Frequency estimation using time domain methods based on robust differential operators

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***Abstract*—Given a band limited signal which over some disjoint intervals of time I_n behaves as a corresponding linear combination $f_n(t)$ of up to N damped sinusoids, we present a method which detects intervals I_n, determines the number of the sinusoidal components over each interval and estimates their frequencies, with high accuracy and in the presence of noise which is not necessarily white. Intervals I_n can have very short duration of just a dozen Nyquist rate intervals, thus insufficient for use of the Fourier transform based methods. Our method operates entirely in the time domain; to be applicable, the signal must be sampled at twice the Nyquist rate. It is based on analyzing local signal behavior using special, numerically robust linear differential operators, called the chromatic derivatives, which were introduced relatively recently, and which hold an unexplored promise in signal and image processing.**

I. Introduction

Estimation of frequencies of several sinusoids in the presence of noise usually relies on evaluation of some form of the Fourier transform of a section of the signal. However, "viewing" the signal through a window distorts its Fourier transform, by convolving it with the Fourier transform of the window. As a consequence, the resulting side lobes of strong sinusoidal components and the noise can mask the main lobes of the weaker components.

Unlike the Fourier transform methods which allow spectral analysis of a vast class of signals, our approach uses a method which is applicable only to signals which, over intervals of interest, are sums of a small number of (possibly damped) sinusoids. Such highly specific method allows accurate estimation of frequencies of the sinusoidal components even if their duration is only a dozen or so Nyquist rate intervals and in the presence of colored noise, provided that the signal is sampled at twice the Nyquist rate. The method is based on the following well known fact: *A function f is a linear combination of n exponentially damped and phase shifted sinusoids over an interval I if and only if f satisfies on I a homogeneous linear differential equation with constant coefficients of order $2n$.*

Thus, we design our algorithm by refining the following basic idea. We sequentially examine sections of the signal over short intervals of time, looking for the smallest $n \leq N$ for which there is a differential equation of order $2n$ which is satisfied by such section of the signal, modulo an error which is commensurate with the level of the noise present. *Note that the fact that the signal is analyzed over short intervals of time has no negative consequences similar to those associated with the use of windowing in the Fourier transform based methods, because whether a signal $f(t)$ satisfies a differential equation at a particular instant in time is a purely local feature of the signal, determined by its behavior in an arbitrarily small neighborhood around this instant.*

The above strategy, however, immediately raises a concern: if we are looking for sections of the signal which are sums of, say, four sine waves, this would involve evaluating differential operators of order eight, applied to a noisy signal. As is well known, numerical differentiation of such high order results in insurmountable numerical problems. *This is where the chromatic derivatives crucially intervene*. Chromatic derivatives are special linear differential operators with constant coefficients whose numerical evaluation is highly accurate and noise robust, even for operators of very high orders (> 20).

II. A brief summary of chromatic derivatives

As is well known, truncations of the expansion of a π-band limited signal of finite energy, $f \in \mathbf{BL}(\pi)$, provided by the Sampling Theorem, $f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,(t-n)$, do not provide good local signal approximations, because the values of $\mathrm{sinc}\,(t-n)$ decay slowly as $|n|$ grows. Thus, to achieve a good local approximation, a very large number of the Nyquist rate samples $f(n)$ are required.

On the other hand, $\mathbf{BL}(\pi)$ signals are analytic functions which can be locally represented by the truncations of the Taylor expansion $f(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \; t^n/n!$. Note that Taylor's expansion is local in nature, because the values of the derivatives $f^{(n)}(0)$ are determined by the values of the signal in an arbitrarily small neighborhood of zero.

However, Taylor's formula has found very limited use in signal processing. This is due to the fact that an accurate evaluation of derivatives of higher orders from discrete noisy samples of a signal is essentially impossible. Moreover, the functions used in the expansion, i.e., the monomials $t^n/n!$, do not correspond to band limited signals; the approximation is unbounded, it converges neither uniformly nor in L_2 and its error increases rapidly when moving away from the center of expansion. Chromatic derivatives and chromatic expansions were introduced in [1] and [2], respectively, to provide a numerically feasible framework for numerical differentiation and for local approximation of band limited signals which do not suffer from the above problems. They were first published in [3]–[5]; their properties were examined in detail in [6]–[8].

1) Chromatic derivatives: Chromatic derivatives are linear differential operators with constant coefficients obtained from suitably chosen families of orthonormal polynomials.[1] Thus, let polynomials $P_n(\omega)$ satisfy

$$\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)w(\omega)d\omega = \delta(m-n), \tag{1}$$

where $w(\omega)$ is a non-negative symmetric weight function. We define linear differential operators associated with such family of *orthonormal* polynomials by the operator polynomials

$$\mathcal{K}^n = (-j)^n P_n\left(j\,\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{2}$$

Thus, $\mathcal{K}^n$ is obtained by replacing ω^k in $P_n(\omega)$ by $j^k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(t)$. It is easy to verify that

$$\mathcal{K}^n_t[\mathrm{e}^{j\,\omega t}] = j^n P_n(\omega)\,\mathrm{e}^{j\,\omega t}. \tag{3}$$

Thus, if $f \in \mathbf{BL}(\pi)$ and $\widehat{f}(\omega)$ is its Fourier transform, then

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n P_n(\omega)\widehat{f}(\omega)e^{j\,\omega t}d\omega. \tag{4}$$

Polynomials $P_n(\omega)$ satisfy that for every $a < \pi$ there exists $M > 0$ such that $|P_n(\omega)| < M$ for all n and all $|\omega| \leq a$. In comparison, if we normalize the "standard" derivatives so that the magnitudes of their frequency responses are bounded uniformly in n, we get

$$\frac{f^{(n)}(t)}{\pi^n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n \left(\frac{\omega}{\pi}\right)^n \widehat{f(\omega)}\mathrm{e}^{j\,\omega t}d\omega. \tag{5}$$

Figure 1 compares the plots of the transfer functions $(\omega/\pi)^n$ of the normalized derivatives $1/\pi^n\ \mathrm{d}^n/\mathrm{d}t^n$ (modulo a factor of j^n) with the plots of the transfer functions $P^L_n(\omega)$ of the chromatic derivatives $\mathcal{K}^n$ associated with the (normalized and re-scaled) Legendre polynomials (right).[2] Plots on the left reveal why numerical evaluation of higher order derivatives from signal samples makes no practical sense: multiplication of the Fourier transform of a signal by the transfer function of a derivative of high order essentially obliterates the spectrum of the signal, leaving only its edges, which in practice contain mostly noise. Note also that the graphs of the transfer functions of the normalized derivatives of high orders and of the same parity cluster together tightly, becoming essentially indistinguishable.

In comparison, Figure 1 (right) shows that the transfer functions of the chromatic derivatives $\mathcal{K}^n$ form a family of well separated, interleaved and increasingly refined comb-like filters. Instead of obliterating, such operators encode the spectral features of the signal and for this reason we call operators $\mathcal{K}^n$ *the chromatic derivatives.*

Chromatic derivatives replace the usual base $\{f, f', f'', \ldots\}$ of the vector space of linear differential operators with an orthonormal base which has many remarkable properties.

First of all, chromatic derivatives can be evaluated using FIR filters operating on samples of the signal taken at twice the Nyquist rate. Such filters can be designed using the

[1]See [8] for the details regarding which families of orthogonal polynomials produce satisfactory families of chromatic derivatives.

[2]Such polynomials are obtained by taking for $w(\omega)$ appearing in (1) the constant weight function $w(\omega) = (2\pi)^{-1}$.

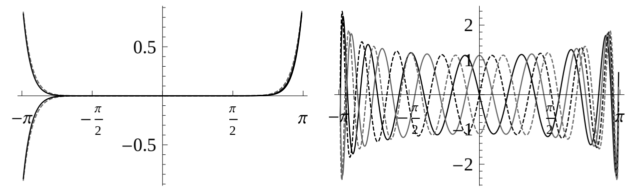

Fig. 1. Graphs of $(\omega/\pi)^n$ (left) and $P^L_n(\omega)$ (right) for $n = 15$ to $n = 18$.

Remez exchange method and are both very accurate and noise robust. For example, a 129 tap transversal filter approximating differential operator $\mathcal{K}^{15}$ corresponding to the re-scaled and normalized Legendre polynomial of order 15, with pass-band occupying 90% of the interval $[-\pi/2, \pi/2]$ and the transition region extending 10% of the bandwidth $\pi/2$ on each side of the boundaries $-\pi/2$ and $\pi/2$, outside the transition region has an error of approximation smaller than 1.3×10^{-4}.

2) Chromatic approximations: Chromatic derivatives can be used to produce local approximations of band limited signals which do not suffer from the mentioned shortcomings of the Taylor expansion; we formulate here only the special case corresponding to the Legendre polynomials normalized and rescaled to $[-\pi, \pi]$; for the general case see [7] or [8].

Proposition 2.1: Let $\mathcal{K}^n$ be the chromatic derivatives associated with the Legendre polynomials, and let $f(t)$ be any function analytic on $\mathbb{R}$; then for all $t \in \mathbb{R}$,

$$f(t) = \sum_{n=0}^{\infty}(-1)^n\ K^n[f](u)\ K^n[\mathrm{sinc}\,](t-u).$$

If in addition $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on $\mathbb{R}$ and in the L_2 norm.

The series in (6) is called *the chromatic expansion of f associated with the Legendre polynomials*; a truncation of this series to first $n+1$ terms is called a *chromatic approximation* of f of order n. Just as Taylor's approximation, a chromatic approximation is also a local approximation: its coefficients are the values of differential operators $\mathcal{K}^m[f](u)$ at a single instant u, and the values of its derivatives of orders up to n at instant u are equal to the values of the corresponding chromatic derivatives of $f(t)$. However, unlike the monomials in the Taylor formula, expansion functions $\mathcal{K}^n[\mathrm{sinc}\,](t)$ are band limited signals, and the above theorem indicates that the approximation error accumulates much slower than the error of the Taylor approximation of the same order.

III. Frequency Estimation

Assume that a $\mathbf{BL}(\pi/2)$ signal $f(t)$ is sampled at integers, thus at twice the Nyqist rate, and that such samples $f(i)$ are corrupted by a zero mean WSS stochastic noise $\nu(i)$, with an autocorrelation function $r(k)$. The power spectral density of the noise is then $S(\omega) = \sum_{k=-\infty}^{\infty} r(k)\mathrm{e}^{-j\,\omega k}$. Let $\rho^2 = \int_{-\pi/2}^{\pi/2} S(\omega)d\omega$; Thus, ρ is equal to the RMS value of the noise component which is within the bandwidth of $f(t)$.

Let $\chi_{\pi/2}(\omega) = 1$ for $|\omega| \leq \pi/2$ and zero otherwise, and let $P_n(\omega)$ be the family of polynomials orthonormal on $[-\pi, \pi]$ with respect to the weight function $w(\omega) =$

$1/(2\pi\rho^2)S(\omega)\chi_{\pi/2}(\omega)$, i.e., such that

$$\tfrac{1}{2\pi}\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)S(\omega)\chi_{\pi/2}(\omega)d\omega = \delta(m-n)\rho^2. \quad (6)$$

This is equivalent to saying that $P_n(\omega/2)$ are orthonormal on $[-\pi,\pi]$ with respect to the weight function $w^*(\omega) = 1/(2\pi\rho^2)S(\omega/2)$. Let $\mathcal{K}^n$ be the chromatic derivatives which correspond to the polynomials $P_n(\omega/2)$ via (2).

Let $B = \sum_{n=0}^{N} X_n\mathcal{K}^n$ be any differential operator of order N, represented in the base of chromatic derivatives $\mathcal{K}^n$, such that its coefficients satisfy $\sum_{n=0}^{N} X_n^2 = 1$. Let also $\kappa^n[f](t) = \sum_{p=-L}^{L}\lambda_p^n f(t+p)$ be a $2L+1$ tap transversal filter approximation of $\mathcal{K}^n[f](t)$, and let $b = \sum_{n=0}^{N} X_n\kappa^n$ be the corresponding FIR approximation of B. Then (4) implies

$$\sum_{p=-L}^{L}\lambda_p^n \mathrm{e}^{-j\,\omega p} \approx j^n P_n(\omega)\chi_{\pi/2}(\omega) \quad (7)$$

for all $|\omega| < \pi$ and thus (6) implies

$$\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{p=-L}^{L}\sum_{q=-L}^{L}\lambda_p^n\lambda_q^m\sum_{k=-\infty}^{\infty} r(k)\mathrm{e}^{-j\,\omega(q-p-k)}d\omega \approx j^{n-m}\delta(m-n)\rho^2. \quad (8)$$

Integrating term by term and collecting the non zero terms,

$$\sum_{p=-L}^{L}\sum_{q=-L}^{L}\lambda_p^n\lambda_q^m r(q-p) \approx \delta(m-n)\rho^2. \quad (9)$$

Since $E[\nu(t+p)\nu(t+q)] = r(q-p)$, (9) impies

$$E\left[\sum_{p=-L}^{L}\sum_{q=-L}^{L}\lambda_p^n\nu(t+p)\lambda_q^m\nu(t+q)\right] \approx \delta(m-n)\rho^2,$$

i.e., $$E\left[\kappa^n(\nu)(t)\kappa^m(\nu)(t)\right] \approx \delta(m-n)\rho^2. \quad (10)$$

Thus, if the filters are chosen to correspond to the power spectral density of the noise, then the noise errors of filters evaluating chromatic derivatives of different orders are essentially uncorrelated and have an RMS value equal to the RMS value of the $\pi/2$ in-band component of the noise.

Since $b[\nu](t)$ is a finite linear combination of samples of $\nu(t)$ and since ν is of zero mean, $b[\nu](t)$ is also of zero mean. Using linearity of b, it is easy to see that this yields

$$E\left[(b[f+\nu](t))^2\right] = (b[f](t))^2 + E\left[(b[\nu](t))^2\right].$$

This implies, with assumption that $\sum_{n=0}^{N} X_n^2 = 1$, (10) and $E\left[b[\nu](t)^2\right] = E\left[\sum_{n=0}^{M}\sum_{m=0}^{M} X_nX_m\kappa^n[\nu](t)\kappa^m[\nu](t)\right]$,

$$E\left[(b[f+\nu](t))^2\right] = (b[f](t))^2 + \rho^2. \quad (11)$$

This is a remarkable fact, because it shows that, if the chromatic derivatives are chosen to correspond to the power spectral density of the noise via (6), then the RMS impact of the noise on any differential operator $B = \sum_{n=0}^{M} X_n\mathcal{K}^n$ *is independent of the values of the coefficients* X_n, and is always equal to the RMS value of the in band component of the noise.

Let Q be any fixed natural number, t a fixed instant of time and $\vec{X} = (X_0,\ldots,X_N)$. If we set

$$R(\vec{X}) = 1/(Q+1)\sum_{p=0}^{Q}(b[f+\nu](t+p))^2, \quad (12)$$

then, by (11),

$$E\left[R(\vec{X})\right] = 1/(Q+1)\sum_{p=0}^{Q}(b[f](t+p))^2 + \rho^2. \quad (13)$$

Clearly, $E\left[R(\vec{X})\right] \geq \rho^2$ and, since the transversal filters κ^n are close approximations of differential operators $\mathcal{K}^n$, $E\left[R(\vec{X})\right] \approx \rho^2$ if and only if $f(t)$ satisfies the differential equation $B[f](x) = 0$ at instants $x = t,\ldots t+Q$.

Thus, to determine if f satisfies a differential equation of order N over $[t, t+Q]$, we should find $\vec{X}$ which minimizes the value of $R(\vec{X})$ and see if such value is approximately equal to ρ^2. To find the minimum of $R(\vec{X})$ subject to the constraint $\sum_{n=0}^{N} X_n^2 = 1$, we apply the Lagrangian multipliers and set to zero the partial derivatives with respect to a new variable β and variables $X_0,\ldots,X_N$, of the objective $R(\vec{X}) - \beta\sum_{n=0}^{N} X_n^2$. Letting $f_\nu(t) = f(t)+\nu(t)$, this gives the following equations: $\|\vec{X}\|^2 = 1$, plus for each m, $0 \leq m \leq N$,

$$-\beta X_m + \sum_{p=0}^{Q}\sum_{n=0}^{N} X_n\kappa^n[f_\nu](t+p)\kappa^m[f_\nu](t+p) = 0. \quad (14)$$

Let $C[m,n] = \frac{1}{1+Q}\sum_{p=0}^{Q}\kappa^n[f_\nu](t+p)\kappa^m[f_\nu](t+p)$ and let $C = (C[m,n])_{m,n=1..N}$ be the corresponding matrix; then the above equations become $C\,\vec{X} = \beta\vec{X}$, i.e, β is an eigenvalue of C and $\vec{X}$ is the corresponding eigenvector of a unit norm. Note that (12) implies $R(\vec{X}) = \sum_{m=0}^{N}\sum_{n=0}^{N} X_mX_nC[m,n]$; thus, $R(\vec{X}) = \langle C\,\vec{X},\vec{X}\rangle$. Since $R(\vec{X}) \geq \rho^2 > 0$, C is a symmetric positive definite matrix and so all of its eigenvalues are positive reals. If $\vec{X}$ is the unit eigenvector for the eigenvalue β, then $R(\vec{X}) = \langle C\,\vec{X},\vec{X}\rangle = \langle\beta\vec{X},\vec{X}\rangle = \beta$; thus, to minimize $R(\vec{X})$ we must choose the unit eigenvector $\vec{X}$ corresponding to the smallest eigenvalue β_m. Consequently, we conclude that, with high probability, $f(t)$ satisfies a differential equation of the form $\sum_{n=0}^{N} X_n\mathcal{K}^n[f](t) = 0$ over the interval $[t, t+Q]$ just in case the smallest eigenvalue β_m of C is approximately equal to ρ^2, i.e., just in case $\beta_m < c\rho^2$ where $c > 1$ and $c \approx 1$.

If $\beta_m \approx \rho^2$, to find the fundamental solutions of the corresponding differential equation $\sum_{n=0}^{N} X_n\mathcal{K}^n[f](t) = 0$ with the coefficients $(X_0,\ldots,X_N)$ coming from the eigenvector corresponding to the smallest eigenvalue β_m of C, we numerically solve the associated algebraic equation $\sum_{n=0}^{N} X_n(-j)^nP_n(-jz) = 0$. The imaginary part of each conjugate pair of solutions $z_i, \overline{z_i}$ of this equation is the frequency ω_i and the real part is the damping factor δ_i of its fundamental solutions $\mathrm{e}^{\delta_i t}\sin(\omega_i t)$ and $\mathrm{e}^{\delta_i t}\cos(\omega_i t)$; a linear combination of these fundamental solutions gives the i^{th} sinusoidal component of the signal; to obtain the amplitude and the phase of such component (if needed) one must use initial conditions given by $N+1$ particular values of $\mathcal{K}^n[f](u)$ for some $1 \leq n \leq N$, $u \in [t, t+Q]$.

3) Test results: To tests or frequency estimation algorithm, we have used a method described in [10] to join fragments of linear combinations of up to 4 damped or expanding sinusoids of frequencies of up to $.47\pi$. Thus, for such class of sinusoids, sampling at integers is twice the Nyquist rate. These fragments of various durations were spaced at least 32 unit intervals apart. They were joined by fragments of $\pi/2$ band limited signals produced using chromatic approximations (Proposition 2.1), in a way which ensured that the resulting interpolated signal has continuous derivatives of all orders up to fifteen. Such a smooth signal is guaranteed to have a very low content in frequencies above $\pi/2$ due to a version of a classical theorem in Harmonic Analysis, see [10]. In our experiments out of band energy of such signals was less than 10^{-5} of the total signal energy. The signal is then sampled at integers, the samples are corrupted by white Gaussian noise ν, and passed through the filter bank $\{\kappa^n\}_{n=0}^{8}$ corresponding to the Legendre polynomials.[3] Signals with a low out of band content can be accurately differentiated using chromatic derivative filter banks, even in the presence of significant noise.

We then calculate for every sampling point the corresponding values $C[m,n]$ for all $m,n \leq 8$, with a value of Q between 12 and 20, depending on the maximal number of sinusoids to be detected, and form matrices $(C[m,n])_{m,n=0}^{2k}$, $1 \leq k \leq 4$. We now look for the smallest $k \leq 4$, if it exists, such that the smallest eigenvalue of $(C[m,n])_{m,n=0}^{2k}$ is smaller than $1.1\rho^2$.

Figure 2 shows the plots of the values of the square root of the smallest eigenvalue of $(C[m,n])_{m,n=0}^{4}$ (top), and of $(C[m,n])_{m,n=0}^{2}$ (bottom), for the case where each fragment was either a single damped sinusoid or sum of two such sinusoids, with the corresponding time supports represented by the gray rectangles of height 1 or 2, respectively. The horizontal line corresponds to the threshold 1.1ρ for the S/N of 25 db. The above plots show that the algorithm correctly identifies fragments I_n as well as the number of sinusoids over each fragment. More over, it also indicates that we do not need to know in advance the signal to noise ratio, because the regions where the corresponding eigenvalues are consistently small over an interval can be identified and used to determine the signal to noise ratio.

We now summarize the results of our preliminary testing of a very direct implementation of the algorithm, which leaves much room for further improvement.

We first generated a signal consisting of fragments of single sinusoids with supports of length of only eight sampling intervals. Since these fragments are joined into a fifteen times differentiable signal, the interpolation functions provide a close approximation over additional intervals of length about eight sampling intervals on each side of the support interval. Thus the total duration of a fragment is about twenty four sampling intervals. Since we sample twice the Nyquist rate, the total duration over which the signal is a close approximation of a corresponding single sinusoid is only about twelve Nyquist rate intervals. The samples are corrupted with white Gaussian noise, with S/N= 25 db. The algorithm correctly finds the corresponding support intervals, missing on average less than 2% of the total number of intervals and falsely detecting about as many spurious intervals. The frequency of the sinusoids is estimated with an RMS error consistently smaller than 0.01 radians, despite duration of only twelve Nyquist rate intervals; compare such accuracy with the resolution of the FFT over only twelve Nyqist rate intervals.

[3]This is an extremely recent piece of work and we had no time to implement the filters needed to test the case of colored noise.

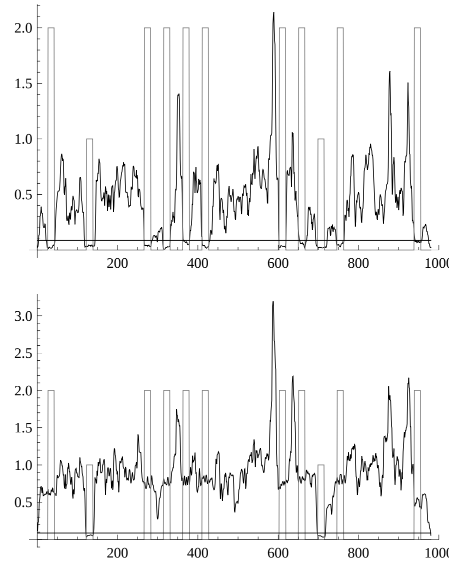

Fig. 2. The smallest eigenvalue of $(C[m,n])_{m,n=0}^{4}$ (top, black) and of $(C[m,n])_{m,n=0}^{2}$ (bottom, black). Markers showing the position and the number of components of fragments in gray.

With signals consisting of linear combinations of at most two sinusoids of equal amplitude and with supports of 16 sampling intervals the number of misses or spurious intervals was less than $\%5$, with the RMS error of frequency estimation again smaller than 0.01 radians. The same results were achieved with signals which are linear combinations of at most three sinusoids and with supports of length of 24 sampling intervals. Finally, signals with four components had supports of 32 sampling intervals, thus with total duration of about 48 samples, i.e., 24 Nyquist rate intervals. In this case the number of misclassified intervals was about 10% and the accuracy of frequency estimation was about 0.02 radians. The programs used in simulations and the references can be found at http://www.cse.unsw.edu.au/˜ignjat/diff.

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