Some Questions Related to Orthogonal Polynomials

I would be very grateful to hear solutions to any of the following questions related to orthogonal polynomials (or ideas how to tackle these problems or even if they "feel" likely to be true or not). The questions have their origin in digital signal processing, see [1] (found at my website). Thanks in advance,
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Notation. A bounded non decreasing function $a(\omega)$ such that $\mu_n = \int_{-\infty}^{\infty} \omega^n \mathrm{d}a(\omega) < \infty$ and $\mu_{2n+1} = 0$ for all $n \geq 0$, is a symmetric moment distribution function. Such functions correspond to symmetric positive definite moment functionals $\mathcal{M}$ mapping polynomials into reals; $\{P_n(\omega)\}_{n\in\mathbb{N}}$ is the family of polynomials ortho**normal** with respect to $\mathcal{M}$, i.e., $\mathcal{M}(P_n(\omega)P_m(\omega)) = \int_{-\infty}^{\infty} P_n(\omega)P_m(\omega)\,\mathrm{d}a(\omega) = \delta(m-n)$, while $\gamma_n > 0$ are the positive reals such that $\{P_n(\omega)\}_{n\in\mathbb{N}}$ satisfy the recurrence

$$P_{n+1}(\omega) = \frac{\omega}{\gamma_n}\,P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\,P_{n-1}(\omega); \tag{1}$$

finally, $L^2_{a(\omega)}$ is the space of functions $f : \mathbb{R} \to \mathbb{C}$ which satisfy $\int_{-\infty}^{\infty} |f(\omega)|^2\,\mathrm{d}a(\omega) < \infty$.

Question 1. *Consider $\mathcal{M}$ for which there exists some $0 \leq p < 1$ such that for the corresponding γ_n,*

$$0 < \liminf_{n\to\infty} \frac{\gamma_n}{n^p} \;\leq\; \limsup_{n\to\infty} \frac{\gamma_n}{n^p} < \infty. \tag{2}$$

Is it possible to characterize such $\mathcal{M}$ purely in terms of the properties of the corresponding $a(\omega)$?

Question 2. *Assume that $f(z)$ is an entire function for which there exist a symmetric moment distribution function $a(\omega)$ and a function $\varphi(\omega) \in L^2_{a(\omega)}$ such that $f(z) = \int_{-\infty}^{\infty} \phi(\omega)\mathrm{e}^{\mathrm{i}\,z\omega}\mathrm{d}a(\omega)$. Are the following two generalizations of the Paley-Wienner theorem true? (or perhaps something very similar)*

(I) *The following are equivalent:*

- *(i) f is of exponential type, i.e., there exist $C, L > 0$ such that $|f(z)| < C\mathrm{e}^{L|z|}$ for all $z \in \mathbb{C}$;*
- *(ii) $a(\omega)$ can be chosen such that $\mathrm{d}a(\omega)$ is finitely supported.*

(thus, we have only weakened the assumption that the restriction of f on $\mathbb{R}$ is in L^2).

(II) *Let $m \geq 1$ be an integer; then the following are equivalent:*

- *(iii) there exist $C, L > 0$ such that $|f(z)| < C\mathrm{e}^{L|z|^m}$ for all $z \in \mathbb{C}$;*
- *(iv) $a(\omega)$ can be chosen such that the corresponding γ_n satisfy (2) for some $0 \leq p \leq 1 - 1/m$.*

Note that (I) is just a special case of (II) for $m = 1$, because γ_n are bounded if and only if $\mathrm{d}a(\omega)$ is finitely supported. In [1] it was proved that (iv) implies (iii), and thus that also (ii) implies (i). If (II) is true, then it would be particularly interesting to answer Question 1 positively.

Question 3. *Assume that the recursion coefficients γ_n satisfy the conditions stated in Question 1; is it true that for all ω in the support $sp(a)$ of the corresponding* $\mathrm{d}a(\omega)$,

$$0 < \lim_{n\to\infty} \frac{1}{n^{1-p}} \sum_{k=0}^{n-1} P_k(\omega)^2 < \infty \quad ? \tag{3}$$

This is true for many classical families of orthogonal polynomials, such as the Legendre and the Chebyshev (for which $p = 0$) as well as for the Hermite polynomials (for which $p = 1/2$); see [1]. Extensive numerical tests with $\gamma_n = n^p$ for many $p \in [0, 1)$ indicate that the conjecture might be true. Unfortunately, recently I was informed by Paul Nevai that the special case for $p = 0$ was an already known and still an open problem.

It is easy to see (Corollary 3.3 in [1]) that for any symmetric positive definite functional $\mathcal{M}$,

$$\rho = \limsup_{n\to\infty} \left(\frac{\mu_n}{n!}\right)^{1/n} = \mathrm{e}\,\limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} < \infty \tag{4}$$

holds if and only if for every α such that $0 \leq \alpha < 1/\rho$, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha|\omega|}\mathrm{d}a(\omega) < \infty, \tag{5}$$

and that for such $\mathcal{M}$ the corresponding polynomials $\{P_n(\omega)\}$ are complete in $L^2_{a(\omega)}$ (Lemma 3.7 in [1]).

Question 4. *Is it possible to characterize functionals $\mathcal{M}$ for which (4) holds purely in terms of the asymptotic behavior of the recursion coefficients γ_n of the corresponding family of orthonormal polynomials?*

Appendix: Why am I interested in the above questions

All of the above questions have relevance for (and an origin in) digital signal processing. They are related to a method for local signal approximation which I came up with while trying to solve some problems occuring in design of switching power amplifiers ("class D" amplifiers) driving reactive loads, such as loudspeakers. Such local approximations might be of interest to mathematicians working on orthogonal polynomials and special functions. Thus, hoping to spark some interest of such experts, besides explaining the reasons for asking the above questions, I also mention a few mathematically relevant facts from [1].

Example. The following example is probably the most useful case for signal processing. Let $\mathbf{BL}(\pi)$ be the space of continuous L^2 functions with the Fourier transform supported within $[-\pi, \pi]$ (i.e., in signal processing terminology, the space of π band limited signals of finite energy), and let $P^{\scriptscriptstyle L}_n(\omega)$ be obtained by normalizing and rescaling the Legendre polynomials, so that $\frac{1}{2\pi}\int_{-\pi}^{\pi} P^{\scriptscriptstyle L}_n(\omega)\, P^{\scriptscriptstyle L}_m(\omega)\mathrm{d}\omega = \delta(m-n)$. We consider linear differential operators

$$\mathcal{K}^n = (-\mathrm{i})^n P^{\scriptscriptstyle L}_n\left(\mathrm{i}\ \frac{\mathrm{d}}{\mathrm{d}t}\right); \tag{6}$$

for such operators $\mathcal{K}^n$ and every $f \in \mathbf{BL}(\pi)$, $\mathcal{K}^n[f](t) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{i}^n\ P^{\scriptscriptstyle L}_n(\omega)\widehat{f}(\omega)\mathrm{e}^{\mathrm{i}\,\omega t}\mathrm{d}\omega.$

Unlike the values of the "standard" derivatives, the values of $\mathcal{K}^n[f](t)$ can be obtained in a numerically extremely accurate and noise robust way from samples of the signal $f \in \mathbf{BL}(\pi)$, taken at a rate somewhat higher than the usual Nyquist rate, using the standard FIR filters. This is true even for differential operators $\mathcal{K}^n$ of very high orders, for example, $n > 20$. While the "standard" derivatives lose information on spectral features of the signal, operators $\mathcal{K}^n$ preserve such features and for that reason we call them the *chromatic derivatives* associated with the Legendre polynomials. Operators $\mathcal{K}^n$ have some remarkable properties, relevant for applications in digital signal processing, given in the propositions below (from [1]).

Proposition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a restriction of an entire function; then the following are equivalent:*
(a) $f \in \mathbf{BL}(\pi)$;
(b) $\sum_{n=0}^{\infty} \mathcal{K}^n[f](0)^2 < \infty$;
(c) for all $t \in \mathbb{R}$ the sum $\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2$ converges, and its values are independent of $t \in \mathbb{R}$.

The following Proposition provides **local representation** of the usual norm, the scalar product and the convolution in $\mathbf{BL}(\pi)$.

Proposition 2. *For all $f, g \in \mathbf{BL}(\pi)$ the following sums do not depend on $t \in \mathbb{R}$, and*

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)^2 &= \int_{-\infty}^{\infty} f(x)^2 dx;\\
\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)\mathcal{K}^n[g](t) &= \int_{-\infty}^{\infty} f(x)g(x)dx;\\
\sum_{n=0}^{\infty} \mathcal{K}^n[f](t)\mathcal{K}^n_t[g(u-t)] &= \int_{-\infty}^{\infty} f(x)g(u-x)dx.
\end{aligned}$$

The next proposition introduces the *chromatic expansion* of an analytic function, which is a form of Taylor's expansion, with differential operators $\mathcal{K}^n$ replacing the derivatives and the spherical Bessel functions replacing the monomials $t^n/n!$. Thus, unlike truncations of the Taylor's expansion, truncations of chromatic expansions belong to $\mathbf{BL}(\pi)$. They also have much better convergence properties which makes them useful for local signal representation in digital signal processing.

Proposition 3. *Let $j_n(z)$ be the spherical Bessel functions of the first kind; then:*

(a) for every entire function $f(z)$ and for all $z \in \mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\mathcal{K}^n\left[\frac{\sin(\pi z)}{\pi z}\right] = \sum_{n=0}^{\infty} \mathcal{K}^n[f](0)\ \sqrt{2n+1}\ j_n(\pi z);$$

(b) if $f \in \mathbf{BL}(\pi)$, then the series converges both uniformly on $\mathbb{R}$ and in L^2.

General Theory. The above properties of the Legendre polynomials and the associated spherical Bessel functions have analogues for very general families of orthogonal polynomials and their associated "special functions".

Assume that a symmetric positive definite moment functional $\mathcal{M}$ for some $\rho \geq 0$ satisfies (4) (or, equivalently, (5)); let $\{P_n(\omega)\}_{n\in\mathbb{N}}$ be the family of the corresponding orthonormal polynomials and $a(\omega)$ the corresponding symmetric moment distribution function. We again define the corresponding differential operators by $\mathcal{K}^n = (-\mathrm{i})^n P_n\left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\right)$. We denote by $L^2_{\mathcal{M}}$ the space of function which are analytic on the strip $S_\rho = \{z \in \mathbb{C} \,:\, |\mathrm{Im}(z)| < 1/(2\rho)\}$ and which also satisfy $\sum_{n=0}^{\infty} |\mathcal{K}^n[f](0)|^2 < \infty$.

One can show that $f \in L^2_{\mathcal{M}}$ if and only if f is analytic on the strip S_ρ and there exists $\phi(\omega) \in L^2_{a(\omega)}$ such that $f(z) = \int_{-\infty}^{\infty} \varphi(\omega)\mathrm{e}^{\mathrm{i}\,\omega z}\mathrm{d}a(\omega)$. We now define $\boldsymbol{m}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega z}\mathrm{d}a(\omega)$, and the corresponding chromatic expansion $\mathrm{CE}^{\mathcal{M}}[f,u](t)$ of a function $f(z) \in C^\infty$, centered at $u \in \mathbb{C}$, as the formal series

$$\mathrm{CE}^{\mathcal{M}}[f,u](t) \;=\; \sum_{k=0}^{\infty} \mathcal{K}^k[f](u)\, \mathcal{K}^k_u[\boldsymbol{m}(t-u)] \;=\; \sum_{k=0}^{\infty} (-1)^k \mathcal{K}^k[f](u)\, \mathcal{K}^k[\boldsymbol{m}](t-u).$$

It is easy to see that $\mathcal{K}^n[\boldsymbol{m}](z) = \int_{-\infty}^{\infty} \mathrm{i}^n P_n(\omega)\mathrm{e}^{\mathrm{i}\,\omega z}\mathrm{d}a(\omega)$. Recently Tom Koornwinder has kindly informed me that such functions, corresponding to a family of polynomials $\{P_n(\omega)\}_{n\in\mathbb{N}}$ orthogonal with respect to some $a(\omega)$, were previously introduced and investigated by Giorgio Mantica. Giorgio called them Fourier Bessel functions and they arise in some problems in quantum mechanics (e.g., quantum intermittency), see [4–6]. He studied their asymptotic behavior; note that, in relation to this, Lemma 5.7. in [1] implies that, if $\mathcal{M}$ satisfies (2) for some $0 \leq p < 1$ and if $m \geq 1/(1-p)$ is an integer, then there exists $K > 0$ such that

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{|Kz|^n}{n!^{1-p}}\,\mathrm{e}^{|Kz|^m}$$

for all $n \in \mathbb{N}$. Since $\boldsymbol{m}^{(n)}(0) = \mu_n$, (4) is necessary to guarantee that $\boldsymbol{m}$ is analytic on a non-trivial domain. On the other hand, (4) is also sufficient for the following theorem in which $\boldsymbol{m} \in L^2_{\mathcal{M}}$ plays the role which $\mathrm{sinc}\,(z) = \sin(\pi z)/(\pi z)$ plays in the case of the Legendre polynomials.

Proposition 4. *Assume that $\mathcal{M}$ satisfies* (2) *and that $f \in L^2_{\mathcal{M}}$; then for all $u \in \mathbb{R}$, the chromatic expansion $\mathrm{CE}^{\mathcal{M}}[f,u](z)$ of $f(z)$ converges to $f(z)$ uniformly on every closed strip $S \subset S_\rho$.*

Note that this is a generalization of Proposition 3 (b). One can prove (see below) that if $\mathrm{d}a(\omega)$ is finitely supported then the chromatic expansion $\mathrm{CE}^{\mathcal{M}}[f,u](z)$ of every analytic function $f(z)$ converges to $f(z)$, and the convergence is uniform on every compact set. However, I could not obtain such a generalization of 3 (a) assuming only (2). In fact, I could not determine when a chromatic expansion converges locally (i.e., point-wise but possibly non-uniformly on the entire $\mathbb{R}$), purely in terms of the properties of the corresponding moment distribution function $a(\omega)$. For that reason I was forced to introduce in [2] the notion of a *weakly bounded moment functional*, which is (a slight generalization of) condition (2). The following Proposition follows directly from Corollary 5.10.(2) in [1]:

Proposition 5. *Let $\mathcal{M}$ be such that* (2) *holds for some $0 \leq p < 1$, let $u \in \mathbb{C}$ be arbitrary and let $f(z)$ be any entire function. If*

$$\lim_{n\to\infty} \frac{|f^{(n)}(u)|^{1/n}}{n^{1-p}} = 0 \tag{7}$$

then the chromatic expansion $\mathrm{CE}^{\mathcal{M}}[f,u](z)$ of $f(z)$ converges for all $z \in \mathbb{C}$ and the convergence is uniform on every compact set.

Note that for $\mathcal{M}$ which satisfy (2) with $p = 0$, condition (7) is satisfied by every entire function.

It turns out that a great deal of the properties of the Bessel functions $J_n(z)$ and the Chebyshev polynomials $T_n(\omega)$ from Watson's classical treatise [7] are just special cases of chromatic expansions which are valid for **all** weakly bounded families of orthonormal polynomials $P_n(\omega)$ and their associated "special functions" $\mathcal{K}^n[\boldsymbol{m}](t)$, and follow essentially trivially from Proposition 5. Here are some examples of such formulas and their generalizations:

$$
\begin{array}{rcl}
\mathrm{e}^{\mathrm{i}\,\omega t} = J_0(z) + 2\sum_{n=1}^{\infty} \mathrm{i}^n T_n(\omega) J_n(t) & \to & \mathrm{e}^{\mathrm{i}\,\omega t} = \sum_{n=0}^{\infty} \mathrm{i}^n P_n(\omega) \mathcal{K}^n[\boldsymbol{m}](t) \\
J_0(t+u) = J_0(u)J_0(t) + 2\sum_{n=1}^{\infty}(-1)^n J_n(u) J_n(t) & \to & \boldsymbol{m}(t+u) = \sum_{n=0}^{\infty}(-1)^n \mathcal{K}^n[\boldsymbol{m}](u)\mathcal{K}^n[\boldsymbol{m}](t) \\
J_0(t)^2 + 2\sum_{k=1}^{\infty} J_n(t)^2 = 1 & \to & \sum_{k=1}^{\infty} \mathcal{K}^n[\boldsymbol{m}](t)^2 = 1 \\
J_0(t) + 2\sum_{n=1}^{\infty} J_{2n}(t) = 1 & \to & \boldsymbol{m}(t) + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\right)\mathcal{K}^{2n}[\boldsymbol{m}](t) = 1
\end{array}
$$

If we are interested in expansions of entire functions, then the inequality $p < 1$ in (2) is sharp, because if $p = 1$ then the corresponding $\boldsymbol{m}(z)$ need not be entire; see [1]. This, together with the fact that in (I) of Question 2 (ii) implies (i) and that in (II) (iv) implies (iii), are some of the reasons why I feel that Question 2 might have a positive answer.

Note that the condition (4) for the uniform convergence of chromatic expansions of functions in $L^2_{a(\omega)}$ can be expressed by (5) purely in terms of the properties of the corresponding $a(\omega)$. However, I do not know if one can also express the condition (2) for the local convergence of chromatic expansions by properties of $a(\omega)$ alone. If this is not possible, one would like to at least express condition (2) in terms of the asymptotic behavior of the coefficients γ_n, so that the conditions for both types of convergence of chromatic expansions are formulated in the same terms. This explains my interest in Questions 1 and 4.

The notion of chromatic derivatives could be extended even further, see very interesting work of Ahmed Zayed in [8] and a related general Question 1 in [1]. Chromatic expansions have also been generalized to several dimensions in [3].

Periodic functions do not belong to the space $L^2_{\mathcal{M}}$; for example $\sum_{n=0}^{\infty} |\mathcal{K}^n[\mathrm{e}^{\mathrm{i}\,\omega t}]|^2 = \mathrm{e}^{2\,\mathrm{i}\,\omega t}\sum_{n=0}^{\infty} P_n(\omega)^2$ diverges. However, there are some interesting (and useful) nonseparable Hilbert spaces in which pure harmonic oscillations have finite norms. Following [1], assume that $\mathcal{M}$ satisfies (2) and let $\mathcal{C}^{\mathcal{M}}$ be the vector space of real analytic functions such that the sequence $\nu_n^f(t) = 1/(n+1)^{1-p}\sum_{k=0}^{n} \mathcal{K}^k[f](t)^2$ converges uniformly on every compact $I \subset \mathbb{R}$.

Proposition 6 (6.2. in [1])**.** *Let* $f, g \in \mathcal{C}^{\mathcal{M}}$ *and* $\sigma_n^{fg}(t) = 1/(n+1)^{1-p}\sum_{k=0}^{n} \mathcal{K}^k[f](t)\mathcal{K}^k[g](t)$*; then the sequence* $\{\sigma_n^{fg}(t)\}_{n\in\mathbb{N}}$ *converges to a constant function. In particular,* $\{\nu_n^f(t)\}_{n\in\mathbb{N}}$ *also converges to a constant.*

Corollary 7 (6.3. in [1])**.** *Let* $\mathcal{C}_0^{\mathcal{M}}$ *be the space of functions* $f(t)$ *such that* $\lim_{n\to\infty}\nu_n^f(t) = 0$*; then in the quotient space* $\mathcal{C}_2^{\mathcal{M}} = \mathcal{C}^{\mathcal{M}}/\mathcal{C}_0^{\mathcal{M}}$ *we can introduce a scalar product and a corresponding norm by the following formulas whose right hand sides are independent of* t*:*

$$
(8) \qquad \langle f, g\rangle^{\mathcal{M}} = \lim_{n\to\infty} \frac{1}{(n+1)^{1-p}} \sum_{k=0}^{n} \mathcal{K}^k[f](t)\,\mathcal{K}^k[g](t); \qquad \|f\|^{\mathcal{M}} = \sqrt{\langle f, f\rangle^{\mathcal{M}}} = \lim_{n\to\infty}\sqrt{\nu_n^f(t)}.
$$

In the cases of the Legendre, Chebyshev and Hermite polynomials, for every ω in the support $\mathrm{sp}(a)$ of the corresponding $\mathrm{d}a(\omega)$ the pure harmonic oscillations of the form $f_\omega(t) = e^{\mathrm{i}\,\omega t}$ have a finite norm $\|f_\omega\|^{\mathcal{M}}$, and for **every** two distinct and positive $\omega_1, \omega_2 \in \mathrm{sp}(a)$, the corresponding $f_{\omega_1}, f_{\omega_2}$ are mutually orthogonal (thus, in case of the Hermite polynomials, every two pure harmonic oscillations with distinct positive frequencies have finite positive norms and are mutually orthogonal). We conjectured in [1] that this is true for every $\mathcal{M}$ which for some $0 \le p < 1$ satisfies (2), and the conjecture from Question 3 just asserts that for every $\omega \in \mathrm{sp}(a)$ the corresponding f_ω has a positive finite norm.

The norm $\|f\|^{\mathcal{M}}$ has practical significance because functions $\nu_n^f(t)$ behave as envelopes of $f(t)$, and for small values of n (e.g., 2 - 5) they closely approximate the envelope obtained by the usual signal processing method based on the Hilbert Transform. As n increases, such envelopes become flatter, approaching a constant. The quotient $\sqrt{\nu_n^{f'}(t)/\nu_n^f(t)}$ for small values of n is a good approximation of the local frequency of the signal, again as defined in signal processing via the Hilbert transform, and as n increases such local frequency gradually "delocalizes" (averages over neighborhoods of t of increasing size).

Finally, I hereby solemnly promise that anyone who gives me any serious comment on these questions and comes to Sydney I will take him/her for a fabulous dinner. Seriously.

References

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