Families of Orthogonal Differential Operators for Signal Processing

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Abstract

This report is intended to be a convenient reference for working in the theory of chromatic derivatives, due to Ignjatovic. We first list the main definitions and formulas used in the theory. The main body of the paper presents the formulas for a number of special classes of chromatic derivatives, and describes the application of each. We also present some methods for numerically deriving additional chromatic derivative classes.

Contents

1 Introduction

This document is a reference guide for people wishing to work with orthogonal differential operators, as developed by Ignjatovic [1]. In the first section we review some theory and the associated formulas for families of differential operators for the Fourier transform. The middle two sections of this paper contain a formulary in which we record the operators and associated functions for each of the most important families of differential operators that can be used by machines processing industrially-important signals. Finally, we finish by recording some numerical methods useful in obtaining functions and formula constants associated with arbitrary families of such operators.

The application of an operator L to a signal f will be written $L[f]$, or occasionally simply as Lf. $\mathcal{F}[f(t)](\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ represents the Fourier transform of f, and $\mathcal{F}^{-1}[F(\omega)](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$ represents the inverse Fourier transform of F. We will use $F(\omega)$ for $\mathcal{F}[f(t)]$ throughout.

2 Review of Chromatic Differentiation and Formulary Contents

For each of the families we record the following information:

1. The names that are used for the family of differential operators.
2. The *frequency-domain weight*, $W(\omega)$, which controls the frequency domain of operation of the family of operators; i.e., the filtering the differential operators do when applied to a signal. The weight essentially determines all of the other formulas for a given family as well as implying differential operator characteristics such as:
    (a) the energy of the signals that the operators can process; ranging from certain infinite energy signals to restricted kinds of finite energy signals.
    (b) the kinds of signals that the differential operators naturally predict just outside of the domain of ordinary local representation. I.e., signals with the same spectrum as the weight are naturally predicted using the associated differential operators.
    (c) how real and imaginary parts of signals interact when representing them using the family of differential operators.

    $W(\omega)$ must satisfy

    $$\int_{-\infty}^{\infty} W(\omega) d\omega = 2\pi \tag{1}$$

    and $\int_{-\infty}^{\infty} W(\omega)\omega^n d\omega < \infty$ for every n.

3. The *frequency-domain polynomials*, $P_n(\omega)$ for every n each of degree n, which are complex-valued orthonormal polynomials with respect to the frequency-domain weight.[1] In detail, we require

    $$\int_{-\infty}^{\infty} P_k(\omega) P_m^*(\omega) W(\omega) d\omega = \begin{cases} 2\pi & k = m \\ 0 & k \neq m \end{cases} \tag{2}$$

    Throughout this formulary, these polynomials are chosen by applying the Gram-Schmidt procedure to $1, x, x^2, x^3, \ldots$, modulo the above requirement. These polynomials determine the transfer function of chromatic differentiation:[2]

    $$K_n[f](t) = \mathcal{F}^{-1}\left[P_n(\omega)F(\omega)\right](t) \tag{3}$$

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[1] Although Legendre's P polynomials form one of the classes of polynomials, we are using "P_n" here in a completely generic sense.

[2] Throughout this paper, we implicitly assume that for any f considered, $F(\omega) = 0$ wherever $W(\omega) = 0$, for the given W. This is significant in equation (3) because, for bandlimited W, K_n does not act outside the bandlimit, even though P_n is non-zero. (We could restrict the range of the polynomials, but the current treatment is analogous to standard differentiation.)

Note that when W displays even symmetry, $(W(-\omega) = W(\omega))$, the indicated orthogonalization procedure produces functions P_{2k} with even symmetry and P_{2k+1} with odd symmetry $(P_{2k+1}(-\omega) = -P_{2k+1}(\omega))$. Given this, we can choose coefficients which will guarantee conjugate symmetry for every n $(P_n(-\omega) = P_n^*(\omega))$, in which case the chromatic derivatives of real signals are real. This is done for all families discussed in this formulary.

4. The explicit *time-domain functions* which are associated with the chromatic representation of each family. These are the functions

$$B_n(t) = (-1)^n \mathcal{F}^{-1}\left[P_n(\omega)W(\omega)\right](t) \tag{4}$$

which are used to interpolate a signal locally after the family's differential operators have acted upon the signal. They may be equivalently presented as

$$\begin{array}{rcl} B_0(t) & = & \mathcal{F}^{-1}\left[W(\omega)\right](t) \\ B_n(t) & = & (-1)^n K_n[B_0](t) \end{array} \tag{5}$$

5. The *chromatic derivative definition*, an inductive definition of the family's differential operators, which is closely related to (and straightforwardly derivable from) the recursively generated frequency-domain polynomials.

6. The *chromatic derivative operator iteration* formula which governs how a chromatic derivative acts upon another chromatic derivative within the same family. Expressing products of frequency-domain polynomials as $P_m P_k = \sum_{j=0}^{m+k} \alpha_{k,m,j} P_j$, we can express

$$\begin{array}{rcl} K_m K_k f & = & \mathcal{F}^{-1}\left[P_m \mathcal{F}\left[\mathcal{F}^{-1}\left[P_k F\right]\right]\right] \\ & = & \mathcal{F}^{-1}\left[P_m P_k F\right] \\ & = & \mathcal{F}^{-1}\left[\displaystyle\sum_{j=0}^{m+k} \alpha_{k,m,j} P_j F\right] \\ & = & \displaystyle\sum_{j=0}^{m+k} \alpha_{k,m,j} K_j f \end{array}$$

Linearization formulas are given for classical families of orthogonal polynomials in the compact form $\tilde{P}_m \tilde{P}_k = \sum_{j=0}^{min(m,k)} \beta_{k,m,j} \tilde{P}_{m+k-2j}$. We thus express iteration of chromatic differentiation as

$$K_m[K_k[f]] = \sum_{j=0}^{min(m,k)} a_{k,m,j} K_{m+k-2j}[f] \tag{6}$$

where the $a_{k,m,j}$ will differ from the classical $\beta_{k,m,j}$ in that we use normalized, complex versions of the classical orthogonal polynomials.

Other pertinent formulas also appear, such as the chromatic derivatives (CD's) of important functions or special time-domain function orthogonality relations.

The primary application for chromatic differentiation is signal representation and approximation. Signal representation is obtained by decomposing the Fourier kernel, which shows how the time and frequency-domain functions are connected.

$$e^{i\omega t} = \sum_{k=0}^{\infty} P_k(\omega) B_k(t) \tag{7}$$

Signals can thus be represented by their chromatic derivatives taken at an arbitrarily chosen "central" point t_0 as follows: [3]

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t_0}e^{i\omega(t-t_0)}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t_0}\left[\sum_{k=0}^{\infty} P_k(\omega)B_k(t-t_0)\right]d\omega \\
&= \sum_{k=0}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} P_k(\omega)F(\omega)e^{i\omega t_0}d\omega\right]B_k(t-t_0) \\
f(t) &= \sum_{k=0}^{\infty} K_k[f](t_0)B_k(t-t_0) \qquad (8)
\end{aligned}$$

A local approximation of a signal is given by considering only $K_0 f$ through $K_M f$. The error introduced by such an approximation is given by the following:

$$\left| f(t) - \sum_{k=0}^{M} K_k[f](t_0)B_k(t-t_0)\right| \leq \sqrt{A}\sqrt{1 - \sum_{k=0}^{M} B_k(t-t_0)^2} \tag{9}$$

where $A = \int_{-\infty}^{\infty} \frac{1}{W(\omega)} \mid \hat{f}(\omega) \mid^2 d\omega$ is "windowed" energy (i.e., energy relative to the frequency domain weight W).

3 Bandlimited Signal Representation and Signal Processing

3.1 Bessel J / Chebyshev T

This is one of the most important families of operators because of the simplicity of its formulas and because of the many properties that Chebyshev T polyno-

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[3] we ignore the required argument for convergence of the infinite sum, which can be found in [2]; a different type of general argument appears in [1]

mials and Bessel J functions possess. In particular the Bessel J / Chebyshev T CD operators are useful in the following situations:

- wherever we need to approximate a signal or transfer function in the frequency domain since Chebyshev T polynomials provide an optimal minimax approximation; e.g. they can be profitably used in filtering.
- wherever we need to or are able to solve time-domain differential equations having solutions which are Bessel functions, of which there are many in physical phenomena; e.g. in antenna design.
- when we need to process signals which locally look like they contain infinite energy; e.g. transient-like signals in audio applications.

For this section, we let $c_0 = 1$ and $c_k = \sqrt{2}$ for positive integers ($k > 0$).

Frequency Domain Weight

$$W^T(\omega) = \begin{cases} \frac{2}{\pi}(1 - (\frac{\omega}{\pi})^2)^{-\frac{1}{2}} & -\pi \le \omega \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Frequency Domain Polynomials

$$P_k^T(\omega) = i^k c_k T_k(\frac{\omega}{\pi})$$

where T_k are Chebyshev polynomials of the first kind.

Time Domain Functions

$$B_k^T(t) = c_k J_k(\pi t)$$

where J_k are Bessel functions of the first kind.

Chromatic Derivative Definition

$$\begin{aligned} K_0^T[f] &= f \\ K_1^T[f] &= \frac{\sqrt{2}}{\pi}\frac{\partial f}{\partial t} \\ K_2^T[f] &= \frac{2}{\pi}\frac{\partial}{\partial t}K_1^T[f] + \sqrt{2}K_0^T[f] \\ K_n^T[f] &= \frac{2}{\pi}\frac{\partial}{\partial t}K_{n-1}^T[f] + K_{n-2}^T[f] \text{ for } n \ge 3 \end{aligned}$$

Chromatic Derivative Operator Iteration

$$T_m(x)T_k(x) = \frac{1}{2}(T_{m+k}(x) + T_{m-k}(x))$$

implies

$$K_m^T[K_k^T[f]] = \frac{1}{2}\left(\left(\frac{c_m c_k}{c_{m+k}}\right)K_{m+k}^T[f] + (-1)^{min(m,k)}\left(\frac{c_m c_k}{c_{|m-k|}}\right)K_{|m-k|}^T[f]\right)$$

and

$$K_k^T[B_m^T] = \frac{(-1)^k}{2}\left(\left(\frac{c_m c_k}{c_{m+k}}\right)B_{m+k}^T + (-1)^{min(m,k)}\left(\frac{c_m c_k}{c_{|m-k|}}\right)B_{|m-k|}^T\right).$$

Orthogonality Relation

$$\int_{-\infty}^{\infty} B_m^T(t)B_n^T(t)\frac{dt}{|t|} = \begin{cases} \infty & m=n=0 \\ \frac{2}{n} & m=n>0 \\ 0 & m \neq n \end{cases}$$

CD's of Sinc

$$K_m^T[\text{sinc}](0) = \begin{cases} 1 & m=0 \\ 0 & m \text{ odd} \\ \sum_{j=0}^{m/2}(-1)^{j+\frac{m}{2}}2^{m-2j+1}\frac{\sqrt{2}m\binom{m-j}{j}}{(m-j)(m-2j+1)} & m \geq 2 \text{ even} \end{cases}$$

CD's of Sine and Cosine $(0 \leq a \leq \pi)$

$$\begin{aligned} K_m^T[\sin(at)](0) &= \begin{cases} (-1)^{(m-1)/2}P_m^T(a) & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \\ K_m^T[\cos(at)](0) &= \begin{cases} (-1)^{m/2}P_m^T(a) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} \end{aligned}$$

3.2 Double Bessel / Chebyshev U

The double Bessel / Chebyshev U differential operators are important themselves and also indicate the possibilities available in other ultrashperical polynomials which use frequency-domain weights which emphasize lower frequencies inside the bandlimit. In particular, these operators are useful:

- wherever we need to approximate a signal or transfer function in the frequency domain since Chebyshev U polynomials provide an optimal least first power approximation; e.g. they can be profitably used in filtering.
- whenever we wish to approximate waveforms which dampen quickly; an inherent property of double Bessel functions.

Frequency Domain Weight

$$W^U(\omega) = \begin{cases} \frac{4}{\pi}(1-(\frac{\omega}{\pi})^2)^{\frac{1}{2}} & -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Frequency Domain Polynomials

$$P_k^U(\omega) = i^k U_k(\frac{\omega}{\pi})$$

where U_k are Chebyshev polynomials of the second kind.

Time Domain Functions

$$B_k^U(t) = J_k(\pi t) + J_{k+2}(\pi t)$$

where J_k are Bessel functions of the first kind. We refer to the sum $J_k + J_{k+2}$ as a "double Bessel" function.

Chromatic Derivative Definition

$$\begin{aligned} K_0^U[f] &= f \\ K_1^U[f] &= \frac{2}{\pi}\frac{\partial f}{\partial t} \\ K_n^U[f] &= \frac{2}{\pi}\frac{\partial}{\partial t}K_{n-1}^U[f] + K_{n-2}^U[f] \text{ for } n \geq 2 \end{aligned}$$

Chromatic Derivative Operator Iteration

$$\begin{aligned} K_m^U[K_k^U[f]] &= \sum_{j=0}^{\min(k,m)} (-1)^j K_{k+m-2j}^U[f] \\ K_k^U[B_m^U] &= \sum_{j=0}^{\min(k,m)} (-1)^{k+j} B_{k+m-2j}^U \end{aligned}$$

Orthogonality Relation

$$\int_{-\infty}^{\infty} B_m^U(\pi t) B_n^U(\pi t)|t|dt = \begin{cases} 4(m+1) & m = n \\ 0 & m \neq n \end{cases}$$

CD's of Sinc

$$K_m^U[\text{sinc}](0) = \begin{cases} 1 & m = 0 \\ 0 & m \text{ odd} \\ \sum_{j=0}^{m/2} (-1)^{j+\frac{m}{2}} 2^{m-2j} \frac{\binom{m-j}{j}}{(m-2j+1)} & k \geq 2 \text{ and even} \end{cases}$$

CD's of Sine and Cosine $(0 \leq a \leq \pi)$

$$\begin{aligned} K_m^U[\sin(at)](0) &= \begin{cases} (-1)^{(m-1)/2} P_m^U(a) & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \\ K_m^U[\cos(at)](0) &= \begin{cases} (-1)^{m/2} P_m^U(a) & m \text{ even} \\ 0 & m \text{ odd} \end{cases} \end{aligned}$$

3.3 Spherical Bessel j / Legendre P

The spherical Bessel j / Legendre P family of differential operators are very useful because of the flat frequency weighting they have. Such operators are useful:

- whenever general bandlimited signals are being processed, whether for filtering, transmission, or compression.
- wherever we need to approximate a signal or transfer function in the frequency domain since Legendre P polynomials provide an optimal least squares approximation; e.g. they can be profitably used in filtering.
- whenever we wish to generalize the sinc functions which are the harmonic interpolants of signals; sinc is the zeroth time-domain function of the spherical Bessels. Thus, this family of differential operators can be used to represent locally any signal of finite energy.

Frequency Domain Weight

$$W^P(\omega) = \begin{cases} 1 & -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Frequency Domain Polynomials

$$P_k^P(\omega) = \sqrt{2k+1}P_k(\frac{\omega}{\pi})$$

where P_k are the Legendre (spherical) polynomials.

Time Domain Functions

$$B_k^P(t) = \sqrt{2k+1}j_k(\pi t) = \sqrt{2k+1}\frac{J_{k+\frac{1}{2}}(\pi t)}{\sqrt{2t}}$$

where j_k are spherical Bessels of the first kind.

Chromatic Derivative Definition

$$\begin{aligned} K_0^P[f] &= f \\ K_1^P[f] &= \frac{\sqrt{3}}{\pi}\frac{\partial f}{\partial t} \\ K_n^P[f] &= \left(\frac{(2n-1)\sqrt{2n+1}}{\pi n\sqrt{2n-1}}\right)\frac{\partial}{\partial t}K_{n-1}^P[f] + \left(\frac{(n-1)\sqrt{2n-1}}{n\sqrt{2n-3}}\right)K_{n-2}^P[f] \\ & \text{for } n \geq 2. \end{aligned}$$

Chromatic Derivative Operator Iteration

$$K_m^P[K_k^P[f]] = \sum_{j=0}^{\min(m,k)} (-1)^j a_{k,m,j}\frac{\sqrt{2m+1}\sqrt{2k+1}}{\sqrt{2(k+m-2j) = 1}}K_{k+m-2j}^P[f]$$

for constants $a_{k,m,j}$ computable from the Ferrers-Adams linearization formula: [4]

$$\begin{aligned}
P_m(x)P_k(x) &= \sum_{j=0}^{\min(m,k)} \left(\frac{2m+2k+1-4j}{2m+2k+1-2j}\right) \cdot \left(\frac{\binom{m+k-j}{k}}{\binom{m+k-j}{k}_{\frac{1}{2}}}\right) \\
&\quad \left(\frac{\binom{k}{j}}{\binom{k}{j}_{\frac{1}{2}}}\right) \cdot P_{m+k-2j}(x) \\
&= \sum_{j=0}^{\min(k,m)} a_{k,m,j} P_{m+k-2j}(x)
\end{aligned}$$

$$K_k^P[B_m^P] = \sum_{j=0}^{\min(k,m)} (-1)^{j+k} a_{k,m,j} \frac{\sqrt{2m+1}\sqrt{2k+1}}{\sqrt{2(k+m-2j)+1}} B_{k+m-2j}^P$$

Orthogonality Relation

$$\int_{-\infty}^{\infty} B_m^P(t) B_n^P(t)dt = \begin{cases} \frac{\sqrt{(2m+1)(2n+1)}\Gamma(\frac{m+1}{2})}{\sqrt{\pi}\Gamma(\frac{m+2}{2})} & m = n \\ 0 & m \neq n \end{cases}$$

CD's of Sinc

$$K_m^P[\text{sinc}](0) = \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise} \end{cases}$$

CD's of Sine and Cosine $(0 \le a \le \pi)$

$$\begin{aligned}
K_m^P[\sin(at)](0) &= \begin{cases} (-1)^{(m-1)/2} P_m^P(a) & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \\
K_m^P[\cos(at)](0) &= \begin{cases} (-1)^{m/2} P_m^P(a) & m \text{ even} \\ 0 & m \text{ odd} \end{cases}
\end{aligned}$$

3.4 Ultrabessel / Ultraspherical $(\nu > -\frac{1}{2}, \nu \neq 0)$

The ultrabessel/ultraspherical differential operators can be used for locally processing most useful, real-valued, bandlimited signals. Such operators include the previously mentioned families of bandlimited operators as special cases.

- Probably the most useful of the remaining families in the ultraspherical group of families is where ν is relatively large; such weights would be useful for processing and generating signals containing mostly low frequencies (but with some higher frequency components). Families with rather simple governing equations are those with $\nu = \frac{1}{2}n$ for n a positive integer.

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[4] where $\binom{z}{w}_\nu = \frac{(\nu)_z}{(\nu)_w(\nu)_{z-w}}$ is the "fractional z choose w" function and where $(\nu)_z = \frac{\Gamma(\nu+z)}{\Gamma(\nu)}$ is the Pochhammer symbol

- These can be particularly useful for transmission modulation in severely attentuating channels which force one to transmit most information in the lower frequencies. For this application we would select the ultraspherical weight which would provide the closest match to what the water-filling version of Shannon's theorem would advise.

Frequency Domain Weight

$$W^{C^\nu}(\omega) = \begin{cases} \frac{\nu\Gamma(\nu)^2}{\pi 2^{-2\nu}\Gamma(2\nu)}(1-(\frac{\omega}{\pi})^2)^{\nu-\frac{1}{2}} & -\pi \le \omega \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Frequency Domain Polynomials

$$P_k^{C^\nu}(\omega) = \sqrt{\frac{(k+\nu)k!\Gamma(2\nu)}{\nu\Gamma(k+2\nu)}}C_k^\nu(\frac{\omega}{\pi})$$

where C_k^ν are the ultraspherical polynomials, also known as the Gegenbauer polynomials.

Time Domain Functions

$$B_k^{C^\nu}(t) = \left(\frac{J_{k+\nu}(\pi t)}{t^\nu}\right)\cdot\left(\frac{2^\nu(k+\nu)\Gamma(\nu)}{\pi^\nu}\right)\sqrt{\frac{\nu\Gamma(k+2\nu)}{k!(k+\nu)\Gamma(2\nu)}}$$

We refer to $\sqrt{\frac{\pi}{2}}\frac{J_{k+\nu}(\pi t)}{t^\nu}$ as "ultrabessel" functions.

Chromatic Derivative Definition

$$\begin{aligned} K_0^{C^\nu}[f] &= f \\ K_1^{C^\nu}[f] &= \sqrt{2(\nu+1)}\left(\frac{1}{\pi}\right)\frac{\partial f}{\partial t} \\ K_n^{C^\nu}[f] &= \sqrt{\frac{(n+\nu)(n-1+\nu)}{n(n-1+2\nu)}}\left(\frac{2}{\pi}\right)\frac{\partial}{\partial t}K_{n-1}^{C^\nu}[f] \\ &\quad +\sqrt{\frac{(n+\nu)(n-1)(n-2+2\nu)}{n(n-1+2\nu)(n-2+\nu)}}K_{n-2}^{C^\nu}[f] \text{ for } n \ge 2. \end{aligned}$$

Chromatic Derivative Operator Iteration

$$\begin{aligned} K_m^{C^\nu}[K_k^{C^\nu}[f]] &= \sum_{j=0}^{\min(m,k)}(-1)^j a_{k,m,j}\cdot K_{k+m-2j}^{C^\nu}[f]\cdot \\ &\quad \sqrt{\frac{(m+\nu)(k+\nu)m!k!\Gamma(2\nu)\Gamma(k+m-2j+2\nu)}{(k+m-2j+\nu)\nu(k+m-2j)!\Gamma(m+2\nu)\Gamma(k+2\nu)}} \end{aligned}$$

for constants $a_{k,m,j}$ computable from the Dougall-Rogers linearization formula: [5]

$$\begin{aligned} C_m^{\nu}(x)C_k^{\nu}(x) &= \sum_{j=0}^{\min(m,k)} \left(\frac{m+k+\nu-2j}{m+k+\nu-j}\right) \cdot \left(\frac{\binom{m+k-2j}{k-j}}{\binom{m+k-j}{k}_{\nu}}\right) \\ &\quad \left(\frac{\binom{m+k-j+2\nu-1}{j}}{\binom{k}{j}_{\nu}}\right) \cdot C_{m+k-2j}^{\nu}\,(x) \\ &= \sum_{j=0}^{\min(m,k)} a_{k,m,j} \cdot C_{m+k-2j}^{\nu}\,(x) \end{aligned}$$

$$\begin{aligned} K_k^{C^{\nu}}[B_m^{C^{\nu}}] &= \sum_{j=0}^{\min(k,m)} (-1)^{j+k} a_{k,m,j}\, B_{k+m-2j}^{C^{\nu}} \cdot \\ &\quad \sqrt{\frac{(m+\nu)(k+\nu)m!k!\Gamma(2\nu)\Gamma(k+m-2j+2\nu)}{(k+m-2j+\nu)\nu(k+m-2j)!\Gamma(m+2\nu)\Gamma(k+2\nu)}} \end{aligned}$$

Moving Between Ultraspherical CD Types Since it is the case that: [6]

$$\begin{aligned} C_n^{\nu}(x) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\nu)_{n-k}(\nu-\mu)_k(n+\mu-2k)}{(\mu+1)_{n-k}k!\mu}\, C_{n-2k}^{\mu}(x) \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{n,k}^{\nu,\mu} C_{n-2k}^{\mu}(x) \end{aligned}$$

we see that:

$$\begin{aligned} K_n^{C^{\nu}}[f] &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{n,k}^{\nu,\mu}\, K_{n-2k}^{C^{\mu}}[f] \cdot \\ &\quad \sqrt{\frac{(n+\nu)\mu n!\Gamma(n-2k+2\mu)\Gamma(2\nu)}{(n-2k+\mu)\nu(n-2k)!\Gamma(n+2\nu)\Gamma(2\mu)}} \end{aligned}$$

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[5] where $\binom{z}{w}_{\nu} = \frac{(\nu)_z}{(\nu)_w(\nu)_{z-w}}$ is the "fractional z choose w" function; where $(\nu)_z = \frac{\Gamma(\nu+z)}{\Gamma(\nu)}$ is the Pochhammer symbol; and where $\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$ is the ordinary generalized "z choose w" function.

[6] where $(x)_k = x \cdot (x+1) \cdots (x+k-1)$ for k a positive integer and $(x)_0 = 1$

4 Non-Bandlimited Signal Representation and Processing

4.1 Gaussian Monomial / Hermite H

When considering the difficult problem of processing non-bandlimited signals, the Gaussian mononmial / Hermite H differential operators have many attractive features because of the fact that their sharply decreasing frequency-domain weight makes them somewhat bandlimited in practice. Thus, the operators are useful:

- whenever there are mainly bandlimited signals to be processed but where we need access to higher frequencies for a specific reason; e.g. for dealing with nonlinearities in transmission.
- when we are processing signals with a lot of bandlimited content but where higher frequencies must be dealt with as central to the signal; e.g. in video processing.
- where we need to process 2-dimensional or higher order signals, because the form of the time and frequency domain functions is very convenient for generalizations of the formulas given below.

Frequency Domain Weight

$$W^H(\omega) = 2\sqrt{\pi}e^{-\omega^2}$$

Frequency Domain Polynomials

$$P_k^H(\omega) = \frac{1}{\sqrt{2^k k!}} H_k(\omega)$$

where the H_k are Hermite polynomials.

Time Domain Functions

$$B_k^H(t) = \frac{1}{\sqrt{2^k k!}} t^k e^{-t^2/4}$$

which are Gaussian monomials.

Chromatic Derivative Definition

$$\begin{aligned}
K_0^H[f] &= f \\
K_1^H[f] &= \sqrt{2}\frac{\partial f}{\partial t} \\
K_n^H[f] &= \frac{1}{\sqrt{2(n-1)}}\frac{\partial}{\partial t}K_{n-1}^H[f] + \sqrt{\frac{(n-1)}{(n-2)}} \cdot K_{n-2}^P[f] \text{ for } n \geq 2
\end{aligned}$$

Chromatic Derivative Operator Iteration

$$K_m^H[K_k^H[f]] = \sum_{j=0}^{\min(m,k)} (-1)^j a_{k,m,j} \cdot \sqrt{\frac{(m+k-2j)!}{m!k!4^j}} K_{k+m-2j}^H[f]$$

for constants $a_{k,m,j}$ computable from the Nielsen linearization formula:

$$\begin{aligned} H_m(x)H_k(x) &= \sum_{j=0}^{\min(m,k)} \binom{m}{j}\binom{k}{j} 2^j j!\, H_{m+k-2j}(x) \\ &= \sum_{j=0}^{\min(m,k)} a_{k,m,j} H_{m+k-2j}(x) \end{aligned}$$

$$K_k^H[B_m^H] = \sum_{j=0}^{\min(k,m)} (-1)^{j+k} a_{k,m,j} \sqrt{\frac{(m+k-2j)!}{m!k!4^j}}\, B_{k+m-2j}^H$$

4.2 SechTanh / Hyperbolic

The sechtanh/hyperbolic differential operator family can be useful when we need to process highly non-bandlimited signals, but where we do not need all of the power of wavelets and/or where we wish to retain the ability to process functions locally (using a fundamentally *local* approximation). Such operators can be used:

- to calculate rough, but wide-ranging, frequency spectrum values of unknown signals.
- to represent high frequency signals approximately using only (relatively) slow integrations to collect the interpolation coefficients.

Frequency Domain Weight

$$W^L(\omega) = \operatorname{sech}\left(\frac{1}{2}\omega\right)$$

Frequency Domain Polynomials

$$P_k^L(\omega) = L_k(\omega/\pi)$$

where L_k are what we call the hyperbolic polynomials which are orthogonal to weight $\operatorname{sech}(\pi x/2)$, and generated by the recursion:[7]

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x \\ L_n(x) &= -\frac{1}{n}x \cdot L_{n-1}(x) - \left(\frac{n-1}{n}\right) L_{n-2}(x) \qquad [n \geq 2] \end{aligned}$$

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[7] Compare to the classical Laguerre polynomials

Time Domain Functions

$$B_k^L(t) = (-1)^k \operatorname{sech}(\pi t) \tanh^k(\pi t) = S_k(t)$$

(hyperbolic secant and hyperbolic tangent)

Chromatic Derivative Definition

$$\begin{aligned} K_0^L[f] &= f(t) \\ K_1^L[f] &= -\frac{1}{\pi}\frac{\partial}{\partial t} f \\ K_n^L[f] &= -\frac{1}{\pi n}\frac{\partial}{\partial t} K_{n-1}^L[f] + \left(\frac{n-1}{n}\right) K_{n-2}^L[f] \end{aligned}$$

Chromatic Derivative Operator Iteration

$$K_m^L[K_k^L[f]] = \sum_{j=0}^{\min(m,k)} (-1)^j a_{k,m,j} \cdot K_{k+m-2j}^L[f]$$

for constants $a_{k,m,j}$ computable as follows:

1. $a_{m,k,k} = a_{k,m,k} = \binom{m}{k}$ for all $0 \leq k \leq m$;
2. by induction on (m, k) we simultaneously define for $m, k > 0$

$$a_{k,m,0} = a_{m,k,0} = a_{k-1,m,0} + a_{k,m-1,0};$$

3. by induction on (m, k, j) we simultaneously define for $j < \min(k, m)$

$$a_{k,m,j} = a_{m,k,j} = a_{k-1,m,j} + a_{k-1,m-1,j-1} + a_{k,m-1,j}.$$

We obtain:

$$K_k^L[B_m^L] = \sum_{j=0}^{\min(k,m)} (-1)^{j+k} a_{k,m,j}\, B_{k+m-2j}^L$$

5 The General Case Numerically

In this section we outline how to find families of differential operators and their associated chromatic expansions when we are given only a real valued frequency domain weight to start with. Naturally, the task is not so straightforward as was the case with the classical orthogonal polynomials because there will not be closed-form formulas and definitions to refer to when building up the necessary library of tools we need to efficiently do local signal processing with chromatic derivatives.

Note that the suggestions below are not intended for real-time computation of families of chromatic derivatives; the computations are intended to be done off-line and will generally take a considerable amount of time to attain good numerical accuracy. However, given an even, real-valued weight with a bandlimit—an important case because it covers all real bandlimited signals (and therefore all complex signals for which the real and imaginary parts have identical spectra)—it is possible to extract a procedure from those given below which produces good approximate frequency and time-domain functions and differential operators during a reasonably short real-time training period. However, this requires starting with auxiliary polynomials in order to define such families, as we outline in §5.1.2 and §5.2.1.

5.1 Obtaining the Frequency-Domain Orthonormal Polynomials

When given a general real-valued weight; a measure on L^2 over the frequency domain from $-\infty$ to $+\infty$ (or a restricted frequency range, if given a bandlimited weight); we follow the steps below to produce the corresponding system of orthonormal polynomials.

It is useful here to note that any set of polynomials $P_i(x)$ orthogonal with respect to a given real-valued weight $W(x)$; $\int_{-\infty}^{+\infty} P_i(x)P_j(x)W(x)dx = 2\pi\delta_{i,j}$; can be generated recursively from $P_0(x)$ and $P_1(x)$ by knowing the constants in the recursion equation: $P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x)$ (where $c_n > 0$, $n \geq 1$ and all constants are real) that accompanies every such sequence of orthogonal polynomials. We aim to find such a set of polynomials through integration (using the weight) using standard orthogonalization techniques and from which we can extract the above recursion constants.

5.1.1 Basic method

1. First, normalize the weight: $\int_{-\infty}^{+\infty} W(x)dx = 2\pi$.

2. To compute up to the N^{th}-order Taylor differential operator, compute the following integrals for all of the positive integers $n > 0$ up to $2N$: $m_n = \int_{-\infty}^{+\infty} x^n W(x)dx$. Such values must exist for the weight to be a suitable one for generating differential operators, and if computed once then the following steps are all just algebraic. If the weight is even ($W(x) = W(-x)$ for all x) then we only have to integrate the weight with even monomials.

3. Define an inner product on the space $\mathbb{R}^{2N}$, which represents the space of polynomial coefficients, as follows: $\langle \vec{p}, \vec{q} \rangle = \sum_{i,j=0}^{N} p_i q_j m_{i+j}$.

4. Then, in the general case in which the weight is not an even function use the modified Gram-Schmidt method with respect to the above inner product, starting with the unit vectors $\langle 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0 \rangle$, to generate the orthogonal polynomial coefficients for n-order polynomials ($n > 1$). At the end we can normalize the polynomials easily by again using the inner product.

5. In the case where $W(x)$ is an even function, we note that this means that we can break the modified Gram-Schmidt process down into two parts (even and odd order polynomials), which could be useful if N is large: the modified Gram-Schmidt process takes on the order of $(N+1)^3$ flops.

6. Finally, get the recursion coefficients that iteratively define the polynomials by solving for them directly using the orthonormal functions or by computing the coefficients of the equation in the next subsection.

Thus, we will have determined both the orthonormal polynomials and the recursion equations which determine the chromatic derivatives; the CD recursion equation is taken from the polynomial recursion equation by just inserting some i's and replacing the x's by partial derivatives.

5.1.2 Alternative Methods

There are two modifications to the above basic methods which involve using analytic expressions for the purpose of increased numerical accuracy.

The first alternative for calculating orthogonal polynomials can be employed if we are computing the polynomials using symbolic computation software, such as Mathematica, and the weight $W(x)$ is in relatively tractable analytic form. In this case we compute the m_n as above, *symbolically*, along with the associated inner product, and just use the following three-term recursion directly after defining $P_0(x) = 1$ and $P_1(x) = x - m_1$:

$$P_{n+1}(x) = \left(x - \left(\frac{\langle xP_n(x), P_n(x)\rangle}{\langle P_n(x), P_n(x)\rangle}\right)\right) P_n(x) - \left(\frac{\langle P_n(x), P_n(x)\rangle}{\langle P_{n-1}(x), P_{n-1}(x)\rangle}\right) P_{n-1}(x)$$

where $\langle\cdot\rangle$ is the inner product associated with the weight W. This will produce orthogonal polynomials which can then be orthonormalized using the inner product (see, e.g., [3, §3.6]). As an example of using this method, we have produced very accurate recursion coefficients for the orthonormalized polynomials associated with the (non-normalized) bandlimited weight function:

$$W(x) = \begin{cases} 1 & |x| < \pi \\ 1 - \frac{1}{\delta}(\frac{|x|}{\pi} - 1) & \pi \le |x| < \pi(1+\delta) \\ 0 & \text{otherwise} \end{cases}$$

for any small $\delta > 0$.

For the second alternative, one can substitute auxiliary orthogonal polynomials for the monomials used in computing m_n in the basic method. Assume that we are given the analytically-known polynomials $R_k(x)$ which are orthogonal with respect to a weight $W_0(x)$—we shall call these the auxiliary polynomials. Then define the N^{th}-order differential operators $m_n = \int_{-\infty}^{+\infty} R_n(x)W(x)dx$. This definition can provide superior numerical accuracy compared to the Taylor differential operator, especially in the bandlimited case, if we select the auxiliary orthogonal polynomials so that $W_0(x)$ is close to $W(x)$. Then, proceeding with the basic method (steps 3 and 4-or-5), we end up with a set of coefficients which tell us which linear mixtures of the auxiliary polynomials form polynomials orthogonal with respect to the original weight $W(x)$. Finally, it is easy to transform these "polynomials in polynomials" into ordinary polynomials since we can, using the recursion generation equation of the auxiliary orthogonal polynomials $R_k(x)$, recursively build a matrix which converts linear combinations of auxiliary polynomials into polynomials orthogonal with respect to $W(x)$.[8] Then, only step 6 of the basic method, normalization, remains.

5.2 Obtaining the Time-Domain Chromatic Expansion Functions

In the general case there is not much to be done in this step except to simply compute the time-domain functions by performing a numerical inverse Fourier transform for each basis function:

$$B_n(t) = (-1)^n \mathcal{F}^{-1} \left[P_n(\omega) W(\omega) \right]$$

for a certain number of t values around 0. The integrations will need to be approximated as the limit of increasing bandlimits if there is no bandlimit on the weight.

If there is a bandlimit on the weight we might be able to do the calculation by performing an FFT on the polynomials times the weight, though in that case we would have to insure at least three things:

- Check the performance of the procedure if the weight will produce functions of infinite energy (using FFTs in this situation can be unstable).
- Use various windows when performing the FFTs to insure consistent solutions are being generated.
- Compute the functions for a very extended range around 0 in order to be able to accurately interpolate the functions in between Nyquist points since those are the only places at which the FFT defines the time domain functions.

Alternatively, given that we want to produce time-domain functions which obey the equation: $K_n[B_0] = (-1)^n B_n$, if it is possible to compute just B_0 using a Fourier transform and approximate B_0 very accurately using functions which are repeatedly differentiable, especially if they are symbolically differentiable, then we can use the recursive form of the CD definition to compute the higher order time-domain functions from the 0th-order time-domain function.

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[8] For example, if $P_0(x) = 1$, $P_1(x) = a_0 x$, and $P_{n+1}(x) = a_n x P_n(x) - b_n P_{n-1}(x)$ then we can inductively show that if $x^n = \sum_{m=0}^{n} \gamma_m^n P_m(x)$, then $x^{n+1} = \left(\frac{\gamma_n^n}{a_n}\right) P_{n+1}(x) + \sum_{m=1}^{n} \left[\left(\frac{\gamma_{m-1}^n}{a_{m-1}}\right) + \left(\frac{b_{m+1}\gamma_{m+1}^n}{a_{m+1}}\right) \right] P_m(x) + \left(\frac{\gamma_1^n b_1}{a_1}\right)$ where each term contributes to the sum where its subscripts are > 0.

5.2.1 Time Domain Basis Functions Using Auxiliary Orthogonal Polynomials

If the frequency-domain polynomials P_k were obtained from an auxiliary set of polynomials R_k orthogonal with respect to some weight W_0, as in §5.1.2, then another option is available to us for creating the basic time-domain functions, provided that the basis functions D_k for the weight W_0 are known. In this case, each $P_n(\omega)$ was obtained as $\sum_{m=0}^{n} \beta_m R_m$. We assume also that we have obtained the linearization formula for the auxiliary polynomials $R_k R_m = \sum_{p=0}^{2\min(m,k)} \gamma_p R_{m+k-p}$ as described in the next subsection. In order to obtain chromatic differentiation up to degree N, we first approximate the ratio of the weights using a linear combination of the auxiliary polynomials:[9] $\frac{W(x)}{W_0(x)} = \sum_{k=0}^{2N} \alpha_k R_k$. The basis functions for W can then be computed as follows:

$$\begin{aligned}
B_n &= (-1)^n \mathcal{F}^{-1}\left[W P_n\right] \\
&= (-1)^n \mathcal{F}^{-1}\left[\left(\sum_{k=0}^{2N} \alpha_k R_k\right) W_0 \sum_{m=0}^{n} \beta_m R_m\right] \\
&= \sum_{k=0}^{2N} \sum_{m=0}^{n} \alpha_k \beta_m \mathcal{F}^{-1}\left[R_k R_m W_0\right] \\
&= \sum_{k=0}^{2N} \sum_{m=0}^{n} \sum_{p=0}^{2\min(m,k)} \alpha_k \beta_m \gamma_p \mathcal{F}^{-1}\left[R_{k+m-p} W_0\right] \\
&= \sum_{k=0}^{2N} \sum_{m=0}^{n} \sum_{p=0}^{2\min(m,k)} \alpha_k \beta_m \gamma_p D_{k+m-p}
\end{aligned}$$

5.3 Obtaining the Other Formulas

The only other fundamental equation for which we need to generate coefficients is the CD iteration formula, a formula very useful in adaptive filtering and modulation among other areas. All that is lacking are the coefficients given by the linearization formula associated with the orthogonal polynomials:

$$P_m(x) P_k(x) = \sum_{i=0}^{2\min(m,k)} d_{k,m,i} P_{k+m-i}(x)$$

Such linear coefficients are always guaranteed to exist: we could solve for the linearization coefficients from the polynomial coefficients by just using the equation above, multiplying out the left-hand side, and solving for the $d_{k,m,i}$ in terms

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[9] If we are trying to create the time-domain functions on line then it is easy to make this approximation if the two weights are first approximated by the auxilliary polynomials. Then, we can compute the ratio of the two weights in terms of auxilliary polynomials by using a generalized Pade approximation, which is rather easy. (See [4])

of the polynomial coefficients. However, such sets of equations tend to be poorly conditioned, the more so as k and m grow, and so a more stable solution method should be sought. Note that even for the classical orthogonal polynomials, producing closed forms of the linearization equations as we display in the formulary above are non-trivial undertakings. And even then some of those forms are not optimal for numerically calculating the coefficients we need.[10]

However, there is a recursive way to generate these tables by using a recursion formula invented by Richard Askey [5]:

1. We can rearrange the basic recursion formula

   $$P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x)$$

   by requiring it to be renormalized so that the leading coefficient of each polynomial is always 1, which we will call *term-normalized* orthogonal polynomials. Then we can rewrite the basic recursion as follows (renaming all constants):

   $$P_1(x)P_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x)$$

   This has the form of a linearization formula—the easiest non-trivial case.

2. Then, it is fairly easy to show, by twice using the above basic, "linearized" recursion formula, that

   $$P_{l+1}P_n - P_l P_{n+1} = (a_n - a_l)P_l P_n + (b_n - b_l)P_l P_{n-1} + b_l[P_l P_{n-1} - P_{l-1}P_n]$$

   We can then compute the $[P_l P_{n-1} - P_{l-1}P_n]$ term by using this recursion formula again. Notice that the indices of the coefficients decrease with each such iteration while the form of the recursion stays constant as we keep reusing it in order to get down to the basic recursion formula of step 1.

3. Thus, we can use the Askey recursion to compute tables of as many linearization coefficients as needed by starting with $l = 1$ and computing the coefficients from the basic polynomial recursion for many n. Then, we use the second recursion to iterate with higher values for l, solving for $P_{l+1}P_n$, all the while collecting in tables the coefficients $d_{k,m,i}$ which fall out of the recursion. Such a procedure produces numerically accurate values provided that our basic orthogonal polynomial recursion coefficients are accurate.

4. Finally, we need to transform these high-order term-normalized (to 1) orthogonal polynomial linearization constants into linearization constants for the orthonormal polynomials we created. But this is straightforward

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[10] Note that in solving for the linearization formulas, we usually do not care about the positivity of the coefficients $d_{k,m,i}$, a property which is crucial for the standard use of these formulas which is associated with polynomial approximations.

by computing the constants α_k which orthonormalize the high-order term-normalized P_k polynomials and then noticing that

$$P_m(x)P_k(x) = \sum_{i=0}^{2\min(m,k)} \frac{\alpha_{k+m-1}}{\alpha_k \alpha_m} d'_{k,m,i} P_{k+m-i}(x)$$

where P_k are the orthonormalized polynomials and $d'_{k,m,i}$ are the constants we found using the above Askey linearization recursion on the term-normalized orthogonal polynomials.

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