 3. (Hermite polynomials/Gaussian monomial functions) Let $H_n(\omega)$ be the Hermite polynomials; then the polynomials given by $P_n^H(\omega) = (2^n\,n!)^{-1/2}\,H_n(\omega)$ satisfy

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}P_n^H(\omega)P_m^H(\omega)\;\mathrm{e}^{-\omega^2}\mathrm{d}\omega = \delta(n-m).$$

The corresponding function defined by (34) is $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\,t^n\mathrm{e}^{-t^2/4}/\sqrt{2^n\,n!}$. The corresponding recursion coefficients are given by $\gamma_n = \sqrt{(n+1)/2}$. The chromatic expansion of $f(t)$ is just the Taylor expansion of $f(t)\,\mathrm{e}^{t^2/4}$, multiplied by by $\mathrm{e}^{-t^2/4}$. For analytic functions satisfying $\lim_{n\to\infty}|f^{(n)}(0)|^{1/n}/\sqrt{n} = 0$ the chromatic expansion of $f(t)$ point-wise converges to $f(t)$ (see [15]). If $\int_{-\infty}^{\infty}|\widehat{f(\omega)}|^2\;\mathrm{e}^{\omega^2}\mathrm{d}\omega < \infty$, then the chromatic expansion of $f(t)$ also converges uniformly.

More examples can be found in [4], [3] and [15].

B. Weakly bounded moment functionals

To study pointwise convergence of chromatic expansions, we introduce a very broad class of moment functionals, containing functionals which correspond to many classical families of orthogonal polynomials. For such moment functionals the corresponding coefficients γ_n in the three term recurrence relations (16) and (18) are such that the sequences $\{\gamma_n\}$ and $\{\gamma_n/\gamma_{n-1}\}$ are bounded from below by a positive number, and such that $\{\gamma_n\}$ grow sub-linearly. For simplicity, in the definition below these conditions are defined using a single constant M.

Definition 2.3: A moment functional $\mathcal{M}$ and the corresponding families $\{P_n(\omega)\}_{n\in\mathbb{N}}$ and $\{\mathcal{K}^n\}^{\mathcal{M}}_{n\in\mathbb{N}}$ are *weakly bounded* if, for some $M, r > 0$ and $0 \le p < 1$, the following hold:

$$\begin{aligned} \frac{1}{M} \le \gamma_n &\le M(n+r)^p; &(41)\\ \frac{\gamma_n}{\gamma_{n+1}} &\le M^2. &(42) \end{aligned}$$

If (41) holds with $p = 0$, we say that $\mathcal{M}$ and the corresponding families $\{P_n(\omega)\}_{n\in\mathbb{N}}$ and $\{\mathcal{K}^n\}^{\mathcal{M}}_{n\in\mathbb{N}}$ are *bounded.*
Thus, families in Example 1 and Example 2 are bounded; the family in Example 3 is not bounded but is weakly bounded with $p = 1/2$. One can show that, if $\boldsymbol{m}(t)$ is to be analytic on $\mathbb{R}$, then the upper bound in (41) is sharp, see Example 4 in [15].

The "standard" and the chromatic derivatives are related by the following equations:

$$\mathrm{d}^n = \sum_{k=0}^{n}(-1)^k\,(\mathrm{d}^n\circ\mathcal{K}^k)[\boldsymbol{m}](0)\;\mathcal{K}^k; \qquad (43)$$

$$\mathcal{K}^n = \sum_{k=0}^{n}\mathcal{K}^n\left[t^k/k!\right](0)\;\mathrm{d}^k. \qquad (44)$$

Weakly bounded moment functionals allow a convenient estimation of the coefficients appearing in the right hand sides of (43) and (44).

Lemma 2.4 ([15]): Let $\mathcal{M}$ be weakly bounded, and let M, p and r be as in (41) and (42). Then the following two inequalities hold for all k and n:

$$|(\mathcal{K}^n \circ \mathrm{d}^k)[\boldsymbol{m}](0)| \quad \leq \quad (2M)^k (k+r)!^p; \tag{45}$$

$$\left|\mathcal{K}^n \left[\frac{t^k}{k!}\right](0)\right| \quad \leq \quad (2M)^n. \tag{46}$$

Using this Lemma we can get an estimate for the modulus of the complex analytic function $\boldsymbol{m}(z) = \sum_{n=0}^{\infty} \mu_n z^n$.

Lemma 2.5 ([15]): Let $\mathcal{M}$ and $p < 1$ be as in Lemma 2.4; then there exists $K > 0$ and an analytic function $\theta(z)$, such that for every n and every $z \in \mathbb{C}$,

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{(K|z|)^n}{n!^{1-p}}\, \theta(z). \tag{47}$$

Using this estimate we can get the following Proposition.

Proposition 2.6 ([15]): Let $\mathcal{M}$ be weakly bounded, $p < 1$ as in (41), $f(z)$ an entire function and $u \in \mathbb{C}$. If

$$\lim_{n\to\infty} |f^{(n)}(u)|^{1/n}/n^{1-p} = 0 \tag{48}$$

then for all $z \in \mathbb{C}$

$$f(z) = \sum_{j=0}^{\infty} (-1)^j \mathcal{K}^j[f](u)\, \mathcal{K}^j[\boldsymbol{m}](z-u). \tag{49}$$

The convergence is uniform on every disc of finite radius.

Thus, if $\mathcal{M}$ is bounded (i.e., $p = 0$), then (48) holds for every analytic function and thus so does (49). Also, if $\mathcal{M}$ is weakly bounded, then $\sin \omega t$ and $\cos \omega t$ satisfy condition (48) and we get the following generalization of the well known expansion of $\mathrm{e}^{j\omega t}$ using the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions $J_n(t)$, $\mathrm{e}^{j\,\omega t} = J_0(t) + 2\sum_{n=1}^{\infty} j^n T_n(\omega) J_n(t)$.

Corollary 2.7: Let $P_n(\omega)$ be the family of orthonormal polynomials associated with a weakly bounded moment functional $\mathcal{M}$, and $\{\mathcal{K}^n\}^{\mathcal{M}}_{n\in\mathbb{N}}$ the corresponding family of the chromatic derivatives; then for every t,

$$\mathrm{e}^{j\omega t} = \sum_{n=0}^{\infty} j^n P_n(\omega)\, \mathcal{K}^n[\boldsymbol{m}](t). \tag{50}$$

A great deal of well known properties of the Bessel functions are just the special cases of more general chromatic expansions. For example, the chromatic expansion of the constant function $f(t) \equiv 1$ yields

$$\boldsymbol{m}(z) + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\right) \mathcal{K}^{2n}[\boldsymbol{m}](z) = 1,$$

with γ_n the recursion coefficients from (16), which generalizes

$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1.$$

Similarly, by expanding $\boldsymbol{m}(z)$ into chromatic series around u, we get

$$\boldsymbol{m}(z+u) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[\boldsymbol{m}](u) \mathcal{K}^n[\boldsymbol{m}](z),$$

which generalizes

$$J_0(z+u) = J_0(u)J_0(z) + 2\sum_{n=1}^{\infty} (-1)^n J_n(u) J_n(z).$$

III. Non-separable Inner product spaces

We now define new non-separable inner product spaces in which all signals from $L_2^{\mathcal{M}}$ are identified with the null signal $\boldsymbol{n}(t) = 0$, and in which pure harmonic oscillations have finite positive norms and are pairwise orthogonal. For technical details of the proofs we refer the reader to [19].

Definition 3.1: Assume again that $\mathcal{M}$ is weakly bounded and let $0 \leq p < 1$ be as in (41). We denote by $\mathcal{C}^{\mathcal{M}}$ the vector space of analytic functions such that the sequence

$$\nu_n^f(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2 \tag{51}$$

converges uniformly on every finite interval.

Proposition 3.2: Let $f, g \in \mathcal{C}^{\mathcal{M}}$ and

$$\sigma_n^{fg}(t) = \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t) \mathcal{K}^k[g](t); \tag{52}$$

then the sequence $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ converges to a constant function.

This proposition is proved by using (20) to show that $\liminf_{n\to\infty} |\frac{d}{dt}\sigma_n^{fg}(t)| = 0$. This is sufficient because $\sigma_n^{fg}(t)$ converges uniformly on every finite interval, and thus $\frac{d}{dt}\left[\lim_{n\to\infty} \sigma_n^{fg}(t)\right] = \lim_{n\to\infty} \frac{d}{dt}\sigma_n^{fg}(t) = 0$; see [19] for details.

Corollary 3.3: Let $\mathcal{C}_0^{\mathcal{M}}$ be the vector space consisting of analytic functions $f(t)$ such that $\lim_{n\to\infty} \nu_n^f(t) = 0$; then in the quotient space $\mathcal{C}_2^{\mathcal{M}} = \mathcal{C}^{\mathcal{M}}/\mathcal{C}_0^{\mathcal{M}}$ we can introduce a scalar product by the following formula whose right hand side is independent of t:

$$\langle f, g\rangle^{\mathcal{M}} = \lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\, \mathcal{K}^k[g](t). \tag{53}$$

The corresponding norm on $\mathcal{C}_2^{\mathcal{M}}$ is denoted by $\|\cdot\|^{\mathcal{M}}$. Clearly, $L_2^{\mathcal{M}} \subset \mathcal{C}_0^{\mathcal{M}}$.

Proposition 3.4 ([19]): Let $\mathcal{M}$ be weakly bounded and $0 \leq p < 1$ be as in (41); if

$$\lim_{n\to\infty} \frac{|\mathcal{K}^n[f](0)|^{1/n}}{n^{1-p}} = 0, \tag{54}$$

then

$$f(t) = \sum_{j=0}^{\infty} (-1)^j \mathcal{K}^j[f](0)\, \mathcal{K}^j[\boldsymbol{m}](t), \tag{55}$$

with the series converging uniformly on every finite interval. Note that this Proposition follows directly from Lemma 2.5, because we get that

$$|\mathcal{K}^n[f](0)\, \mathcal{K}^n[\boldsymbol{m}](t)|^{1/n} \quad < \quad \frac{|\mathcal{K}^n[f](0)|^{1/n}}{n^{1-p}}\, K|z|\, \theta(z)^{1/n}$$

which implies the claim.

Corollary 3.5: If $f \in \mathcal{C}_2^{\mathcal{M}}$ then the chromatic expansion of f converges to f uniformly on every finite interval.

This Corollary follows from the fact that, since $1/(n+1)^{1-p} \sum_{k=0}^{n} \mathcal{K}^k[f](t)^2$ converges to $(\|f\|^{\mathcal{M}})^2 < \infty$, for all sufficiently large n we have

$$\begin{aligned} |\mathcal{K}^n[f](t)|^{1/n} &\leq \left(\sum_{k=0}^{n} \mathcal{K}^k[f](t)^2\right)^{1/(2n)} \\ &< \left(2\,\|f\|^{\mathcal{M}}\right)^{1/n}(n+1)^{(1-p)/(2n)}, \end{aligned}$$

which implies (54).

Since $\mathcal{K}^n[\boldsymbol{m}](t) \in L_2^{\mathcal{M}}$,

$$\left\|\sum_{j=0}^{n}(-1)^j \mathcal{K}^j[f](0)\,\mathcal{K}^j[\boldsymbol{m}](t)\right\|^{\mathcal{M}} = 0 \qquad (56)$$

for all n. Thus, in general, the chromatic series does not converge to f in $\mathcal{C}_2^{\mathcal{M}}$. In fact, there can be no such expansion valid for every $f \in \mathcal{C}_2^{\mathcal{M}}$, converging in $\mathcal{C}_2^{\mathcal{M}}$ to f, because, in general, space $\mathcal{C}_2^{\mathcal{M}}$ is non separable, as the following two examples show.

Example 2 - continued. (Chebyshev polynomials and their associated space $\mathcal{C}_2^{T}$)

For this case the corresponding space $\mathcal{C}_2^{\mathcal{M}}$ will be denoted by $\mathcal{C}_2^{T}$, and in (41) we have $p = 0$. Thus, the scalar product on $\mathcal{C}_2^{T}$ is defined by $\langle f,g\rangle^T = \lim_{n\to\infty} 1/(n+1)\sum_{k=0}^{n}\mathcal{K}^k[f](t)\mathcal{K}^k[g](t)$.

Proposition 3.6 ([19]): Functions $f_\omega(t) = \sqrt{2}\sin\omega t$ and $g_\omega(t) = \sqrt{2}\cos\omega t$ for $0 < \omega < \pi$ form an orthonormal system of vectors in $\mathcal{C}_2^{T}$.

Example 3 - continued. (Hermite polynomials and their associated space $\mathcal{C}_2^{H}$)

The corresponding space $\mathcal{C}_2^{\mathcal{M}}$ in this case is denoted by $\mathcal{C}_2^{H}$, and in (41) we have $p = 1/2$. Thus, the scalar product in $\mathcal{C}_2^{H}$ is defined by $\langle f,g\rangle^H = \lim_{n\to\infty} 1/\sqrt{n+1}\ \sum_{k=0}^{n}\mathcal{K}^k[f](t)\mathcal{K}^k[g](t)$.

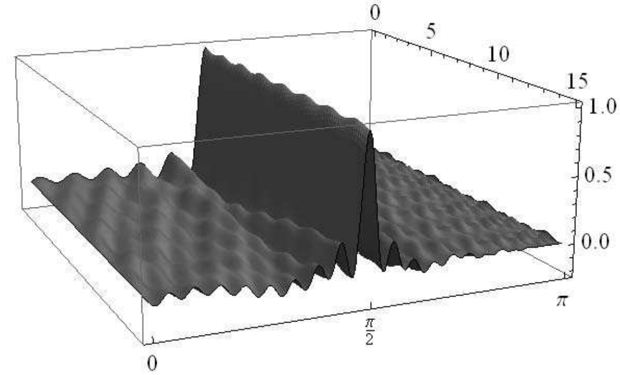
Proposition 3.7 ([19]): For all $\omega > 0$ functions $f_\omega(t) = \sin\omega t$ and $g_\omega(t) = \cos\omega t$ form an orthogonal system of vectors in $\mathcal{C}_2^{H}$, and $\|f_\omega\|^{\mathcal{M}} = \|g_\omega\|^{\mathcal{M}} = \mathrm{e}^{\omega^2/2}/\sqrt[4]{2\pi}$.
Note that in this case, unlike the case of the family associated with the Chebyshev polynomials, the norm of a pure harmonic oscillation of unit amplitude depends on its frequency.

Both Proposition 3.6 and Proposition 3.7 are proved by direct calculations, using basic properties of the corresponding families of orthogonal polynomials; see [19]. In the same manner one can verify that similar propositions hold for other classical families of orthogonal polynomials, such as the Legendre polynomials.

Conjecture 3.8: Assume that for some $p < 1$ the recursion coefficients γ_n in (16) are such that γ_n/n^p converges to a finite positive limit. Then the corresponding space $\mathcal{C}_2^{\mathcal{M}}$ is non-separable.

A. Local energy estimation of a signal

We now consider the mapping $f, g \mapsto \sigma_n^{fg}$, where σ_n^{fg} is given by (52). This mapping is linear in f and g and is in a sense an approximation of the scalar product (53). The value of σ_n^{fg} is "nearly independent" of t, meaning that if n is large, the value of σ_n^{fg} change slowly with t. Figure 6 shows plots of $\sigma_{63}^{fg}(t)$ and $\sigma_{1023}^{fg}(t)$, for fixed $f(t) = \sin\pi t/2$ and for $g_\omega(t) = \sin\omega t$ with $0 < \omega < \pi$, and for $0 < t < 15$. Note that



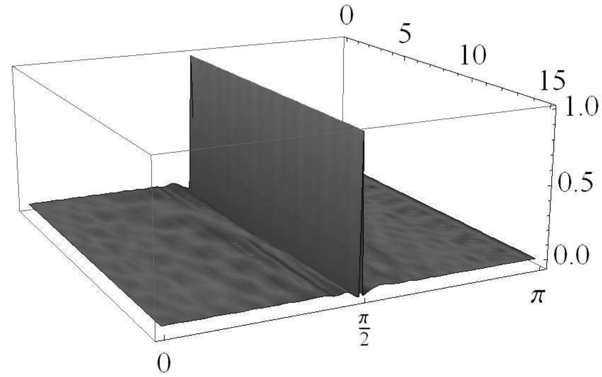


Fig. 6. $\sigma_{63}^{fg}(t)$ (top) and $\sigma_{1023}^{fg}(t)$ (bottom), for $f(t) = \sin\pi t/2$ and for $g_\omega(t) = \sin\omega t$ with $0 < \omega < \pi$, and for $0 < t < 15$.

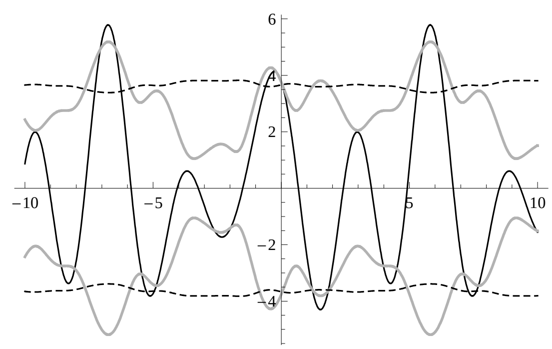


Fig. 7. A signal $f(t)$ (black) and the corresponding $\nu_2^f(t)$ (grey) and $\nu_{16}^f(t)$ (dashed).

$\sigma_{1023}^{fg}(t)$ is essentially independent of t. Both figures clearly display a Gibbs phenomenon around the central frequency $\pi/2$. Similarly, $\nu_n^f(t)$ is an approximation of the corresponding norm $\|\cdot\|^{\mathcal{M}}$. Figure 7 shows a signal $f(t)$ (black), and the corresponding $\nu_2^f(t)$ (grey) and $\nu_{16}^f(t)$ (dashed). Thus, for low values of n, $\nu_n^f(t)$ behaves as an envelope of the signal which "flattens out" as the degree n increases and which can be used for estimation of local energy of the signal. In a forthcoming paper we will show how $\nu_n^f(t)$ can be used as a metric for the norm of the error signal in an adaptive equalizer and discuss the impact of using such metric on the convergence rate of such an equalizer.

IV. Local approximations with trigonometric functions

Assuming that $\mathcal{M}$ is weakly bounded, we fix a natural number N and let ω_k, $1 \leq k \leq N$, be the positive zeros of $P_{2N}(\omega)$. We set $f_{\omega_k}(t) = \sin\omega_k t$ and $g_{\omega_k}(t) = \cos\omega_k t$. Let $\mathcal{C}_2^{\mathcal{M}}(2N)$ be the vector space spanned by functions $B_{2N}^{\mathcal{M}} = \{f_{\omega_k}(t), g_{\omega_k}(t) : 1 \leq k \leq N\}$. Let $H(n,m,t) = \sum_{j=0}^{2N-1} \mathcal{K}^j[f_{\omega_n}](t)\mathcal{K}^j[f_{\omega_m}](t)$; then

$$\begin{aligned} H(n,m,t) &= \textstyle\sum_{k=0}^{N-1} P_{2k}(\omega_n)\, P_{2k}(\omega_m) \cos\omega_n t\ \cos\omega_m t \\ &+ \textstyle\sum_{k=0}^{N-1} P_{2k+1}(\omega_n)\, P_{2k+1}(\omega_m) \sin\omega_n t\ \sin\omega_m t. \quad (57) \end{aligned}$$

Since $P_{2N}(\omega_n) = 0$ for all $1 \leq n \leq N$, (21) implies that $\sum_{k=0}^{2N-1} P_k(\omega_n)P_k(\omega_m) = 0$ for all $n \neq m$. Since polynomials $P_n(\omega_n)$ contain only powers of the same parity as n this implies that also

$$\textstyle\sum_{k=0}^{N-1} P_{2k}(\omega_n)\, P_{2k}(\omega_m) = 0; \quad (58)$$

$$\textstyle\sum_{k=0}^{N-1} P_{2k+1}(\omega_n)\, P_{2k+1}(\omega_m) = 0, \quad (59)$$

and consequently $H(n,m,t) = 0$ for all $n \neq m$ and all t.

On the other hand (22) implies that for all $1 \leq n \leq N$

$$\textstyle\sum_{k=0}^{N-1} P_{2k+1}(\omega_n)^2 = \sum_{k=0}^{N-1} P_{2k}(\omega_n)^2, \quad (60)$$

and (57) implies

$$H(n,n,t) = \tfrac{1}{2} \textstyle\sum_{k=0}^{2N-1} P_k(\omega_n)^2. \quad (61)$$

Thus, if we define $\langle f,g\rangle_{2N}^{\mathcal{M}} = \sum_{j=0}^{2N-1} \mathcal{K}^j[f](t)\mathcal{K}^j[g](t)$, then this sum does not depend on t for $f,g \in \mathcal{C}_2^{\mathcal{M}}(2N)$ and is a scalar product on $\mathcal{C}_2^{\mathcal{M}}(2N)$ such that $B_{2N}^{\mathcal{M}}$ is an orthogonal (but not necessarily orthonormal) base. We now construct an orthonormal base that has properties analogous to the properties of the base $\{\mathcal{K}^n[\boldsymbol{m}](t)\}_{n\in\mathbb{N}}$.

Let ω_{N+k}, $1 \leq k \leq N$, be the corresponding negative zeroes of $P_{2N}(\omega)$; by the Gauss quadrature formula, there exist numbers $A_k > 0$, $1 \leq k \leq 2N$, such that $\mathcal{M}(Q(\omega)) = \sum_{k=1}^{2N} A_k Q(\omega_k)$ for every polynomial $Q(\omega)$ which is of degree at most $2N(2N-1)$. Also,

$$\textstyle\sum_{j=0}^{2N-1} P_j(\omega_k)^2 = \frac{1}{A_k}; \quad (62)$$

see, for example, 6.2 (b) on page 34 of [20].

We now define function

$$\boldsymbol{m}_{2N}(t) = \textstyle\sum_{k=1}^{2N} A_k \cos\omega_k t. \quad (63)$$

Clearly, $\boldsymbol{m}_{2N} \in \mathcal{C}_2^{\mathcal{M}}(2N)$. A straightforward calculation using (58) – (61) shows that for every fixed $u \in \mathbb{R}$ vectors $\{\mathcal{K}^n[\boldsymbol{m}_{2N}](t-u)\}_{n\leq 2N-1}$ form an orthonormal base of $\mathcal{C}_2^{\mathcal{M}}(2N)$, such that $(\mathcal{K}^n \circ \mathcal{K}^m)[\boldsymbol{m}_{2N}](0) = \delta(m-n)$. Thus, if we define

$$\mathrm{CA}_{2N}^{\mathcal{M}}[f,u](t) = \sum_{n=0}^{2N-1} (-1)^n \mathcal{K}^n[f](u)\mathcal{K}^n[\boldsymbol{m}_{2N}](t-u),$$

then $\mathrm{d}^k/\mathrm{d}t^k[f](u) = \mathrm{d}^k/\mathrm{d}t^k \left[\mathrm{CA}_{2N}^{\mathcal{M}}[f,u](t)\right]\big|_{t=u}$. Consequently, $\mathrm{CA}_{2N}^{\mathcal{M}}[f,u](t)$ is also a local approximation of $f(t)$ around $t = u$.

Conjecture 4.1: Let $\mathcal{M}$ be as in Conjecture 3.8. If $f \in \mathcal{C}_2^{\mathcal{M}}$ then $\mathrm{CA}_{2N}^{\mathcal{M}}[f,u](t)$ converges to $f(t)$ for every $t \in \mathbb{R}$.

V. Conclusion

We hope that the above remarkable features of chromatic derivatives and chromatic expansions show that they are promising new tool in signal representation. The author would welcome suggestions from the signal processing community regarding their potential applications.

References

[1] A. Ignjatovic, "Signal processor with local signal behavior," 2000, US Patent 6115726. (Provisional Patent Disclosure was filled October 3, 1997. Patent application was filled May 28, 1998. The patent was issued September 5, 2000).

[2] A. Ignjatovic and N. Carlin, "Method and a system of acquiring local signal behavior parameters for representing and processing a signal," 2001, US Patent 6313778. (Provisional Patent Disclosure was filled July 9, 1999; patent application was filled July 9, 2000. The patent was issued November 6, 2001).

[3] M. Cushman and T. Herron, "The general theory of chromatic derivatives," Kromos Technology, Los Altos, California, Tech. Rep., 2001.

[4] T. Herron and J. Byrnes, "Families of orthogonal differential operators for signal processing," Kromos Technology, Los Altos, California, Tech. Rep., 2001.

[5] T. Herron, "Towards a new transform domain adaptive filtering process using differential operators and their associated splines," in *Proc. ISPACS*, Nashville, November 2001.

[6] M. Cushman, "Image compression," Kromos Technology, Los Altos, California, Tech. Rep., 2001.

[7] ——, "A method for approximate reconstruction from filterbanks," in *Proc. SIAM Conference on Linear Algebra in Signals, Systems and Control*, Boston, August 2001.

[8] M. Cushman, M. J. Narasimha, and P. P. Vaidyanathan, "Finite-channel chromatic derivative filter banks," *IEEE Signal Processing Letters*, vol. 10, no. 1, 2003.

[9] A. Papulis, "Generalized sampling expansion," *IEEE Transactions on Circuits and Systems*, vol. 24, 1977.

[10] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Prentice-Hall, 1993.

[11] M. J. Narasimha, A. Ignjatovic, and P. P. Vaidyanathan, "Chromatic derivative filter banks," *IEEE Signal Processing Letters*, vol. 9, no. 7, 2002.

[12] P. P. Vaidyanathan, A. Ignjatovic, and S. Narasimha, "New sampling expansions of band limited signals based on chromatic derivatives," in *Proc. 35th Asilomar Conference on Signals, Systems, and Computers*, Monterey, California, November 2001.

[13] J. Byrnes, "Local signal reconstruction via chromatic differentiation filter banks," in *Proc. 35th Asilomar Conference on Signals, Systems, and Computers*, Monterey, California, November 2001.

[14] G. Walter and X. Shen, "A sampling expansion for non bandlimited signals in chromatic derivatives," *IEEE Transactions on Signal Processing*, vol. 53, 2005.

[15] A. Ignjatovic, "Local approximations based on orthogonal differential operators," *Journal of Fourier Analysis and Applications*, vol. 13, no. 3, 2007.

[16] G. Walter, "Chromatic Series and Prolate Spheroidal Wave Functions," *Journal of Integral Equations and Applications*, vol. 20, no. 2, 2006.

[17] F. B. Hildebrand, *Introduction to Numerical Analysis*. Dover Publications, 1974.

[18] A. Oppenheim and A. Schafer, *Discrete–Time Signal Processing*. Prentice Hall, 1999.

[19] A. Ignjatovic, "Chromatic derivatives and associated function spaces," *manuscript available on the web*, 2009.

[20] T. S. Chihara, *An introduction to Orthogonal Polynomials*. Gordon and Breach, 1978.

[21] G. Freud, *Orthogonal Polynomials*. Pergamon Press, 1971.

[22] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1966.