Local Approximations Based on Orthogonal Differential Operators

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ABSTRACT. Let $\mathcal{M}$ be a symmetric positive definite moment functional and let $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ be the family of orthonormal polynomials that corresponds to $\mathcal{M}$. We introduce a family of linear differential operators $\mathcal{K}^n = (-\mathrm{i})^n P_n^{\mathcal{M}}(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t})$, called the chromatic derivatives associated with $\mathcal{M}$, which are orthonormal with respect to a suitably defined scalar product. We consider a Taylor type expansion of an analytic function $f(t)$, with the values $f^{(n)}(t_0)$ of the derivatives replaced by the values $\mathcal{K}^n[f](t_0)$ of these orthonormal operators, and with monomials $(t-t_0)^n/n!$ replaced by an orthonormal family of "special functions" of the form $(-1)^n\mathcal{K}^n[\boldsymbol{m}](t-t_0)$, where $\boldsymbol{m}(t) = \sum_{n=0}^{\infty}(-1)^n\mathcal{M}(\omega^{2n})\,t^{2n}/(2n)!$. Such expansions are called the chromatic expansions. Our main results relate the convergence of the chromatic expansions to the asymptotic behavior of the coefficients appearing in the three term recurrence satisfied by the corresponding family of orthogonal polynomials $P_n^{\mathcal{M}}(\omega)$. Like the truncations of the Taylor expansion, the truncations of a chromatic expansion at $t = t_0$ of an analytic function $f(t)$ approximate $f(t)$ locally, in a neighborhood of t_0. However, unlike the values of $f^{(n)}(t_0)$, the values of the chromatic derivatives $\mathcal{K}^n[f](t_0)$ can be obtained in a noise robust way from sufficiently dense samples of $f(t)$. The chromatic expansions have properties which make them useful in fields involving empirically sampled data, such as signal processing.

1. Introduction

Let $\mathbf{BL}(\pi)$ denote the set of π-band limited signals of finite energy, i.e., the set of continuous L_2 functions with a Fourier transform supported within the interval $[-\pi,\pi]$. Nyquist's expansion, $f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,\pi(t-n)$, with

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$\operatorname{sinc} t = \sin t/t$, represents a signal $f \in \mathbf{BL}(\pi)$ using its samples at all integers. This makes Nyquist's expansion *global in nature.* On the other hand, every $f \in \mathbf{BL}(\pi)$ is also an analytic function that can be represented by its Taylor series, using the values of all derivatives of f at a single instant, say t_0. Since the values of the derivatives of f at t_0 are determined by the values of f in an arbitrarily small neighborhood of t_0, Taylor's expansion is *local in nature.* Consequently, these two expansions are complementary, and this is reflected in a very different behavior of the error terms of approximations obtained by truncating the two corresponding series.

Unlike the Nyquist expansion that has a most fundamental role in digital signal processing, the Taylor expansion has found very limited practical use there, for several reasons. First of all, numerical differentiation is noise sensitive. As put in [10], *"... numerical differentiation should be avoided whenever possible, particularly when the data are empirical and subject to appreciable errors of observation."* Secondly, truncations of the Nyquist series of a function $f \in \mathbf{BL}(\pi)$ belong to $\mathbf{BL}(\pi)$ and converge to f; thus, the action of a continuous linear operator A on a signal $f \in \mathbf{BL}(\pi)$ can be approximated by the action of A on truncations of the Nyquist series representing f. However, nothing similar is true of the Taylor expansion.

In this paper we develop the theory of *chromatic derivatives* and the associated *chromatic expansions* from a new perspective and in a more general way than was done in the past, either by the present author who introduced these notions in [11], or by other contributors; see our references. Chromatic expansions are of equally local nature as Taylor's, but do not suffer from any of the problems mentioned above. They can provide a numerically robust framework for dealing with phenomena that are most naturally described using differential operators.

2. Basic Notions

2.1 Moment functionals

Let $\mathcal{M}_\omega : \mathcal{P}_\omega \to \mathbb{R}$ be a linear functional on the vector space $\mathcal{P}_\omega$ of real polynomials in the variable ω. Such $\mathcal{M}_\omega$ is called a moment functional and $\mu_n = \mathcal{M}_\omega(\omega^n)$ is the moment of $\mathcal{M}_\omega$ of order n. For all $m, n \geq 0$, let

$$\Delta_n^m = \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \mu_1 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ \mu_{n+m} & \cdots & \mu_{2n+m} \end{vmatrix}. \tag{2.1}$$

Thus, Δ_n^0 is the Hankel determinant of order n, and we also denote it by Δ_n. We will only consider moment functionals satisfying the following three conditions:

1. $\mathcal{M}_\omega$ is symmetric, i.e., $\mu_{2n+1} = 0$ for all n.
2. $\mathcal{M}_\omega$ is positive definite; thus, $\Delta_n > 0$ and $\mu_{2n} > 0$ for all n.
3. $$\limsup_{n\to\infty} \left(\frac{\mu_n}{n!}\right)^{1/n} = \mathrm{e}\ \limsup_{n\to\infty} \frac{\mu_n^{1/n}}{n} < \infty. \tag{2.2}$$

Given a functional $\mathcal{M}_\omega : \mathcal{P}_\omega \to \mathbb{R}$ satisfying these three conditions, we introduce a corresponding linear functional $\mathcal{M}_t : \mathcal{D}_t \to \mathbb{R}$ on the vector space $\mathcal{D}_t$ of linear differential operators with constant real coefficients, $A_t = \alpha_0 + \alpha_1 \frac{\mathrm{d}}{\mathrm{d}t} + \cdots + \alpha_n \frac{\mathrm{d}^n}{\mathrm{d}t^n}$, by $\mathcal{M}_t(\mathrm{d}^n/\mathrm{d}t^n) = \mathrm{i}^n \mu_n$; since $\mathcal{M}_\omega$ is symmetric, $\mathcal{M}_t$ is real valued.[1] We will call $\mathcal{M}_t$ a moment functional on $\mathcal{D}_t$, and $\mathcal{M}_t(\mathrm{d}^n/\mathrm{d}t^n)$ is the moment of $\mathcal{M}_t$ of order n. When no confusion arises, we drop the subscripts and use $\mathcal{M}$ for both $\mathcal{M}_\omega$ and $\mathcal{M}_t$, and $\mathcal{D}$ for $\mathcal{D}_t$.

Let $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ be the family of orthonormal polynomials that corresponds to a symmetric positive definite moment functional $\mathcal{M}$; thus, for all m, n, $\mathcal{M}(P_m^{\mathcal{M}}(\omega)\, P_n^{\mathcal{M}}(\omega)) = \delta(m-n)$. We associate with $\mathcal{M}$ a family of linear differential operators $\{\mathcal{K}^n\}_{n\in\mathbb{N}}^{\mathcal{M}}$ defined by the operator polynomial

$$\mathcal{K}^n = (-\mathrm{i})^n P_n^{\mathcal{M}}\left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{2.3}$$

Such operators have real coefficients because $\mathcal{M}$ is symmetric. Since polynomials $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ are orthonormal, for all $n \geq 0$ there exist $\gamma_n > 0$ such that, if we also set $\gamma_{-1} = 0$ and $P_{-1}^{\mathcal{M}}(\omega) \equiv 0$, the three term recurrence

$$P_{n+1}^{\mathcal{M}}(\omega) = \frac{1}{\gamma_n}\,\omega\, P_n^{\mathcal{M}}(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\, P_{n-1}^{\mathcal{M}}(\omega) \tag{2.4}$$

holds for all $n \geq 0$; see e.g. [2]. On the other hand, operators $\{\mathcal{K}^n\}_{n\in\mathbb{N}}^{\mathcal{M}}$ can be shown to satisfy the recurrence

$$\mathcal{K}^{n+1} = \frac{1}{\gamma_n}\,(\mathrm{d} \circ \mathcal{K}^n) + \frac{\gamma_{n-1}}{\gamma_n}\,\mathcal{K}^{n-1}, \tag{2.5}$$

with the same coefficients γ_n as in (2.4), and that for all m and n,

$$\mathcal{M}[\mathcal{K}^n \circ \mathcal{K}^m] = (-1)^n \delta(m-n). \tag{2.6}$$

[1] We use the square brackets to indicate the arguments of operators acting on various function spaces. If A is a linear differential operator, and if a function $f(t, \vec{w})$ has parameters $\vec{w}$, we write $A_t[f]$ to distinguish the variable t of differentiation; if $f(t)$ contains only variable t, we write $A[f]$ for $A_t[f]$ and $\mathrm{d}^k[f]$ for $\mathrm{d}^k f/\mathrm{d}t^k$.

Let $f, g \in C^{\infty}$; then the equality

$$\mathrm{d}\left[\sum_{m=0}^{n} \mathcal{K}^m[f]\,\mathcal{K}^m[g]\right] = \gamma_n\left(\mathcal{K}^{n+1}[f]\,\mathcal{K}^n[g] + \mathcal{K}^n[f]\,\mathcal{K}^{n+1}[g]\right) \tag{2.7}$$

corresponds to the Christoffel – Darboux identity for orthogonal polynomials and can be proved in a similar way, using (2.5) to form a telescopic sum.

From the corresponding properties of orthogonal polynomials, see e.g. [2], one can show that

$$\mathcal{K}^n = \frac{(-\,\mathrm{i})^n}{(\Delta_n\Delta_{n-1})^{1/2}} \begin{vmatrix} \mu_0 & \dots & \mu_n \\ \mu_1 & \dots & \mu_{n+1} \\ \vdots & & \vdots \\ \mu_{n-1} & \dots & \mu_{2n-1} \\ \mathrm{d}^0 & \dots & \mathrm{i}^n\mathrm{d}^n \end{vmatrix}. \tag{2.8}$$

Assume that $\mathcal{M}$ satisfies conditions 1 – 3; we define

$$\boldsymbol{m}(t) = \sum_{k=0}^{\infty} \mathrm{i}^k \mu_k \frac{t^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \mu_{2k} \frac{t^{2k}}{(2k)!}. \tag{2.9}$$

Inequality (2.2) ensures that the interval of convergence of $\boldsymbol{m}(t)$ is nontrivial. Since $\boldsymbol{m}^{(n)}(t) = \sum_{k=0}^{\infty} \mathrm{i}^n \mu_{n+k}\,(\mathrm{i}\,t)^k/k!$, we have $\boldsymbol{m}^{(n)}(0) = \mathrm{i}^n\mu_n$; thus,

$$\mathcal{M}(A) = A[\boldsymbol{m}](0) \tag{2.10}$$

for every $A \in \mathcal{D}$. Using (2.6) we get

$$\mathcal{M}[\mathcal{K}^n \circ \mathcal{K}^m] = (\mathcal{K}^n \circ \mathcal{K}^m)[\boldsymbol{m}](0) = (-1)^n\delta(m-n). \tag{2.11}$$

One can easily verify that (2.8) and (2.9) imply that for all k, n,

$$k < n \rightarrow (\mathrm{d}^k \circ \mathcal{K}^n)[\boldsymbol{m}](0) = 0; \tag{2.12}$$

$$(\mathrm{d}^{n+2k+1} \circ \mathcal{K}^n)[\boldsymbol{m}](0) = 0; \tag{2.13}$$

$$(\mathrm{d}^{n+2k} \circ \mathcal{K}^n)[\boldsymbol{m}](0) = \frac{(-1)^{n+k}\Delta_n^{2k}}{(\Delta_n\Delta_{n-1})^{1/2}}. \tag{2.14}$$

Thus,

$$\mathcal{K}^n[\boldsymbol{m}](t) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\Delta_n^{2k}}{(\Delta_n\Delta_{n-1})^{1/2}} \frac{t^{n+2k}}{(n+2k)!}. \tag{2.15}$$

2.2 Examples

We now give a few examples that show that for several families of classical orthogonal polynomials the corresponding $\boldsymbol{m}(t)$ are some familiar special functions. The first example below was introduced in [11]; the other examples were introduced in [3, 8]. The claims are easy consequences of well known properties of orthogonal polynomials and special functions involved.

Example 1. *(Chebyshev polynomials of the first kind/Bessel functions)* For the family of orthonormal polynomials obtained by normalizing the Chebyshev polynomials of the first kind, $T_n(\omega)$, by setting $P_0^{\mathcal{M}}(\omega) \equiv 1$ and $P_n^{\mathcal{M}}(\omega) = \sqrt{2}\ T_n(\omega)$ for $n > 0$, the corresponding function (2.9) is $\boldsymbol{m}(t) = J_0(t)$, and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{2}\,J_n(t)$, where $J_n(t)$ is the Bessel function of the first kind of order n. In the recurrence relation (2.5) the coefficients are given by $\gamma_0 = 1/\sqrt{2}$ and $\gamma_n = 1/2$ for $n > 0$.

Example 2. *(Legendre polynomials/Spherical Bessel functions)* Let $L_n(\omega)$ be the Legendre polynomials; if we let $P_n^{\mathcal{M}}(\omega) = \sqrt{2n+1}\,L_n(\omega)$ then $\boldsymbol{m}(t) = \operatorname{sinc} t$ and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\sqrt{(2n+1)\pi}j_n(t)$, where $j_n(t)$ is the spherical Bessel function of the first kind of order n, i.e., $j_n(t) = J_{n+1/2}(t)/\sqrt{2\,t}$. The corresponding recursion coefficients are given by $\gamma_n = (n+1)/\sqrt{4(n+1)^2-1}$. If we set $P_n^{\mathcal{M}}(\omega) = \sqrt{2n+1}\,L_n(\omega/\pi)$, then $\boldsymbol{m}(t) = \operatorname{sinc}\pi t$.

Example 3. *(Hermite polynomials/Gaussian monomial functions)* Let $H_n(\omega)$ be the Hermite polynomials; then the polynomials given by $P_n^{\mathcal{M}}(\omega) = (2^n\,n!)^{-1/2}\,H_n(\omega)$ are orthonormal. The corresponding function defined by (2.9) is $\boldsymbol{m}(t) = \mathrm{e}^{-t^2/4}$ and $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n\,t^n\mathrm{e}^{-t^2/4}/\sqrt{2^n\,n!}$. The corresponding recursion coefficients are given by $\gamma_n = \sqrt{(n+1)/2}$.

Example 4. *(Herron family)* Let $\{E_n\}_{n\in\mathbb{N}}$ be the Euler numbers; thus, $\operatorname{sech} t = \sum_{n=0}^{\infty} E_{2n}\,t^{2n}/(2n)!$ for $|t| < \pi/2$. If the family of orthonormal polynomials is given by $L_0(\omega) \equiv 1$, $L_1(\omega) = \omega$, and $L_{n+1}(\omega) = \omega/(n+1)L_n(\omega) - n/(n+1)L_{n-1}(\omega)$, then $\boldsymbol{m}(t) = \sum_{n=0}^{\infty} E_{2n}\,t^{2n}/(2n)!$ and the interval of convergence of $\boldsymbol{m}(t)$ is finite, $I_{\boldsymbol{m}} = (-\pi/2, \pi/2)$. In this case $\mathcal{K}^n[\boldsymbol{m}](t) = (-1)^n \operatorname{sech} t\,\tanh^n t$ and $\gamma_n = n+1$ for all $n \geq 0$. This example is a slight modification of an example due to Herron [8].

Note that in Example 1 and Example 2 the corresponding functions $\boldsymbol{m}(t)$ have finitely supported Fourier transforms, while in Example 2 and Example 3 the corresponding $\boldsymbol{m}(t)$ belong to L_2.

2.3 Numerical robustness of orthonormal differential operators

Using (2.4) and (2.5) we get

$$\mathcal{K}_t^n[\mathrm{e}^{\mathrm{i}\omega t}] = \mathrm{i}^n P_n^{\mathcal{M}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{2.16}$$

Thus, if $f(t)$ has a Fourier transform $\widehat{f(\omega)}$ such that the corresponding Fourier integral can be differentiated with respect to the variable t under the integral sign, then the Fourier transform of $\mathcal{K}^n[f](t)$ is $\mathrm{i}^n P_n^{\mathcal{M}}(\omega)\widehat{f(\omega)}$. This fact gives special properties to orthonormal differential operators acting over spaces of such functions, that the "ordinary" derivatives $\mathrm{d}^n/\mathrm{d}t^n$ do not have.

First of all, unlike the values of the derivatives $f^{(n)}(t)$, the values of orthogonal differential operators $\mathcal{K}^n[f](t)$ can be obtained in an accurate and noise robust way, either from sufficiently dense samples of $f(t)$, or, if $f(t)$ is an analog signal, by sampling the output of a bank of analog filters that correspond to operators $\mathcal{K}^n$. To explain this, we consider Example 1 corresponding to the Chebyshev polynomials and compare the behavior of the symbols, or, in signal processing terminology, the transfer functions $(\mathrm{i}\,\omega)^n$ of the standard derivatives $\mathrm{d}^n/\mathrm{d}t^n$ of orders $n = 13$ and $n = 15$, with the transfer functions $\sqrt{2}\,\mathrm{i}^n\,T_n(\omega)$ of the chromatic derivatives $\mathcal{K}^n$ of the same orders.

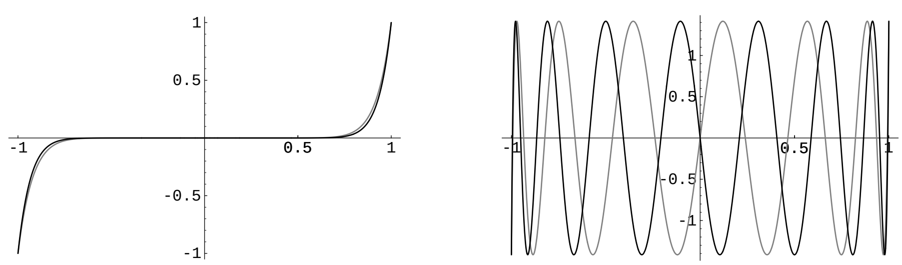

FIGURE 1

The left graph on Figure 1 shows the plots of ω^{13} (gray) and ω^{15} (black); the right graph shows the plots of $\sqrt{2}\,T_{13}(\omega)$ (gray) and $\sqrt{2}\,T_{15}(\omega)$ (black). While the transfer functions of the derivatives $\mathrm{d}^n/\mathrm{d}t^n$ of the same parity cluster tightly together and obliterate all but the edges of the spectrum (the Fourier transform) of a $\mathbf{BL}(1)$ signal, the transfer functions of the chromatic derivatives $\mathcal{K}^n$ form a family of well separated, interleaved and increasingly refined comb filters, that instead of obliterating, *encode* the spectral features of the signal. For this reason we call orthonormal differential operators the *chromatic derivatives.*

Figure 2 (left) compares the transfer function of the "standard" derivative of order fifteen (gray), restricted to the bandwidth $[-\pi/4, \pi/4]$, with the

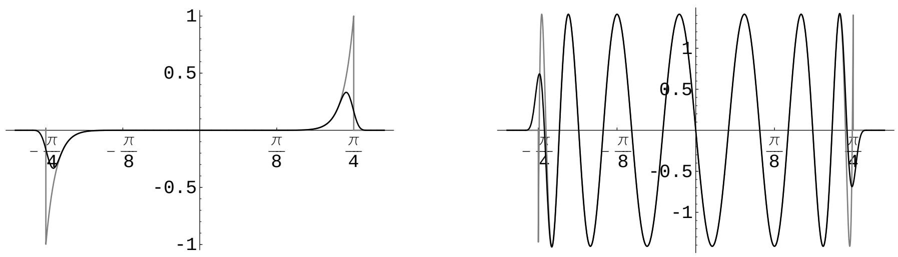

FIGURE 2

transfer function of its digital transversal filter implementation (black); Figure 2 (right) compares the transfer function of the chromatic derivative of the same order (gray) with the transfer function of its digital transversal filter implementation (black). Filters in both implementations have 256 taps spaced four taps per Nyquist rate interval (four times oversampling); their pass band occupies 90% of the interval $[-\pi/4, \pi/4]$ and their transition band extends 10% on each side of their band limit $\pi/4$. Both filters were designed using the Remez exchange algorithm; see e.g. [16]. Within both the pass band and the stop band the error of approximation of these filters is less than 10^{-4}.

The right graph on Figure 2 shows that, if the signal is moderately oversampled, it is possible to design digital filter banks which, for an input signal of bandwidth only 10% narrower than the band limit of the filterbank, provide both accurate and noise robust values of the chromatic derivatives of the input signal up to quite high orders. Such filter banks accurately encode the spectral features of the input signal and can be used in practical applications in fields that involve empirically sampled data, such as signal processing. On the other hand, from the left graph on Figure 2 it is clear that digital filters approximating the derivatives $\mathrm{d}^n/\mathrm{d}t^n$ of high orders provide essentially no useful information.

2.4 Chromatic expansions

The *chromatic expansion* centered at $t = u$ of an analytic function $f(t)$ is the formal series

$$\mathrm{CE}^{\mathcal{M}}[f, u](t) = \sum_{n=0}^{\infty} (-1)^n \, \mathcal{K}^n[f](u) \, \mathcal{K}^n[\boldsymbol{m}](t-u). \tag{2.17}$$

The truncation of the chromatic expansion to the first $n+1$ terms is denoted by $\mathrm{CA}^{\mathcal{M}}[f, n, u](t)$ and is called the *chromatic approximation* of $f(t)$ of order n, centered at $t = u$,

$$\mathrm{CA}^{\mathcal{M}}[f, n, u](t) = \sum_{k=0}^{n} (-1)^k \, \mathcal{K}^k[f](u) \, \mathcal{K}^k[\boldsymbol{m}](t-u). \tag{2.18}$$

The main objective of §3 is to study when a chromatic expansion of an analytic function $f(t)$ pointwise converges to $f(t)$, while in §4 we study when a chromatic expansion converges uniformly.

From (2.11) it follows that the chromatic approximation $\mathrm{CA}^{\mathcal{M}}[f,n,u](t)$ of order n of an analytic function $f(t)$ for all $m \leq n$ satisfies

$$\mathcal{K}_t^m[\mathrm{CA}^{\mathcal{M}}[f,n,u](t)]\big|_{t=u} = \sum_{k=0}^{n} (-1)^k \mathcal{K}^k[f](u)\, (\mathcal{K}^m \circ \mathcal{K}^k)[\boldsymbol{m}](0) = \mathcal{K}^m[f](u).$$

Since $\mathcal{K}^m$ is a linear combination of derivatives $\mathrm{d}^k/\mathrm{d}t^k$ for $k \leq m$, this implies that for all $m \leq n$,

$$f^{(m)}(u) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}[\mathrm{CA}^{\mathcal{M}}[f,n,u](t)]\big|_{t=u} = \sum_{k=0}^{n}(-1)^k\, \mathcal{K}^k[f](u)\, (\mathrm{d}^m \circ \mathcal{K}^k)[\boldsymbol{m}](0). \tag{2.19}$$

Similarly, since $\mathrm{d}^m/\mathrm{d}t^m \left[\sum_{k=0}^{n} f^{(k)}(u)(t-u)^k/k!\right]\big|_{t=u} = f^{(m)}(u)$ for all $m \leq n$, we also have that for all $m \leq n$,

$$\mathcal{K}^m[f](u) = \mathcal{K}_t^m \left[\sum_{k=0}^{n} f^{(k)}(u)(t-u)^k/k!\right]\Bigg|_{t=u} = \sum_{k=0}^{n} f^{(k)}(u)\, \mathcal{K}^m \left[t^k/k!\right](0). \tag{2.20}$$

Equations (2.19) and (2.20) for $m = n$ relate the standard and the chromatic bases of $\mathcal{D}$,

$$\mathrm{d}^n = \sum_{k=0}^{n}(-1)^k\, (\mathrm{d}^n \circ \mathcal{K}^k)[\boldsymbol{m}](0)\, \mathcal{K}^k; \tag{2.21}$$

$$\mathcal{K}^n = \sum_{k=0}^{n} \mathcal{K}^n \left[t^k/k!\right](0)\, \mathrm{d}^k. \tag{2.22}$$

Since for $j > k$ all powers of t in $\mathcal{K}^k\left[t^j/j!\right]$ are positive, we have

$$j > k \quad \rightarrow \quad \mathcal{K}^k\left[t^j/j!\right](0) = 0. \tag{2.23}$$

Note that (2.23) and (2.12) imply that the matrices $[(-1)^k\, \mathcal{K}^k[\boldsymbol{m}^{(i)}](0)]_{i,k\leq n}$ and $[\mathcal{K}^k\left[t^j/j!\right](0)]_{j,k\leq n}$ are triangular, while (2.21) and (2.22) imply that they are inverses of each other.

We now compare the chromatic approximations of analytic functions with the Taylor approximations. The first equality in (2.19) shows that a chromatic approximation, just like Taylor's approximation, is a local approximation. Referring to Example 2 (Legendre polynomials/Spherical Bessel functions), Figure 3 compares the behavior of the chromatic approximation (gray) of a signal $f \in \mathbf{BL}(\pi)$ (black) with the behavior of the Taylor approximation of $f(t)$ (dashed). Both approximations are of order sixteen,

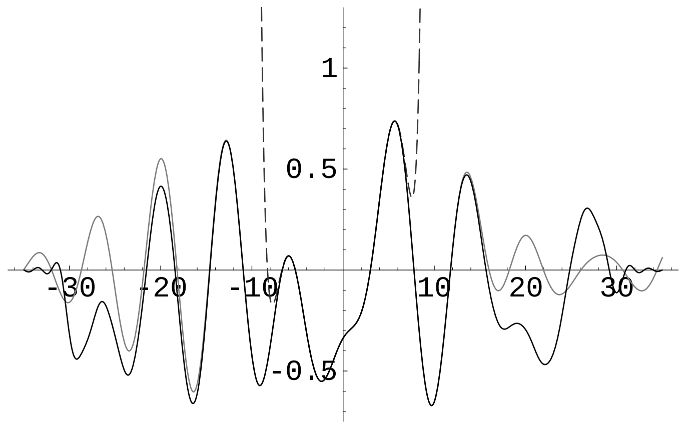

FIGURE 3

and the signal $f(t)$ is defined using Nyquist's representation with samples $\{f(n) : |f(n)| < 1, -30 \leq n \leq 30\}$ that were randomly generated. Figure 3 reveals that, when approximating a signal in $\mathbf{BL}(\pi)$, the chromatic approximation has a much gentler error accumulation when moving away from the point of expansion than the Taylor approximation of the same order. Also, unlike the monomials appearing in the Taylor formula, the "special functions" $\mathcal{K}^n[\boldsymbol{m}](t)$ appearing in a chromatic expansion satisfy $|\mathcal{K}^n[\boldsymbol{m}](t)| \leq 1$ for all real t (Corollary 5).

In particular, in case of Example 2 we show that the chromatic expansion $\mathrm{CA}^{\mathcal{M}}[f, u](t)$ of every analytic function $f(t)$ pointwise converges to $f(t)$ (Corollary 3), and if $f(t)$ is an analytic function that belongs to L_2, then the convergence is also uniform (Proposition 3). Such features of the chromatic approximations, together with the numerical robustness of the chromatic derivatives, make the chromatic approximations applicable in fields involving empirically sampled data.

2.5 Weakly bounded moment functionals

We now consider a broad class of symmetric positive definite moment functionals that contains functionals that correspond to many classical families of orthogonal polynomials. For such functionals the corresponding recursion coefficients $\gamma_n > 0$ appearing in (2.5) satisfy the following three conditions: (i) γ_n are bounded from below by a positive constant; (ii) the growth rate of γ_n is sub-linear in n; and (iii) the quotients γ_n/γ_{n+1} are bounded from above. For technical simplicity in the definition below we use a single constant M in all of these bounds.

Definition 1. Let $\mathcal{M}$ be a symmetric positive definite moment functional and let $\gamma_n > 0$ be such that (2.5) holds for all $n > 0$.

1. $\mathcal{M}$ is *weakly bounded* if there exist some $M \geq 1$, some $0 \leq p < 1$ and some integer r, such that for all $n \geq 0$,

$$\frac{1}{M} \leq \gamma_n \leq M(n+r)^p, \tag{2.24}$$

$$\frac{\gamma_n}{\gamma_{n+1}} \leq M^2. \tag{2.25}$$

2. $\mathcal{M}$ is *bounded* if there exists some $M \geq 1$ such that for all $n \geq 0$,

$$\frac{1}{M} \leq \gamma_n \leq M. \tag{2.26}$$

Thus, every bounded functional $\mathcal{M}$ is also weakly bounded with $p = 0$. Recall that in Example 1 (Chebyshev polynomials/Bessel functions) the recursion coefficients satisfy $\gamma_n = 1/2$ for $n > 1$; in Example 2 (Legendre polynomials/Spherical Bessel functions), $\gamma_n = (n+1)/\sqrt{4(n+1)^2+1}$; consequently, in both cases the corresponding functional $\mathcal{M}$ is bounded. In Example 3 (Hermite polynomials/Gaussian monomial functions), the recursion coefficients satisfy $\gamma_n = \sqrt{(n+1)/2}$; thus, in this case the corresponding moment functional $\mathcal{M}$ is weakly bounded, but not bounded. In Example 4 we have $\gamma_n = n+1$; thus, the corresponding functional $\mathcal{M}$ is not weakly bounded.

In the remaining part of this paper, unless explicitly stated otherwise, all moment functionals involved are assumed to be weakly bounded.

Weakly bounded moment functionals allow a useful estimation of the coefficients in the corresponding equations (2.21) and (2.22).

Lemma 1. *Let $\mathcal{M}$ be weakly bounded, and let M, p and r be as in Definition 1. Then the following two inequalities hold for all k and n:*

$$|(\mathcal{K}^n \circ \mathrm{d}^k)[\boldsymbol{m}](0)| \leq (2M)^k (k+r)!^p; \tag{2.27}$$

$$\left|\mathcal{K}^n\left[\frac{t^k}{k!}\right](0)\right| \leq (2M)^n. \tag{2.28}$$

Proof. By (2.12), it is enough to prove (2.27) for all $n \leq k$. We proceed by induction on k; applying (2.5) to $\mathrm{d}^k[\boldsymbol{m}](t)$ we get

$$|(\mathcal{K}^n \circ \mathrm{d}^{k+1})[\boldsymbol{m}](t)| \leq \gamma_n |(\mathcal{K}^{n+1} \circ \mathrm{d}^k)[\boldsymbol{m}](t)| + \gamma_{n-1}\, |(\mathcal{K}^{n-1} \circ \mathrm{d}^k)[\boldsymbol{m}](t)|.$$

Using the induction hypothesis and (2.12) again, we get for all $n \leq k+1$,

$$\begin{aligned} |(\mathcal{K}^n \circ \mathrm{d}^{k+1})\,[\boldsymbol{m}]\,(0)| &\leq (M(k+1+r)^p + M(k+r)^p)(2M)^k (k+r)!^p \\ &\leq (2M)^{k+1}(k+1+r)!^p. \end{aligned}$$

Similarly, by (2.23), it is enough to prove (2.28) for all $k \leq n$. This time we proceed by induction on n and use (2.5), (2.24) and (2.25) to get

$$\left|\mathcal{K}^{n+1}\left[\frac{t^k}{k!}\right]\right| \leq M\left|\mathcal{K}^n\left[\frac{t^{k-1}}{(k-1)!}\right]\right| + M^2\left|\mathcal{K}^{n-1}\left[\frac{t^k}{k!}\right]\right|.$$

By induction hypothesis and using (2.23) again, we get that for all $k \leq n+1$, $\left|\mathcal{K}^{n+1}\left[\frac{t^k}{k!}\right](0)\right| \leq M\,(2M)^n + M^2(2M)^{n-1} \leq (2M)^{n+1}$. □

Corollary 1. *Let $\mathcal{M}$ be weakly bounded; then for every $\varepsilon > 0$ there exists k_0 such that $\left|(\mathcal{K}^n \circ \mathrm{d}^k)\,[\boldsymbol{m}]\,(0)/k!\right|^{1/k} \leq \varepsilon$ for all $k > k_0$ and all n.*

Proof. Choose K such that $(2M)^k(k+1+r)!^p/k!^p < K^k$ for all $k > 0$. By (2.27),

$$\frac{\left|(\mathcal{K}^n \circ \mathrm{d}^k)\,[\boldsymbol{m}]\,(0)\right|}{k!} < \frac{K^k}{k!^{1-p}} < \left(\frac{K\,e}{k^{1-p}}\right)^k. \tag{2.29}$$

The claim now follows from the fact that $p < 1$. □

Corollary 2. *Let $\boldsymbol{m}(t)$ correspond to a weakly bounded moment functional $\mathcal{M}$; then $\lim_{k\to\infty}\left(\frac{\mu_k}{k!}\right)^{1/k} = \lim_{k\to\infty}\left|\frac{\boldsymbol{m}^k(0)}{k!}\right|^{1/k} = 0$, i.e., inequality (2.2) is satisfied. Moreover, $\boldsymbol{m}(z) = \sum_{n=0}^{\infty} \mathrm{i}^n \mu_n z^n/n!$ is an entire function on $\mathbb{C}$.*

In our Example 4 we have $\gamma_n = n+1$ and thus (2.25) and (2.24) are satisfied with $p = 1$. However, the interval of convergence of $\boldsymbol{m}(t) = \sum_{n=0}^{\infty} E_{2n}\,t^{2n}/(2n)!$ is finite, $I_{\boldsymbol{m}} = (-\pi/2, \pi/2)$. Thus, Example 4 shows that if $\boldsymbol{m}(z)$ is to be an entire function, then the upper bound in (2.24) of Definition 1 of a weakly bounded moment functional is sharp.

Definition 2. Let $\mathcal{M}$ be a weakly bounded moment functional; then $\mathcal{B}_{\mathcal{M}}$ denotes the set of functions that are analytic on $\mathbb{R}$, such that for every compact interval I, $\lim\limits_{n\to\infty}\sup\limits_{t\in I}|\mathcal{K}^n[f](t)|^{1/n} = 0$.

Proposition 1. *$\mathcal{B}_{\mathcal{M}}$ is a vector space that contains $\boldsymbol{m}(t)$ and is closed for differentiation. Thus, $\mathcal{K}^n[\boldsymbol{m}](t) \in \mathcal{B}_{\mathcal{M}}$ for all n.*

Proof. Let I be a compact interval and let L be such that $|t| < L$ for all $t \in I$. Corollary 1 implies that for every $\varepsilon > 0$ there exists k_0 such that for all $n > k_0$ and all k, $|(\mathrm{d}^{n+k} \circ \mathcal{K}^n)[\boldsymbol{m}](0)/(n+k)!|^{1/(n+k)} < \varepsilon/L$. Thus,

$$|\mathcal{K}^n[\boldsymbol{m}](t)| \leq \sum_{k=0}^{\infty}\frac{|(\mathrm{d}^{n+k} \circ \mathcal{K}^n)[\boldsymbol{m}](0)|}{(n+k)!}\,|t|^{n+k} < \sum_{k=0}^{\infty}\varepsilon^{n+k} = \frac{\varepsilon^n}{1-\varepsilon} \tag{2.30}$$

for all $t \in I$, which implies $\boldsymbol{m}(t) \in \mathcal{B}_{\mathcal{M}}$. It is easy to check that $\mathcal{B}_{\mathcal{M}}$ is closed for addition and scalar multiplication. By (2.24), $\limsup_{n\to\infty}(\gamma_n)^{1/n} = 1$. Using (2.5), $|\mathcal{K}^n[f'](t)| = |(d\circ\mathcal{K}^n)[f](t)| \leq \gamma_n\,|\mathcal{K}^{n+1}[f](t)| + \gamma_{n-1}\,|\mathcal{K}^{n-1}[f](t)|$. It is now easy to verify that if $f \in \mathcal{B}_{\mathcal{M}}$ then also $f' \in \mathcal{B}_{\mathcal{M}}$. □

3. Pointwise convergence of chromatic expansions

The past work on chromatic expansions relied heavily on properties of the Fourier transform. In this paper we develop a more elementary treatment, that also applies to functions that do not have a Fourier transform (as a function). Our approach extends the classical treatment of Bessel functions as presented in [23], and is based on notions that have a simple geometric interpretation, presented in §6. We relate these notions to the Fourier transform and the past work on chromatic expansions in §5.

Lemma 2. *Let $\mathcal{M}$ and $p<1$ be as in Lemma 1; then there exists K such that for every integer $k \geq 1/(1-p)$, every n and every $z \in \mathbb{C}$,*

$$|\mathcal{K}^n[\boldsymbol{m}](z)| < \frac{(K|z|)^n}{n!^{1-p}}\, k\,(K|z|+1)^{k-1}\, \mathrm{e}^{(K|z|+1)^k}. \tag{3.1}$$

Proof. Let K be as in the proof of Corollary 1; using the Taylor series for $\mathcal{K}^n[\boldsymbol{m}](z)$, (2.12), the first inequality of (2.29) and our assumptions, we get

$$\begin{aligned} |\mathcal{K}^n[\boldsymbol{m}](z)| &< \sum_{m=0}^{\infty} \frac{|Kz|^{n+m}}{(n+m)!^{1-p}} < \frac{|Kz|^n}{n!^{1-p}} \sum_{m=0}^{\infty} \frac{|Kz|^m}{m!^{1/k}} \\ &< \frac{|Kz|^n}{n!^{1-p}} \sum_{m=0}^{\infty} \frac{(K|z|+1)^{k\lfloor m/k\rfloor+k-1}}{\lfloor m/k\rfloor!} \\ &= \frac{|Kz|^n}{n!^{1-p}}\, k\,(K|z|+1)^{k-1}\, \mathrm{e}^{(K|z|+1)^k}. \end{aligned}$$

□

Proposition 2. *Let $\mathcal{M}$ be weakly bounded, $p<1$ as in (2.24), $f(z)$ an entire function and $u \in \mathbb{C}$. If $\limsup_{n\to\infty} |f^{(n)}(u)/n!^{1-p}|^{1/n} = 0$, then for all $z \in \mathbb{C}$,*

$$f(z) = \sum_{j=0}^{\infty} (-1)^j \mathcal{K}^j[f](u)\, \mathcal{K}^j[\boldsymbol{m}](z-u). \tag{3.2}$$

Proof. It is enough to prove the statement for $u=0$. Let k and K be as in Lemma 2. We define $\theta(z) \equiv k(K|z|+1)^{k-1}\,\mathrm{e}^{(K|z|+1)^k}$ and

$$h(z) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{n+m}\; f^{(n)}(0)\, \mathcal{K}^{n+m}\left[\frac{t^n}{n!}\right](0)\; \mathcal{K}^{n+m}[\boldsymbol{m}](z). \tag{3.3}$$

From our assumptions, (2.28) and (3.1), for every $\varepsilon > 0$ there exists Q such that $|h(z)| < \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q\,\varepsilon^n\, n!^{1-p}\,(2M)^{n+m}\,|Kz|^{n+m}\,\theta(z)/(n+m)!^{1-p}$.

As in the proof of Lemma 2, taking an integer $k \geq 1/(1-p)$, we get

$$\begin{aligned} |h(z)| &< \sum_{n=0}^{\infty} Q(2\,\varepsilon MK|z|)^n (2MK|z|+1)^{k-1} \sum_{m=0}^{\infty} \frac{(2MK|z|+1)^{k\lfloor m/k \rfloor}}{\lfloor m/k \rfloor !}\,\theta(z) \\ &< \sum_{n=0}^{\infty} Q(2\,\varepsilon MK|z|)^n\,\sigma(z), \end{aligned}$$

where $\sigma(z) = k\,(2MK|z|+1)^{k-1}\,\mathrm{e}^{(2MK|z|+1)^k}\,\theta(z)$. This implies that the series defining $h(z)$ converges absolutely and uniformly on every disc of finite radius. Thus, using (2.11) and (2.22), for every natural number s,

$$\begin{aligned} \mathcal{K}^s[h](0) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} f^{(n)}(0)\,\mathcal{K}^{n+m}\left[\frac{t^n}{n!}\right](0)\,(\mathcal{K}^s \circ \mathcal{K}^{n+m})[\boldsymbol{m}](0) \\ &= \sum_{n+m=s}(-1)^{n+m} f^{(n)}(0)\,\mathcal{K}^{n+m}\left[\frac{t^n}{n!}\right](0)\,(\mathcal{K}^s \circ \mathcal{K}^{n+m})[\boldsymbol{m}](0) \\ &= \sum_{n=0}^{s} f^{(n)}(0)\,\mathcal{K}^s\left[\frac{t^n}{n!}\right](0) \;=\; \mathcal{K}^s[f](0). \end{aligned} \tag{3.4}$$

This implies $h^{(s)}(0) = f^{(s)}(0)$ for all s, i.e., $f(z) \equiv h(z)$. On the other hand, the substitution $k = n + m$ in (3.3), regrouping, (2.22) and (2.23) yield

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}(-1)^k f^{(n)}(0)\,\mathcal{K}^k\left[\frac{t^n}{n!}\right](0)\,\mathcal{K}^k[\boldsymbol{m}](z) \\ &= \sum_{k=0}^{\infty}(-1)^k\mathcal{K}^k[f](0)\,\mathcal{K}^k[\boldsymbol{m}](z). \end{aligned}$$

□

Corollary 3. *If $\mathcal{M}$ is bounded, then for every entire function f and all $u, z \in \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty}(-1)^k\mathcal{K}^k[f](u)\,\mathcal{K}^k[\boldsymbol{m}](z-u)$.*

Proof. If $f(z)$ is entire, then for every u, $\lim_{n\to\infty}|f^{(n)}(u)/n!|^{1/n} = 0$. The Corollary now follows from Proposition 2 with $p = 0$. □

Corollary 4. *Let $P_n^{\mathcal{M}}(\omega)$ be the orthonormal polynomials associated with a weakly bounded moment functional $\mathcal{M}$; then $\mathrm{e}^{\mathrm{i}\omega t} = \sum_{n=0}^{\infty}\mathrm{i}^n P_n^{\mathcal{M}}(\omega)\,\mathcal{K}^n[\boldsymbol{m}](t)$.*

Proof. If $p < 1$ then

$$\lim_{n\to\infty}\frac{\sqrt[n]{|\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\sin\omega t]|_{t=0}}}{n^{1-p}} = \lim_{n\to\infty}\frac{\sqrt[n]{|\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\cos\omega t]|_{t=0}}}{n^{1-p}} = \lim_{n\to\infty}\frac{|\omega|}{n^{1-p}} = 0$$

and the claim follows from Proposition 2 and (2.16). □

We note that Corollary 3 generalizes the classical result that every entire function can be expressed as a Neumann series of Bessel functions [23], by replacing the Neumann series with any chromatic expansion that corresponds to a bounded moment functional. Corollary 4, on the other hand, generalizes the well known equality for the Chebyshev polynomials $T_n(\omega)$ and the Bessel functions $J_n(t)$, i.e., $\mathrm{e}^{\mathrm{i}\,\omega t} = J_0(t) + 2\sum_{n=1}^{\infty} \mathrm{i}^n T_n(\omega) J_n(t)$. In a similar fashion, one can obtain many other classical results on Bessel functions from [23], as well as their generalizations. For example, the chromatic expansion of the constant function $f(z) \equiv 1$ yields the equality $\boldsymbol{m}(z) + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{\gamma_{2k-2}}{\gamma_{2k-1}}\right)\mathcal{K}^{2n}[\boldsymbol{m}](z) = 1$, with γ_n the recursion coefficients from (2.4), which generalizes the identity $J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1$.

4. Uniform convergence of chromatic approximations

Definition 3. Let $\mathcal{M}$ be a weakly bounded moment functional; then $L_2^{\mathcal{M}}$ denotes the vector space of analytic functions $f(t)$ such that $\sum_{n=0}^{\infty}\mathcal{K}^n[f](t)^2$ is a continuous function on $\mathbb{R}$.

Since the sum in the above definition has positive terms, by Dini's Theorem $f \in L_2^{\mathcal{M}}$ just in case the sum $\sum_{n=0}^{\infty}\mathcal{K}^n[f](t)^2$ converges uniformly on every compact interval $I \subset \mathbb{R}$.

Lemma 3. *For all $f, g \in L_2^{\mathcal{M}}$ the sum $\Sigma(u) = \sum_{k=0}^{\infty}\mathcal{K}^k[f](u)\,\mathcal{K}^k[g](u)$ converges uniformly on every compact interval I, and is independent of u.*

Proof. Since $\sum_{m=0}^{\infty}\mathcal{K}^m[f](u)^2$ and $\sum_{m=0}^{\infty}\mathcal{K}^m[g](u)^2$ converge uniformly on every compact interval I, $\Sigma(u)$ converges absolutely and uniformly on such I as well. Let $\Sigma^*(n,u) = \sum_{m=0}^{n}\mathcal{K}^m[f](u)\,\mathcal{K}^m[g](u)$. Since $\mathcal{M}$ is weakly bounded, (2.7) and (2.24) imply that for some $M \geq 1$, $0 \leq p < 1$ and r,

$$\mathrm{d}\,[\Sigma^*(n,u)] \leq M(n+r)^p(|\mathcal{K}^{n+1}[f](u)\mathcal{K}^n[g](u)| + |\mathcal{K}^n[f](u)\mathcal{K}^{n+1}[g](u)|). \tag{4.1}$$

Let $S_k(u) = |\mathcal{K}^k[f](u)| + |\mathcal{K}^{k+1}[f](u)| + |\mathcal{K}^k[g](u)| + |\mathcal{K}^{k+1}[g](u)|$; then

$$\sum_{k=0}^{\infty} S_k(u)^2 \leq 4\sum_{k=0}^{\infty}\mathcal{K}^k[f](u)^2 + \mathcal{K}^{k+1}[f](u)^2 + \mathcal{K}^k[g](u)^2 + \mathcal{K}^{k+1}[g](u)^2 < \infty.$$

Thus, the series $\sum_{k=0}^{\infty} S_k(u)^2$ is convergent, and consequently for every fixed u there are infinitely many n such that

$$(|\mathcal{K}^n[f](u)| + |\mathcal{K}^{n+1}[f](u)| + |\mathcal{K}^n[g](u)| + |\mathcal{K}^{n+1}[g](u)|)^2 < \frac{1}{n+r}.$$

For every such n, all four summands must be smaller than $1/\sqrt{n+r}$. This, together with (4.1), implies that for such n, $|d[\Sigma^*(n,u)]| < 2M/(n+r)^{1-p}$, i.e., that $\liminf_{n\to\infty} |d[\Sigma^*(n,u)]| = 0$. Since $\lim_{n\to\infty} \mathrm{d}\,[\Sigma^*(n,u)]$ exists, it must be equal to zero, and consequently $\Sigma(u)$ is constant on $\mathbb{R}$. □

The above Lemma has several useful consequences. First of all, it allows us to define on $L_2^{\mathcal{M}}$ a scalar product and a convolution in a highly local, yet instant independent way (i.e., the two sums in the definition below are independent of u).

Definition 4. Let $f, g \in L_2^{\mathcal{M}}$, then we define

$$\langle f, g\rangle_{\mathcal{M}} = \textstyle\sum_{k=0}^{\infty} \mathcal{K}^k[f](u)\, \mathcal{K}^k[g](u); \tag{4.2}$$

$$(f *_{\mathcal{M}} g)(t) = \textstyle\sum_{k=0}^{\infty} \mathcal{K}^k[f](u)\, \mathcal{K}_u^k[g(t-u)]. \tag{4.3}$$

Let $\|f\|_{\mathcal{M}} = \langle f, f\rangle_{\mathcal{M}}^{1/2}$. If $f \in L_2^{\mathcal{M}}$, then $\|\mathcal{K}^n[f]\|_\infty \le \|f\|_{\mathcal{M}}$ for all n. Thus, for $f \in L_2^{\mathcal{M}}$, the filters that correspond to the operators $\{\mathcal{K}^n\}_{n\in\mathbb{N}}^{\mathcal{M}}$ have outputs uniformly bounded by $\|f\|_{\mathcal{M}}$. This feature is important for applications in fields that involve digital sampling of such filter banks.

Let $\mathcal{B}_{\mathcal{M}}$ be as in Definition 2; clearly $\mathcal{B}_{\mathcal{M}} \subset L_2^{\mathcal{M}}$. Thus, by Proposition 1, $\mathcal{K}^n[\boldsymbol{m}](t) \in L_2^{\mathcal{M}}$ for all n. By Lemma 3 and (2.6), $\sum_{k=0}^{\infty}(\mathcal{K}^k \circ \mathcal{K}^n)[\boldsymbol{m}](t)\,(\mathcal{K}^k \circ \mathcal{K}^m)[\boldsymbol{m}](t) = \sum_{k=0}^{\infty}(\mathcal{K}^k \circ \mathcal{K}^n)[\boldsymbol{m}](0)\,(\mathcal{K}^k \circ \mathcal{K}^m)[\boldsymbol{m}](0) = \delta(m-n)$.

Corollary 5. *For every fixed u, the family of functions $\{\mathcal{K}^n[\boldsymbol{m}](t-u)\}_{n\in\mathbb{N}}$ is an orthonormal set of vectors in $L_2^{\mathcal{M}}$. Similarly, the set of sequences $\{[(\mathcal{K}^n \circ \mathcal{K}^k)[\boldsymbol{m}](u)]_{k\in\mathbb{N}}\}_{n\in\mathbb{N}}$ is an orthonormal set of vectors in l_2. In particular, for all $u \in \mathbb{R}$,*

$$\sum_{k=0}^{\infty} \mathcal{K}^k[\boldsymbol{m}](u)^2 = 1. \tag{4.4}$$

Thus, for all $u \in \mathbb{R}$,

$$|\mathcal{K}^n[\boldsymbol{m}](u)| \le 1. \tag{4.5}$$

Corollary 6. *Let $\mathcal{M}$ be weakly bounded and let $A_1, A_2 \in \mathcal{D}$. Then*

$$\langle A_1, A_2\rangle_{\mathcal{M}}^{\mathcal{D}} = \sum_{n=0}^{\infty} \mathcal{M}[\mathcal{K}^n \circ A_1]\, \mathcal{M}[\mathcal{K}^n \circ A_2] \tag{4.6}$$

is a scalar product on $\mathcal{D}$, and the mapping $A \mapsto A[\boldsymbol{m}](t)$ is a unitary isomorphism between $\mathcal{D}$ and the space spanned by the family $\{\mathcal{K}^n[\boldsymbol{m}](t)\}_{n\in\mathbb{N}}$ with the scalar product $\langle f, g\rangle_{\mathcal{M}}$. Similarly, for every fixed u, the mapping $A \mapsto [(\mathcal{K}^k \circ A)[\boldsymbol{m}](u)]_{k\in\mathbb{N}}$ is a unitary embedding of $\mathcal{D}$ into l_2 and $A \mapsto [(\mathcal{K}^k \circ A)[\boldsymbol{m}](0)]_{k\in\mathbb{N}}$ maps $\{\mathcal{K}^n\}_{n\in\mathbb{N}}^{\mathcal{M}}$ onto the usual base of l_2.

Since $\mathcal{K}^k$ is a linear combination of derivatives of the same parity as k, we have $\mathcal{K}_u^k[g(t-u)] = (-1)^k \mathcal{K}^k[g](t-u)$; thus, by Lemma 3 and (4.3), for every u and t,

$$(f *_{\mathcal{M}} g)(t) = \sum_{k=0}^{\infty} (-1)^n\, \mathcal{K}^k[f](u)\, \mathcal{K}^k[g](t-u). \tag{4.7}$$

By taking $g(t) \equiv \boldsymbol{m}(t)$ in (4.7), we get

$$(f *_{\mathcal{M}} \boldsymbol{m})(t) = \sum_{k=0}^{\infty} (-1)^n \, \mathcal{K}^k[f](u) \, \mathcal{K}^k[\boldsymbol{m}](t-u) = \mathrm{CE}^{\mathcal{M}}[f,u](t). \tag{4.8}$$

Finally, by setting first $u = 0$ and then $u = t$ in (4.7), we get the following useful lemma.

Lemma 4. *For every $f, g \in L_2^{\mathcal{M}}$,*

$$\sum_{k=0}^{\infty} (-1)^k \, \mathcal{K}^k[f](t) \, \mathcal{K}^k[g](0) = \sum_{k=0}^{\infty} (-1)^k \, \mathcal{K}^k[f](0) \, \mathcal{K}^k[g](t). \tag{4.9}$$

Proposition 3. *Let $\mathcal{M}$ be weakly bounded and $f \in L_2^{\mathcal{M}}$; then the chromatic expansion $\mathrm{CE}^{\mathcal{M}}[f,u](t)$ of $f(t)$ converges to $f(t)$ in $L_2^{\mathcal{M}}$ and thus also converges to $f(t)$ uniformly on $\mathbb{R}$.*

Proof. We can assume $u = 0$. Since $f, \boldsymbol{m} \in L_2^{\mathcal{M}}$ and $\mathcal{K}^k[\boldsymbol{m}](0) = 0$ for $k > 0$, using the second inequality of (4.8) and Lemma 4, $\mathrm{CE}^{\mathcal{M}}[f,0](t) = \sum_{k=0}^{\infty} (-1)^k \, \mathcal{K}^k[f](t) \, \mathcal{K}^k[\boldsymbol{m}](0) = f(t)$. The convergence is uniform because $\left| \sum_{j=n+1}^{\infty} (-1)^j \mathcal{K}^j[f](0) \, \mathcal{K}^j[\boldsymbol{m}](t) \right|^2 \leq \sum_{j=n+1}^{\infty} \mathcal{K}^j[f](0)^2 \, \sum_{j=n+1}^{\infty} \mathcal{K}^j[\boldsymbol{m}](t)^2 \leq \sum_{j=n+1}^{\infty} \mathcal{K}^j[f](0)^2$. So, for $m > n$, $\mathcal{K}_t^m[\sum_{j=n+1}^{\infty} (-1)^j \mathcal{K}^j[f](0) \, \mathcal{K}^j[\boldsymbol{m}](t)]\big|_{t=0} = \sum_{k=n+1}^{\infty} (-1)^k \, \mathcal{K}^k[f](0) (\mathcal{K}^m \circ \mathcal{K}^k)[\boldsymbol{m}](0) = \mathcal{K}^m[f](0)$. Since this implies $\left\| \sum_{k=n+1}^{\infty} \mathcal{K}^k[f](0) \, \mathcal{K}^k[\boldsymbol{m}](t) \right\|_{\mathcal{M}} = \left(\sum_{k=n+1}^{\infty} \mathcal{K}^k[f](0)^2 \right)^{1/2}$, we conclude that $\mathrm{CE}^{\mathcal{M}}[f,0](t)$ converges to $f(t)$ in $L_2^{\mathcal{M}}$. □

Note that Proposition 3 and (4.8) imply that for $f \in L_2^{\mathcal{M}}$

$$(f *_{\mathcal{M}} \boldsymbol{m})(t) = f(t). \tag{4.10}$$

Setting $f(t) = \boldsymbol{m}(t)$ in (4.10) and (4.8), and subsequently also setting $u = t/2$ in (4.8), we get

$$\sum_{k=0}^{\infty} (-1)^k \, \mathcal{K}^k[\boldsymbol{m}](t/2)^2 = \boldsymbol{m}(t).$$

This is a generalization of the well known equality for the Bessel functions, $J_0(t/2)^2 + 2\sum_{n=1}^{\infty} (-1)^n J_n(t/2)^2 = J_0(t)$. In a similar manner, we can generalize of many other identities for Bessel functions from [23].

Proposition 4. *$L_2^{\mathcal{M}}$ is a complete space.*

Proof. If $f \in L_2^{\mathcal{M}}$ then $\|f\|_\infty \leq \|f\|_{\mathcal{M}}$; thus, every Cauchy sequence $\{f_k\}_{k \in \mathbb{N}}$ in $L_2^{\mathcal{M}}$ converges uniformly on $\mathbb{R}$ to an analytic function $f(t)$, and for every fixed n, the sequence $\{\mathcal{K}^n[f_k]\}_{k \in \mathbb{N}}$ converges uniformly to $\mathcal{K}^n[f]$. Let $\varepsilon > 0$ and N be arbitrary; then there exists m such that

$(\sum_{k=0}^{N}\mathcal{K}^k[f_m(t)-f_n(t)]^2)^{1/2}<\varepsilon/2$ for all $n>m$ and all t. Let $n>m$ be such that $(\mathcal{K}^k[f_n](t)-\mathcal{K}^k[f](t))^2<\varepsilon^2/(4N+4)$ for all $0\leq k\leq N$ and for all t. Then $(\sum_{k=0}^{N}\mathcal{K}^k[f_m(t)-f(t)]^2)^{1/2}<(\sum_{k=0}^{N}\mathcal{K}^k[f_m(t)-f_n(t)]^2)^{1/2}+(\sum_{k=0}^{N}\mathcal{K}^k[f_n(t)-f(t)]^2)^{1/2}<\varepsilon/2+\varepsilon/2=\varepsilon$. Since N is arbitrary, also $(\sum_{k=0}^{\infty}\mathcal{K}^k[f_m(t)-f(t)]^2)^{1/2}\leq\varepsilon$. Finally, since $\left(\sum_{n=N}^{\infty}\mathcal{K}^n[f](t)^2\right)^{1/2}\leq\left(\sum_{n=0}^{\infty}(\mathcal{K}^n[f](t)-\mathcal{K}^n[f_m](t))^2\right)^{1/2}+\left(\sum_{n=N}^{\infty}\mathcal{K}^n[f_m](t)^2\right)^{1/2}$, and since the sum $\sum_{n=N}^{\infty}\mathcal{K}^n[f_m](t)^2$ converges uniformly on every compact interval I, also $\sum_{n=0}^{\infty}\mathcal{K}^n[f](t)^2$ converges uniformly on every such I. Thus, $f\in L_2^{\mathcal{M}}$, and $\{f_m\}_{m\in\mathbb{N}}$ converges to f in $L_2^{\mathcal{M}}$. □

Corollary 7. *If $\mathcal{M}$ is weakly bounded and the sequence of reals $[a_m]_{m\in\mathbb{N}}$ belongs to l_2, then $g(t)=\sum_{m=0}^{\infty}a_m\,\mathcal{K}^m[\boldsymbol{m}](t)$ belongs to $L_2^{\mathcal{M}}$. For such $g(t)$ we have $\mathcal{K}^n[g](0)=(-1)^n\,a_n$; thus, for every fixed u, the mapping $f\mapsto[\mathcal{K}^k[f](u)]_{k\in\mathbb{N}}$ is a unitary isomorphism between $L_2^{\mathcal{M}}$ and l_2.*

Proof. It is easy to see that $\alpha_n(t)=\sum_{k=0}^{n}a_k\,\mathcal{K}^k[\boldsymbol{m}](t)$ is a Cauchy sequence in $L_2^{\mathcal{M}}$ that also uniformly converges to $\alpha(t)=\sum_{k=0}^{\infty}a_k\,\mathcal{K}^k[\boldsymbol{m}](t)$. By Proposition 4, the sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ converges in $L_2^{\mathcal{M}}$ to $\alpha\in L_2^{\mathcal{M}}$. □

Corollary 8. *$\{\mathcal{K}^n[\boldsymbol{m}](t-u)\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $L_2^{\mathcal{M}}$.*

Proof. Let $f(t)\in L_2^{\mathcal{M}}$ and let u be fixed; by Corollary 5, $\{\mathcal{K}^n[\boldsymbol{m}](t-u)\}_{n\in\mathbb{N}}$ is an orthonormal system in $L_2^{\mathcal{M}}$. By Lemma 3, $\langle f(t),\mathcal{K}^n[\boldsymbol{m}](t-u)\rangle_{\mathcal{M}}=\sum_{j=0}^{\infty}\mathcal{K}^j[f](t)(\mathcal{K}^j\circ\mathcal{K}^n)[\boldsymbol{m}](t-u)\Big|_{t=u}=\sum_{j=0}^{\infty}\mathcal{K}^j[f](u)(\mathcal{K}^j\circ\mathcal{K}^n)[\boldsymbol{m}](0)=(-1)^n\mathcal{K}^n[f](u)$. Thus, we have $\sum_{n=0}^{\infty}\langle f,\mathcal{K}^n[\boldsymbol{m}](t-u)\rangle_{\mathcal{M}}\,\mathcal{K}^n[\boldsymbol{m}](t-u)=\mathrm{CE}^{\mathcal{M}}[f,u](t)=f(t)$. Consequently, the Fourier expansion of $f(t)\in L_2^{\mathcal{M}}$ with respect to the orthonormal system $\{\mathcal{K}^n[\boldsymbol{m}](t-u)\}_{n\in\mathbb{N}}$ is the chromatic expansion of $f(t)$. By Proposition 3 such Fourier expansion converges to $f(t)$ in $L_2^{\mathcal{M}}$. □

Lemma 5. *If $\mathcal{M}$ is bounded, then $L_2^{\mathcal{M}}$ is closed for differentiation.*

Proof. Assuming that $\gamma_n<M$ for all n, we have $(\mathrm{d}\circ\mathcal{K}^n)[f](t)=\gamma_n\,\mathcal{K}^{n+1}[f](t)-\gamma_{n-1}\mathcal{K}^{n-1}[f](t)<M(|\mathcal{K}^{n+1}[f](t)|+|\mathcal{K}^{n-1}[f](t)|)$. Thus, $\sum_{k=0}^{\infty}\mathcal{K}^k[f'](t)^2<2M^2(\sum_{k=0}^{\infty}\mathcal{K}^{k+1}[f](t)^2+\sum_{k=0}^{\infty}\mathcal{K}^{k-1}[f](t)^2)$, and so if $\sum_{k=0}^{\infty}\mathcal{K}^k[f](t)^2$ converges uniformly on every compact interval I, so does $\sum_{k=0}^{\infty}\mathcal{K}^k[f'](t)^2$. □

If $\mathcal{M}$ is only weakly bounded, then $L_2^{\mathcal{M}}$ need not be closed for differentiation. For example, let $\{\mathcal{K}^n\}_{n\in\mathbb{N}}^{\mathcal{M}}$ be defined by the three term recurrence relation (2.5) with $\gamma_n=(2+(-1)^n)(n+1)^{4/5}$; such $\mathcal{M}$ is clearly weakly bounded. Let $g(t)=\sum_{n=0}^{\infty}1/(n+1)^{6/5}\,\mathcal{K}^n[\boldsymbol{m}](t)<\sum_{n=0}^{\infty}1/(n+1)^{6/5}$. By Corollary 7, $g\in L_2^{\mathcal{M}}$. Since the series defining $g(t)$ is uniformly convergent, by (2.5),

$$g'(t) = \sum_{n=0}^{\infty} \frac{\gamma_n\,\mathcal{K}^{n+1}[\boldsymbol{m}](t) - \gamma_{n-1}\,\mathcal{K}^{n-1}[\boldsymbol{m}](t)}{(n+1)^{6/5}} = \sum_{n=0}^{\infty} a_n \mathcal{K}^n[\boldsymbol{m}](t),$$

with $a_n = (2-(-1)^n)/n^{2/5} - (2+(-1)^n)(n+1)^{4/5}/(n+2)^{6/5}$ and the last series converging uniformly. Thus, $\mathcal{K}^m[g'](0) = \sum_{n=0}^{\infty} a_n(\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](0) = (-1)^m a_m$. Since $|a_m| > 1/m^{2/5}$, we get that $\sum_{n=0}^{\infty} \mathcal{K}^n[g'](0)^2 = \sum_{n=0}^{\infty} a_n^2$ diverges, and thus $g' \notin L_2^{\mathcal{M}}$.

Let $\mathcal{S}^u[f](t)$ denote the shift operator, i.e., $\mathcal{S}^u[f](t) = f(t-u)$, and let A be a linear operator on $L_2^{\mathcal{M}}$ that is continuous with respect to the norm $\|f\|_{\mathcal{M}}$. If A is shift invariant, i.e., such that for every fixed h, $(A \circ S^h)[f] = (S^h \circ A)[f]$ for all $f \in L_2^{\mathcal{M}}$, then A also commutes with differentiation on $L_2^{\mathcal{M}}$. Using Lemma 3,

$$\begin{aligned} A[f] &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](u)\, A[\mathcal{K}^n[\boldsymbol{m}](t-u)] \\ &= \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](u)\, \mathcal{K}^n[A[\boldsymbol{m}]](t-u) = f *_{\mathcal{M}} A[\boldsymbol{m}]. \end{aligned} \tag{4.11}$$

Thus, the action of such A on any function in $L_2^{\mathcal{M}}$ is uniquely determined by $A[\boldsymbol{m}]$, which plays the role of the *impulse response* of a *continuous time invariant linear system* in the standard signal processing paradigm based on Nyquist's expansion. This shows that a chromatic expansion has some good features of both Taylor's expansion and Nyquist's expansion. Representing Nyquist's expansion using the shift operator $S^h[f](t) = f(t-h)$, i.e.,

$$f(t) = \sum_{n=-\infty}^{\infty} \mathcal{S}^{-n}[f](0)\, \mathcal{S}^n[\operatorname{sinc} \pi t], \tag{4.12}$$

the chromatic expansion can now be seen as analogous to the Nyquist expansion, with the operators $\mathcal{K}^n$ replacing the shift operators $\mathcal{S}^n$, and with $\boldsymbol{m}(t)$ replacing $\operatorname{sinc} \pi t$. However, while Nyquist's representation of a signal $f \in \mathbf{BL}(\pi)$ uses a set of samples $\{f(n)\}_{n\in\mathbb{N}}$ evenly spread in time that provide information on the global behavior of the signal, the chromatic representation of $f(t)$ uses a set of simultaneous samples $\{\mathcal{K}^n[f](u)\}_{n\in\mathbb{N}}$, taken at a single instant u, and thus is local in nature.

The next proposition shows that among continuous linear operators on $L_2^{\mathcal{M}}$ only shift invariant operators commute with differentiation.

Proposition 5. *Let $\mathcal{M}$ be weakly bounded. If a continuous linear operator $A : L_2^{\mathcal{M}} \to L_2^{\mathcal{M}}$ satisfies $(\mathrm{d}^n \circ A)[\boldsymbol{m}](t) = (A \circ \mathrm{d}^n)[\boldsymbol{m}](t)$ for all n, then A must be shift invariant on $L_2^{\mathcal{M}}$.*

Proof. Such an operator A also satisfies $(A \circ \mathcal{K}^n)[\boldsymbol{m}](t) = (\mathcal{K}^n \circ A)[\boldsymbol{m}](t)$ for all n. Thus, also using Proposition 3 and Lemma 4, for every $f \in L_2^{\mathcal{M}}$,

$$A[f](t) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[A[\boldsymbol{m}]](t) = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[A[\boldsymbol{m}]](0)\, \mathcal{K}^n[f](t).$$

Since operators $\mathcal{K}^n[f](t)$ are shift invariant, A is also shift invariant. □

5. Chromatic derivatives and the Fourier transform

In this section we relate our notions to the Fourier transform and to the past work on chromatic expansions.

Let $\mathcal{M}$ be a weakly bounded moment functional on $\mathcal{P}_\omega$. Since $\mathcal{M}$ is positive definite, there exists a bounded, non-decreasing function $a(\omega)$, called an m–distribution function, such that $\int_{-\infty}^{\infty} \omega^n \,\mathrm{d}a(\omega) = \mu_n$ and for all m and n, $\int_{-\infty}^{\infty} P_n^{\mathcal{M}}(\omega)\, P_m^{\mathcal{M}}(\omega)\, \mathrm{d}a(\omega) = \delta(m-n)$ (see e.g. [7]). More over, our Corollary 2 and a theorem of Riesz [18] imply that such m–distribution function is substantially unique, see e.g. §II.5 in [7], Theorem 5.1.

Let $L^2_{a(\omega)}$ be the vector space consisting of functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that $\|\varphi\|_{a(\omega)} = (\int_{-\infty}^{\infty} |\varphi(\omega)|^2 \,\mathrm{d}a(\omega))^{1/2} < \infty$, with the scalar product defined by the Lebesgue – Stieltjes integral $\langle \alpha, \beta \rangle_{a(\omega)} = \int_{-\infty}^{\infty} \alpha(\omega)\, \overline{\beta(\omega)}\, \mathrm{d}a(\omega)$. Since

$$\int_{-\infty}^{\infty} |(\mathrm{i}\,\omega)^n \varphi(\omega)\, \mathrm{e}^{\mathrm{i}\,\omega t}| \mathrm{d}a(\omega) \le \left(\int_{-\infty}^{\infty} \omega^{2n} \mathrm{d}a(\omega) \int_{-\infty}^{\infty} |\varphi(\omega)|^2 \mathrm{d}a(\omega) \right)^{1/2} < \infty, \tag{5.1}$$

for every $\varphi \in L^2_{a(\omega)}$ we can define a corresponding function $f : \mathbb{R} \to \mathbb{C}$ by

$$f(t) = \int_{-\infty}^{\infty} \varphi(\omega) e^{\mathrm{i}\omega t} \,\mathrm{d}a(\omega), \tag{5.2}$$

and we can differentiate (5.2) under the integral sign any number of times. Using (2.16), we obtain

$$\mathcal{K}^n[f](t) = \int_{-\infty}^{\infty} \mathrm{i}^n \, P_n^{\mathcal{M}}(\omega)\, \varphi(\omega)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega). \tag{5.3}$$

By Corollary 2, if $\mathcal{M}$ is weakly bounded, then $\lim_{n\to\infty} \mu_n^{1/n}/n = 0$; thus, for every $\varepsilon > 0$ there exists n_0 such that $\mu_n < \varepsilon^n n^n$ for all $n > n_0$. Consequently, for all $\varphi \in L^2_{a(\omega)}$ and for $f(t)$ such that (5.2) holds, using (5.1) we get that for $n > n_0$

$$|f^{(n)}(t)| \le \mu_{2n}^{1/2} \, \|\varphi\|_{a(\omega)}^{1/2} \le \varepsilon^n (2n)^n \, \|\varphi\|_{a(\omega)}^{1/2}.$$

This inequality can be used to bound the remainder term of the Taylor approximation of $f(t)$ of order $n-1$, and is easily seen to imply that for all $\varphi \in L^2_{a(\omega)}$ the corresponding $f(t)$ defined by (5.2) must be analytic.

Proposition 6. *Families of orthogonal polynomials $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ that are associated with a weakly bounded moment functional $\mathcal{M}$ are complete in their corresponding spaces $L^2_{a(\omega)}$.*

Proof. Follows from Lemma 2 and Riesz's theorems from [18]; see e.g. Theorems 4.2 and 4.3 in §II.4. of [7]. □

As a corollary, we get the completeness of many classical families of orthogonal polynomials, such as the Hermite polynomials.

Since the family $\{P_n^{\mathcal{M}}(\omega)\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $L^2_{a(\omega)}$, for every $\varphi \in L^2_{a(\omega)}$ and for the corresponding $f(t)$ such that (5.2) holds, (5.3) implies that for almost all ω,

$$\varphi(\omega) = \sum_{n=0}^{\infty} (-\mathrm{i})^n \mathcal{K}^n[f](0)\, P_n^{\mathcal{M}}(\omega). \tag{5.4}$$

Consequently, for every analytic function $f(t)$ there can be at most one function $\varphi(\omega) \in L^2_{a(\omega)}$ (in the sense of $L^2_{a(\omega)}$) such that (5.2) holds; such $\varphi(\omega)$ we call the *$\mathcal{M}$–Fourier–Stieltjes transform* of $f(t)$, and we write $\varphi = \mathcal{F}^{\mathcal{M}}[f]$. More generally, (5.3) implies that for every fixed u, $\varphi(\omega)\,\mathrm{e}^{\mathrm{i}\omega u} = \sum_{n=0}^{\infty}(-\mathrm{i})^n\mathcal{K}^n[f](u)\, P_n^{\mathcal{M}}(\omega)$ for almost all ω. Thus, by Parseval's equality,

$$\sum_{n=0}^{\infty} |\mathcal{K}^n[f](u)|^2 = \left\|\varphi(\omega)\mathrm{e}^{\mathrm{i}\omega u}\right\|^2_{a(\omega)} = \|\varphi(\omega)\|^2_{a(\omega)}\,. \tag{5.5}$$

Let $\mathcal{F}L^2_{a(\omega)}$ be the subspace of $L^2_{a(\omega)}$ consisting of all functions $\varphi \in L^2_{a(\omega)}$ such that the function $f(t)$ defined by (5.2) is real valued. If we define $b(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}a(\omega)$, then $b^{(n)}(0) = \mathrm{i}^n\mu_n = \boldsymbol{m}^{(n)}(0)$. Since $\int_{-\infty}^{\infty} \mathrm{d}a(\omega) = \mu_0 < \infty$, we have $\mathcal{F}^{\mathcal{M}}[b] \in L^2_{a(\omega)}$ and consequently $b(t)$ is analytic. Thus, $b(t) \equiv \boldsymbol{m}(t)$, i.e.,

$$\boldsymbol{m}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}a(\omega). \tag{5.6}$$

Note that (5.5) implies that $\sum_{n=0}^{\infty}|\mathcal{K}^n[f](u)|^2$ does not depend on u. Consequently, if $\varphi(\omega) \in \mathcal{F}L^2_{a(\omega)}$, then $f(t)$ defined by (5.2) is in $L_2^{\mathcal{M}}$, and

$$\|f\|^2_{\mathcal{M}} = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u)^2 = \int_{-\infty}^{\infty} |\varphi(\omega)|^2 \mathrm{d}a(\omega) = \|\varphi\|^2_{a(\omega)}\,. \tag{5.7}$$

On the other hand, if $f \in L_2^{\mathcal{M}}$, then $\sum_{k=0}^{\infty}\mathcal{K}^k[f](0)^2 < \infty$ and the sequence of functions defined by

$$\varphi_n(\omega) = \mathcal{F}^{\mathcal{M}}\left[\sum_{k=0}^{n}(-1)^n\mathcal{K}^n[f](0)\,\mathcal{K}^n[\boldsymbol{m}](t)\right] = \sum_{k=0}^{n}(-\mathrm{i})^n\mathcal{K}^n[f](0)\,P_n^{\mathcal{M}}(\omega)$$

is a Cauchy sequence in $\mathcal{F}L^2_{a(\omega)}$. Consequently, $\{\varphi_n(\omega)\}_{n\in\mathbb{N}}$ converges in the sense of $L^2_{a(\omega)}$ norm, i.e., almost everywhere to $\varphi(\omega)$ given by (5.4), and (5.2) and (5.7) hold. In this way we get the following version of the Plancherel Theorem.

Proposition 7. *If $\mathcal{M}$ is weakly bounded then for every analytic real function f we have $f \in L_2^{\mathcal{M}}$ just in case $\mathcal{F}^{\mathcal{M}}[f] \in \mathcal{F}L^2_{a(\omega)}$ and*

$$\|f\|_{\mathcal{M}} = \left(\sum_{n=0}^{\infty} \mathcal{K}^n[f](u)^2\right)^{1/2} = \left(\int_{-\infty}^{\infty} |\mathcal{F}^{\mathcal{M}}[f](\omega)|^2 \mathrm{d}a(\omega)\right)^{1/2} = \|\mathcal{F}^{\mathcal{M}}[f]\|_{a(\omega)}. \tag{5.8}$$

Thus, the mapping $f \mapsto \mathcal{F}^{\mathcal{M}}[f]$ is a unitary isomorphism between $L_2^{\mathcal{M}}$ and $\mathcal{F}L^2_{a(\omega)}$, and $\langle f, g\rangle_{\mathcal{M}} = \langle \mathcal{F}^{\mathcal{M}}[f], \mathcal{F}^{\mathcal{M}}[g]\rangle_{a(\omega)}$, i.e., $\sum_{n=0}^{\infty} \mathcal{K}^n[f](u)\, \mathcal{K}^n[g](u) = \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[f](\omega)\, \overline{\mathcal{F}^{\mathcal{M}}[g](\omega)}\, \mathrm{d}\omega$.

Similarly, if $f, g \in L_2^{\mathcal{M}}$ then also

$$\begin{aligned}
&\sum_{n=0}^{M} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[g](t) \\
&\quad = \int_{-\infty}^{\infty} \sum_{n=0}^{M} (-1)^n\ \mathcal{K}^n[f](0)\, \mathrm{i}^n P_n^{\mathcal{M}}(\omega)\ \mathcal{F}^{\mathcal{M}}[g](\omega)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega) \\
&\quad = \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[g](\omega)\, \overline{\mathcal{F}^{\mathcal{M}}} \left[\sum_{n=0}^{M} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[\boldsymbol{m}](t)\right] \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}a(\omega).
\end{aligned}$$

Since $\mathcal{F}^{\mathcal{M}}[\sum_{n=0}^{M} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[\boldsymbol{m}](t)]$ converges to $\mathcal{F}^{\mathcal{M}}[f]$ in $\mathcal{F}L^2_{a(\omega)}$,

$$f *_{\mathcal{M}} g = \sum_{n=0}^{\infty} (-1)^n \mathcal{K}^n[f](0)\, \mathcal{K}^n[g](t) = \int_{-\infty}^{\infty} \mathcal{F}^{\mathcal{M}}[g](\omega)\, \mathcal{F}^{\mathcal{M}}[f](\omega)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}a(\omega).$$

Corollary 9. *If $f, g \in L_2^{\mathcal{M}}$ then $\mathcal{F}^{\mathcal{M}}[f *_{\mathcal{M}} g](\omega) = \mathcal{F}^{\mathcal{M}}[g](\omega)\, \mathcal{F}^{\mathcal{M}}[f](\omega)$.*

If $a(\omega)$ is absolutely continuous, then there exists a non-negative weight function $\mathrm{w}(\omega)$ such that almost everywhere $a'(\omega) = \mathrm{w}(\omega)$. In this case $\widehat{f(\omega)} = 2\pi\, \mathrm{w}(\omega)\, \mathcal{F}^{\mathcal{M}}[f](\omega)$, where $\widehat{f(\omega)} = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t$ is the usual Fourier transform of $f(t)$. If the moment functional $\mathcal{M}$ is bounded, then $a(\omega)$ and $\mathrm{w}(\omega)$ are finitely supported; see e.g. [2]. Thus, if the support of $\mathrm{w}(\omega)$ is contained in the finite interval $[-\sigma, \sigma]$, then $\boldsymbol{m}(t) \in \mathbf{BL}(\sigma)$, i.e., $\boldsymbol{m}(t)$ is a σ-band limited signal, and $\boldsymbol{m}(t) = \int_{-\sigma}^{\sigma} \mathrm{w}(\omega)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega$.

In our Example 2 for the modified Legendre polynomials defined by $P_n^{\mathcal{M}}(\omega) = \sqrt{2n+1}\, L_n(\omega/\pi)$, we have $\boldsymbol{m}(t) = \operatorname{sinc} \pi t$. In this case $\mathrm{w}(\omega) = 1$ for $-\pi \le \omega \le \pi$ and zero outside $[-\pi, \pi]$; thus, $L_2^{\mathcal{M}}$ consists of functions with the Fourier transform supported in $[-\pi, \pi]$ that satisfy $\frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{f(\omega)}|^2 \mathrm{d}\omega =$

$\int_{-\infty}^{\infty} f(t)^2 \mathrm{d}t < \infty$, i.e., all π-band limited signals of finite energy, the most important space of signals in signal processing practice.

Thus, the space $\mathbf{BL}(\pi)$ can be spanned either by the integer shifts of the function $\mathrm{sinc}_{\pi}(t) = \mathrm{sinc}\,(\pi t)$, which results in the "global", Nyquist's expansion of a signal $f \in \mathbf{BL}(\pi)$, or by the chromatic derivatives associated with the Legendre polynomials of $\mathrm{sinc}_{\pi}(t)$, which results in the local, chromatic expansion. Since for $f \in \mathbf{BL}(\pi)$ the Nyquist expansion of f converges uniformly, $\mathcal{K}^n[f](t) = \sum_{k=-\infty}^{\infty} f(k)\ \mathcal{K}^n[\mathrm{sinc}_{\pi}](t-k)$ for all n. This equation for $t = 0$ and the chromatic expansion of $f(t)$ provide the transformations between the Nyquist rate samples used in Nyquist's expansion and the values of the chromatic derivatives used in the chromatic expansion:

$$\begin{aligned}
\mathcal{K}^n[f](0) &= \sum_{k=-\infty}^{\infty} (-1)^n\ f(k)\ \mathcal{K}^n[\mathrm{sinc}_{\pi}](k); \\
f(n) &= \sum_{k=-\infty}^{\infty} (-1)^k\ \mathcal{K}^k[f](0)\ \mathcal{K}^k[\mathrm{sinc}_{\pi}](n).
\end{aligned}$$

In our Example 1, $L_2^{\mathcal{M}}$ consists of signals $f(t)$ whose Fourier transform $\widehat{f(\omega)}$ is equal to zero for $|\omega| > 1$, and satisfies $\int_{-1}^{1} \sqrt{1-\omega^2}|\widehat{f(\omega)}|^2 \mathrm{d}\omega < \infty$, see [12]. Thus, the chromatic expansion associated with this example is convenient for representing signals with lots of power in frequencies near the edge of the bandwidth. To represent signals that have low power in the high end of the spectrum, a good choice would be to use the chromatic expansion that corresponds to the Chebyshev polynomials of the second kind, $U_n(\omega)$, because in this case the space $L_2^{\mathcal{M}}$ consists of band limited signals $f(t)$ with the Fourier transform $\widehat{f(\omega)}$ that satisfies $\int_{-1}^{1} |\widehat{f(\omega)}|^2/\sqrt{1-\omega^2}\mathrm{d}\omega < \infty$; see [8]. For non-band limited signals one would use the chromatic expansions that correspond to weakly bounded moment functionals that are not bounded, such as the one described our Example 3, and for signals that are finitely supported in the time domain (or in the space domain, such as images, treated by a natural extension of our theory to two dimensions) one can use chromatic expansions associated with families of the kind described in our Example 4. In [8] it is shown how to custom design families of chromatic derivatives from empirically obtained features of a linear system, for example from the impulse response of a telecommunication channel.

6. A geometric interpretation

Let $\mathcal{M}$ be weakly bounded. By Corollary 5, for every fixed t the set of vectors $\{[(\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t)]_{n\in N}\}_{m\in N}$ is an orthonormal basis of l_2. Similarly, by Lemma 7, for every fixed t the mapping from $L_2^{\mathcal{M}}$ into l_2 defined by

$f \mapsto [\mathcal{K}^n[f](t)]_{n\in\mathbb{N}}$ is a unitary isomorphism between $L_2^{\mathcal{M}}$ and l_2. If t varies, then $\vec{f}(t) = [\mathcal{K}^n[f](t)]_{n\in\mathbb{N}}$ defines a curve in l_2.

Lemma 6. *If $f \in L_2^{\mathcal{M}}$ then $\vec{f}(t) = [\mathcal{K}^n[f](t)]_{n\in\mathbb{N}}$ is a continuous curve in l_2.*

Proof. If $f \in L_2^{\mathcal{M}}$ then $\left\| f - \mathcal{S}^h[f] \right\|_{\mathcal{M}}^2 \leq \sum_{n=0}^{N} (\mathcal{K}^n[f](u) - \mathcal{K}^n[\mathcal{S}^h[f]](u))^2 + 2\sum_{n=N+1}^{\infty} \mathcal{K}^n[f](u)^2 + 2\sum_{n=N+1}^{\infty} \mathcal{K}^n[\mathcal{S}^h[f]](u)^2$. The claim now follows from the facts that on every compact interval $I \subset \mathbb{R}$ the sum $\sum_{n=0}^{\infty} \mathcal{K}^n[f](u)^2$ converges uniformly, and that functions $\{\mathcal{K}^n[f](t)\}_{n\leq N}$ are uniformly continuous on I. □

Lemma 7. *Let $\mathcal{B}_{\mathcal{M}}$ be as in Definition 2 and assume that $g \in \mathcal{B}_{\mathcal{M}}$. Then* $\lim_{h\to 0} \left\| \frac{g(t) - \mathcal{S}^h[g](t)}{h} - g'(t) \right\|_{\mathcal{M}} = 0$. *Thus, the curve $\vec{g}(t) = [\mathcal{K}^n[g](t)]_{n\in\mathbb{N}}$ is everywhere differentiable and $\vec{g}\,'(t) = [\mathcal{K}^n[g'](t)]_{n\in\mathbb{N}}$.*

Proof. Since $\mathcal{B}_{\mathcal{M}} \subset L_2^{\mathcal{M}}$, Proposition 1 implies $g' \in L_2^{\mathcal{M}}$. Thus, if I is a compact interval, then for every $\varepsilon > 0$ there exists N such that $\sum_{n=N+1}^{\infty} \mathcal{K}^n [g'] (u)^2 < \varepsilon$ for all $u \in I$. Also, for every t and h let ξ_n^t be a number between t and $t-h$ such that $(\mathcal{K}^n[g](t) - \mathcal{K}^n[g](t-h))/h = (\mathcal{K}^n[g])'(\xi_n^t) = \mathcal{K}^n[g'](\xi_n^t)$. Since

$$\sum_{n=0}^{\infty} \mathcal{K}^n \left[\frac{g(t) - g(t-h)}{h} - g'(t)\right]^2 = \sum_{n=0}^{\infty} \left(\mathcal{K}^n[g'](\xi_n^t) - \mathcal{K}^n[g'](t)]\right)^2$$
$$< \sum_{n=0}^{N} \left(\mathcal{K}^n[g'](\xi_n^t) - \mathcal{K}^n[g'](t)\right)^2 + 2\sum_{n=N+1}^{\infty} \mathcal{K}^n[g'](\xi_n^t)^2 + 2\sum_{n=N+1}^{\infty} \mathcal{K}^n[g'](t)^2$$

and functions $\mathcal{K}^n [g'] (u)$ are uniformly continuous on every compact interval, the above sum can be made arbitrarily small on every such interval I. □

If we let $\vec{e}_{k+1}(t) = \left[(\mathcal{K}^k \circ \mathcal{K}^n)[\boldsymbol{m}](t)\right]_{n\in\mathbb{N}}$ for $k \geq 0$, since by Proposition 1 $\mathcal{K}^n[\boldsymbol{m}](t) \in \mathcal{B}_{\mathcal{M}}$ for all n, by Lemma 7 $\vec{e}_k(t)$ are differentiable for all k. Since l_2 is complete and $\vec{e}_1(t)$ is continuous, $\vec{e}_1(t)$ also has an antiderivative $\vec{e}_0(t)$. Thus, $\vec{e}_1(t) = \vec{e}_0{}'(t)$, and, from (2.5),

$$\begin{aligned}
\vec{e}_1{}'(t) &= [(\mathrm{d} \circ \mathcal{K}^n)[\boldsymbol{m}](t)]_{n\in\mathbb{N}} = \gamma_0\, \vec{e}_2(t) \\
\vec{e}_k{}'(t) &= -\gamma_{k-2} \left[(\mathcal{K}^{k-2} \circ \mathcal{K}^n)[\boldsymbol{m}](t)\right]_{n\in\mathbb{N}} + \gamma_{k-1} \left[(\mathcal{K}^k \circ \mathcal{K}^n)[\boldsymbol{m}](t)\right]_{n\in\mathbb{N}} \\
&= -\gamma_{k-2}\, \vec{e}_{k-1}(t) + \gamma_{k-1}\, \vec{e}_{k+1}(t).
\end{aligned}$$

This means that $\vec{e}_0(t)$ is a helix in l_2 with curvatures $\kappa_k = \gamma_{k-1}$ ($k \geq 1$); for every t and $k \geq 0$, vectors $\vec{e}_{k+1}(t) = [(\mathcal{K}^k \circ \mathcal{K}^n)[\boldsymbol{m}](t)]_{n\in\mathbb{N}}$ are orthonormal in l_2 (Lemma 5) and they form an orthonormal moving base of the helix $\vec{e}_0(t)$; the above two equations are the corresponding Frenet–Serret formulas.

Since for $f \in L_2^{\mathcal{M}}$ the chromatic expansion of f converges uniformly,

$$\mathcal{K}^m[f](t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u)(\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t-u). \tag{6.1}$$

Consequently, for all t and u, the infinite matrix $[(\mathcal{K}^m \circ \mathcal{K}^n)[\boldsymbol{m}](t-u)]_{m,n=0}^{\infty}$ defines a unitary operator on l_2 that maps the vector $[\mathcal{K}^n[f](u)]_{n\in N}$ that represents f at the instant u into the vector $[\mathcal{K}^n[f](t)]_{n\in N}$ that represents f at the instant t. Thus, as time evolves, the orthonormal moving base $\{\vec{e}_k(t)\}_{k\geq 1}$ slides along the helix $\vec{e}_0(t)$, while the norms and angles between vectors $[\mathcal{K}^n[f](t)]_{n\in N}$ that represent $f \in L_2^{\mathcal{M}}$ at an instant t remain invariant.

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The chromatic expansions for the special case presented in Example 1 were introduced by the present author in [11]. The theory emerged in the course of the author's design of a pulse-width modulation switching power amplifier, which was simulated and tested by D. Maksimovic at the University of Colorado at Boulder. The author also used a similar kind of reasoning to design a communication channel equalizer, which was initially implemented and tested by R. Van Dorn. Subsequently, the theory was generalized and extended to various other systems corresponding to several classical families of orthogonal polynomials by the research team of the author's startup, *Kromos Technology Inc.* [3, 8]. The results were published on the company web site as a series of technical reports[2] as well as in a few conference and journal papers, see the references. The Kromos team designed and implemented a channel equalizer [9], a digital transceiver (unpublished) and an image compression method [4, 5], all employing chromatic approximations. In [6] the chromatic expansions were related to the work of Papoulis [17] and Vaidyanathan [19], and in [15, 20] the theory was cast in the framework commonly used in signal processing. The behavior of the error of the chromatic approximations was studied in [1]; the expansion from Example 3 introduced in [3] was also independently introduced and explored in [21]. A most interesting generalization of the chromatic expansions to the prolate spheroidal wave functions was introduced recently by Gilbert Walter in his forthcoming paper [22]. Finally, the author wishes to thank Tim Herron for a careful reading of an earlier version of the paper, to Gilbert Walter for many e-mail discussions, and to the referee for many suggestions that greatly improved the exposition.

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