

Frequency estimation using time domain methods based on robust differential operators

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Abstract—Given a band limited signal which over some disjoint intervals of time I_n behaves as a corresponding linear combination $f_n(t)$ of up to N damped sinusoids, we present a method which detects intervals I_n , determines the number of the sinusoidal components over each interval and estimates their frequencies, with high accuracy and in the presence of noise which can be colored. Intervals I_n can have very short duration of just a dozen Nyquist rate intervals, hampering the use of the Fourier transform based methods. Our method operates entirely in the time domain; to be applicable, the signal must be sampled at a rate higher than the Nyquist rate. It is based on analyzing local signal behavior using special, numerically robust linear differential operators, called the chromatic derivatives, which were introduced relatively recently, and which hold yet unexplored promise in signal and image processing.

I. INTRODUCTION

Estimation of frequencies of several sinusoids in the presence of noise usually relies on evaluation of some form of the Fourier transform of a section of the signal. However, “viewing” the signal through a window distorts its Fourier transform by convolving it with the Fourier transform of the window. As a consequence, the resulting side lobes of the strong sinusoidal components and the noise can mask the main lobes of the weaker components. Also, if close in frequency, the main lobes of strong components mutually interfere causing bias or can merge into a single lobe.

Unlike the Fourier transform methods which allow spectral analysis of a vast class of signals, our approach uses a method which is applicable only to signals which, over intervals of interest, are sums of a small number of damped sinusoids. The method is based on the following well known fact: *A function f is a linear combination of n exponentially modified and phase shifted sinusoids over an interval $I \subseteq \mathbb{R}$ if and only if f satisfies on I a homogeneous linear differential equation with constant coefficients of order $2n$.*

Thus, we design our algorithm by refining the following basic idea. We sequentially examine sections of the signal over relatively short intervals of time, looking for the smallest $n \leq N$ for which there is a differential equation of order $2n$ which is satisfied by such section of the signal, modulo an error which is commensurate with the level of the noise present. *Note that the fact that the signal is analyzed over short intervals of time has no negative consequences similar to those associated with the use of windowing in the Fourier transform based methods, because whether a signal $f(t)$ satisfies a*

differential equation at a particular instant in time is a purely local feature of the signal, determined by its behavior in an arbitrarily small neighborhood around that instant. As a consequence, as we will demonstrate, our method can discern two sinusoids in a sampled signal over an interval of time even if in the Fourier transform of its sampling distribution the two corresponding lobes have merged into a single lobe.

The above proposal, however, immediately raises a concern: if we are looking for sections of the signal which are sums of, say, four sine waves, this would involve evaluating derivatives of order eight of a noisy signal. As is well known, numerical differentiation of such high order results in insurmountable numerical problems. *This is where the chromatic derivatives crucially intervene.* Chromatic derivatives are special linear differential operators with constant coefficients whose numerical evaluation is highly accurate and noise robust, even for operators of very high orders (> 20).

II. A BRIEF SUMMARY OF CHROMATIC DERIVATIVES

Despite providing a good local representation of a function, Taylor’s formula has found very limited use in signal processing. This is due to the fact that an accurate evaluation of derivatives of higher orders from discrete noisy samples of a signal is essentially impossible. Moreover, the functions used in the expansion, i.e., the monomials $t^n/n!$, do not correspond to band limited signals; the approximation is unbounded, it converges neither uniformly nor in L_2 and its error increases rapidly when moving away from the center of expansion. Chromatic derivatives and chromatic expansions were introduced in [1] and [2], respectively, to provide a numerically feasible framework for numerical differentiation and for local approximation of band limited signals which do not suffer from the above problems. They were first published in [3]–[5]; their properties were examined in detail in [6]–[8].

1) *Chromatic derivatives*: Chromatic derivatives are linear differential operators with constant coefficients obtained from suitably chosen families of orthonormal polynomials; see [8] for the details regarding which families of orthogonal polynomials produce satisfactory families of chromatic derivatives.

Assume that polynomials $P_n(\omega)$ satisfy

$$\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)w(\omega)d\omega = \delta(m - n), \quad (1)$$

where $w(\omega)$ is a non-negative symmetric weight function. We define linear differential operators associated with such a fam-

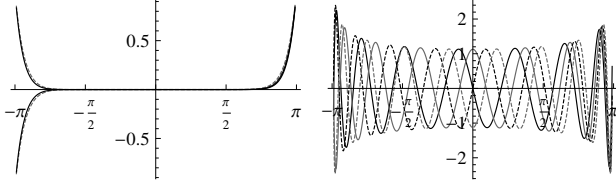


Fig. 1: Graphs of $(\omega/\pi)^n$ (left) and $P_n^L(\omega)$ (right) for $n = 15$ to $n = 18$.

ily of *orthonormal* polynomials by the operator polynomials

$$\mathcal{K}^n = (-j)^n P_n \left(j \frac{d}{dt} \right). \quad (2)$$

Thus, \mathcal{K}^n is obtained by replacing ω^k in $P_n(\omega)$ by $j^k \frac{d^k}{dt^k} f(t)$. It is easy to verify that

$$\mathcal{K}_t^n [e^{j\omega t}] = j^n P_n(\omega) e^{j\omega t}. \quad (3)$$

Consequently, if $f \in \mathbf{BL}(\pi)$ and $\widehat{f}(\omega)$ is its Fourier transform,

$$\mathcal{K}^n [f](t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} j^n P_n(\omega) \widehat{f}(\omega) e^{j\omega t} d\omega. \quad (4)$$

Polynomials $P_n(\omega)$ satisfy that for every $a < \pi$ there exists $M > 0$ such that $|P_n(\omega)| < M$ for all n and all $|\omega| \leq a$. In comparison, if we normalize the “standard” derivatives so that the magnitudes of their frequency responses are bounded uniformly in n , we get

$$\frac{f^{(n)}(t)}{\pi^n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} j^n \left(\frac{\omega}{\pi} \right)^n \widehat{f}(\omega) e^{j\omega t} d\omega. \quad (5)$$

Figure 1 compares the plots of the transfer functions $(\omega/\pi)^n$ of the normalized derivatives $1/\pi^n d^n/dt^n$ (modulo a factor of j^n) with the plots of the transfer functions $P_n^L(\omega)$ of the chromatic derivatives \mathcal{K}^n associated with the (normalized and re-scaled) Legendre polynomials (right). Such polynomials are obtained by taking for $w(\omega)$ appearing in (1) the constant weight function $w(\omega) = (2\pi)^{-1}$. Plots on the left reveal why numerical evaluation of higher order derivatives from signal samples makes no practical sense: multiplication of the Fourier transform of a signal by the transfer function of a derivative of high order essentially obliterates the spectrum of the signal, leaving only its edges, which in practice contain mostly noise. Note also that the graphs of the transfer functions of the normalized derivatives of high orders and of the same parity cluster together tightly, becoming essentially indistinguishable.

In comparison, Figure 1 (right) shows that the transfer functions of the chromatic derivatives \mathcal{K}^n form a family of well separated, interleaved and increasingly refined comb-like filters. Instead of obliterating, such operators encode the spectral features of the signal and for this reason we call operators \mathcal{K}^n *the chromatic derivatives*.

Chromatic derivatives thus replace the usual base $\{f, f', f'', \dots\}$ of the vector space of linear differential operators with an orthonormal base, and it turns out that such a base has many remarkable properties.

First of all, chromatic derivatives can be accurately evaluated using FIR filters operating on samples of the signal taken at a higher than the Nyquist rate; oversampling twice is quite sufficient. These filters can be designed using the

Remez exchange method and are both very accurate and noise robust. For example, a 129 tap transversal filter approximating differential operator \mathcal{K}^{15} corresponding to the re-scaled and normalized Legendre polynomial of order 15, with pass-band occupying 90% of the interval $[-\pi/2, \pi/2]$ and the transition region extending 10% of the bandwidth $\pi/2$ on each side of the boundaries $-\pi/2$ and $\pi/2$, outside the transition region has an error of approximation smaller than 1.3×10^{-4} .

2) *Chromatic approximations*: Chromatic derivatives can be used to produce local approximations of band limited signals which do not suffer from the mentioned shortcomings of the Taylor expansion; we formulate here only the special case corresponding to the Legendre polynomials normalized and rescaled to $[-\pi, \pi]$; for the general case see [7] or [8].

Proposition 2.1: Let \mathcal{K}^n be the chromatic derivatives associated with the Legendre polynomials, and let $f(t)$ be any function analytic on \mathbb{R} ; then for all $t \in \mathbb{R}$,

$$f(t) = \sum_{n=0}^{\infty} (-1)^n K^n [f](u) K^n [\text{sinc}](t - u).$$

If in addition $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on \mathbb{R} and in the L_2 norm.

The series in (6) is called *the chromatic expansion of f associated with the Legendre polynomials*; a truncation of this series to first $n + 1$ terms is called a *chromatic approximation of f of order n* . Just as Taylor’s approximation, a chromatic approximation is also a local approximation: its coefficients are the values of differential operators $\mathcal{K}^m [f](u)$ at a single instant u , and the values of its derivatives of orders up to n at instant u are equal to the values of the corresponding derivatives of $f(t)$. However, unlike the monomials in the Taylor formula, expansion functions $\mathcal{K}^n [\text{sinc}](t)$ are band limited signals, and the above theorem indicates that the approximation error accumulates much slower than the error of the Taylor approximation of the same order; see [8].

III. FREQUENCY ESTIMATION

We now show how chromatic derivatives can be used in frequency estimation. Assume that a $\mathbf{BL}(\pi/2)$ signal $f(t)$ is sampled at integers, thus at twice the Nyquist rate, and that such samples $f(i)$ are corrupted by a zero mean WSS stochastic noise $\nu(i)$, with an autocorrelation function $r(k)$. The power spectrum density of the noise is then $S(\omega) = \sum_{k=-\infty}^{\infty} r(k) e^{-j\omega k}$. Let $\rho^2 = \int_{-\pi/2}^{\pi/2} S(\omega) d\omega$; Thus, ρ is equal to the RMS value of the noise component which is within the bandwidth of $f(t)$.

Let $\chi_{\pi/2}(\omega) = 1$ for $|\omega| \leq \pi/2$ and zero otherwise, and let $P_n(\omega)$ be the family of polynomials orthonormal on $[-\pi, \pi]$ with respect to the weight function $w(\omega) = 1/(2\pi\rho^2)S(\omega)\chi_{\pi/2}(\omega)$, i.e., such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(\omega) P_m(\omega) S(\omega) \chi_{\pi/2}(\omega) d\omega = \delta(m - n) \rho^2. \quad (6)$$

This is equivalent to saying that $P_n(\omega/2)$ are orthonormal on $[-\pi, \pi]$ with respect to the weight function $w^*(\omega) = 1/(2\pi\rho^2)S(\omega/2)$. Let \mathcal{K}^n be the chromatic derivatives which correspond to the polynomials $P_n(\omega/2)$ via (2).

We now present our first heuristics for frequency estimation. Let $B = \sum_{n=0}^N X_n \mathcal{K}^n$ be any differential operator of order N , represented in the base of chromatic derivatives \mathcal{K}^n , such that its coefficients satisfy $\sum_{n=0}^N X_n^2 = 1$. Let also $\kappa^n[f](t) = \sum_{p=-L}^L \lambda_p^n f(t+p)$ be a $2L+1$ tap transversal filter with bandlimit $\pi/2$ approximating $\mathcal{K}^n[f](t)$ for $f \in \mathbf{BL}(\pi/2)$, and let $b = \sum_{n=0}^N X_n \kappa^n$ be the corresponding FIR approximation of B . Then (4) implies

$$\sum_{p=-L}^L \lambda_p^n e^{-j\omega p} \approx j^n P_n(\omega) \chi_{\pi/2}(\omega) \quad (7)$$

for all $|\omega| < \pi$ and consequently (6) implies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{p=-L}^L \sum_{q=-L}^L \lambda_p^n \lambda_q^m \sum_{k=-\infty}^{\infty} r(k) e^{-j\omega(q-p-k)} d\omega \approx j^{n-m} \delta(m-n) \rho^2. \quad (8)$$

Integrating term by term and collecting the non zero terms,

$$\sum_{p=-L}^L \sum_{q=-L}^L \lambda_p^n \lambda_q^m r(q-p) \approx \delta(m-n) \rho^2. \quad (9)$$

Since $E[\nu(t+p)\nu(t+q)] = r(q-p)$, (9) implies

$$E \left[\sum_{p=-L}^L \sum_{q=-L}^L \lambda_p^n \nu(t+p) \lambda_q^m \nu(t+q) \right] \approx \delta(m-n) \rho^2, \quad (10)$$

i.e., $E[\kappa^n(\nu)(t) \kappa^m(\nu)(t)] \approx \delta(m-n) \rho^2$.

Thus, if the filters are chosen to correspond to the power spectrum of the noise, then the errors which are due to noise of the filters evaluating chromatic derivatives of different orders are uncorrelated and have an RMS value equal to the RMS value of the $\pi/2$ in-band component of the noise. Since $b[\nu](t)$ is a finite linear combination of samples of $\nu(t)$ and since ν is of zero mean, $b[\nu](t)$ is also of zero mean. Using linearity of b , it is easy to see that this implies

$$E[b[f+\nu](t)^2] = b[f](t)^2 + E[b[\nu](t)^2].$$

This, the assumption $\sum_{n=0}^N X_n^2 = 1$, (10) and $E[b[\nu](t)^2] = E \left[\sum_{n=0}^M \sum_{m=0}^M X_n X_m \kappa^n[\nu](t) \kappa^m[\nu](t) \right]$ together imply

$$E[b[f+\nu](t)^2] = b[f](t)^2 + \rho^2. \quad (11)$$

This is a remarkable fact, because it shows that, if the chromatic derivatives are chosen to correspond to the power spectrum of the noise via (6), then the RMS impact of the noise on any differential operator $B = \sum_{n=0}^M X_n \mathcal{K}^n$ is independent of the particular values of the coefficients X_n , and is always equal to the RMS value of the in band component of the noise.

Let Q be any fixed natural number, t a fixed instant of time and $\vec{X} = (X_0, \dots, X_N)$. If we set

$$R(\vec{X}) = \frac{1}{Q+1} \sum_{p=0}^Q b[f+\nu](t+p)^2, \quad (12)$$

then, by (11), $E[R(\vec{X})] = \frac{1}{Q+1} \sum_{p=0}^Q b[f](t+p)^2 + \rho^2$. Thus, $E[R(\vec{X})] \geq \rho^2$ and, since the transversal filters κ^n are close approximations of differential operators \mathcal{K}^n , $E[R(\vec{X})] \approx \rho^2$ if and only if $f(t)$ satisfies the differential equation $B[f](x) = 0$ at all instants $x = t, \dots, t+Q$.

Consequently, to determine if f satisfies a differential equation of order N over $[t, t+Q]$, we should find \vec{X} which

minimizes the value of $R(\vec{X})$ and see if such value is approximately equal to ρ^2 . To find the minimum of $R(\vec{X})$ subject to the constraint $\sum_{n=0}^N X_n^2 = 1$, we apply the Lagrangian multipliers and set to zero the partial derivatives with respect to a new variable β and variables X_0, \dots, X_N , of the objective $R(\vec{X}) - \beta \sum_{n=0}^N X_n^2$. Letting $f_\nu(t) = f(t) + \nu(t)$, this gives the following equations: $\|\vec{X}\|^2 = 1$, plus for each m , $0 \leq m \leq N$,

$$-\beta X_m + \sum_{p=0}^Q \sum_{n=0}^N X_n \kappa^n[f_\nu](t+p) \kappa^m[f_\nu](t+p) = 0. \quad (13)$$

Let $C[m, n] = \frac{1}{1+Q} \sum_{p=0}^Q \kappa^n[f_\nu](t+p) \kappa^m[f_\nu](t+p)$ and let $C = (C[m, n])_{m, n=1..N}$ be the corresponding matrix; then the above equations become $C \vec{X} = \beta \vec{X}$, i.e. β is an eigenvalue of C and \vec{X} is the corresponding eigenvector of a unit norm. Note that (12) implies $R(\vec{X}) = \sum_{m=0}^N \sum_{n=0}^N X_m X_n C[m, n]$; thus, $R(\vec{X}) = \langle C \vec{X}, \vec{X} \rangle$. Since $R(\vec{X}) \geq \rho^2 > 0$, C is a symmetric positive definite matrix and thus all of its eigenvalues are positive reals. If \vec{X} is the unit eigenvector for the eigenvalue β , then $R(\vec{X}) = \langle C \vec{X}, \vec{X} \rangle = \langle \beta \vec{X}, \vec{X} \rangle = \beta$; thus, to minimize $R(\vec{X})$ we must choose the unit eigenvector \vec{X}_m which corresponds to the smallest eigenvalue β_m . We conclude that, with high probability, $f(t)$ satisfies a differential equation of the form $\sum_{n=0}^N X_n \mathcal{K}^n[f](t) = 0$ over the interval $[t, t+Q]$ just in case the smallest eigenvalue β_m of C is approximately equal to ρ^2 , i.e., just in case $\beta_m < c\rho^2$ where $c > 1$ and $c \approx 1$.

If $\beta_m \approx \rho^2$, to find the fundamental solutions of the corresponding differential equation $\sum_{n=0}^N X_n \mathcal{K}^n[f](t) = 0$ with the coefficients $\vec{X}_m = (X_0, \dots, X_N)$, we numerically solve the associated algebraic equation $\sum_{n=0}^N X_n (-j)^n P_n(-jz) = 0$. The imaginary part of each conjugate pair of solutions z_i, \bar{z}_i of this equation is the frequency ω_i and the real part is the damping factor δ_i of the fundamental solutions $e^{\delta_i t} \sin(\omega_i t)$ and $e^{\delta_i t} \cos(\omega_i t)$ of the corresponding differential equation; the i^{th} sinusoidal component of the signal is a linear combination of these two fundamental solutions.

This completes our first heuristics; we now present our second heuristics. Signal $f(t)$ satisfies a differential equation of order N over an interval $[t, t+Q]$ just in case the rank of the matrix $K_f = (\kappa^n[f](t+p) : 0 \leq p \leq Q, 0 \leq n \leq N)$, with $Q+1$ rows and $N+1$ columns, is at most N . Let $K_\nu = (\kappa^n[\nu](t+p) : 0 \leq p \leq Q, 0 \leq n \leq N)$ and $K_{f+\nu} = K_f + K_\nu$. Thus, $K_{f+\nu}$ is a perturbation of the matrix K_f caused by the effects of the noise, represented by the matrix K_ν . By (10), the noise matrix K_ν should have $N+1$ mutually (approximately) orthogonal columns and the expected value of its entries should be approximately equal to ρ .

Consequently, given the matrix $K_{f+\nu}$, we should look for the smallest μ for which there exist a matrix K^* of rank N and a matrix K° with $N+1$ orthogonal columns of unit norm each, such that $K_{f+\nu} = K^* + \mu K^\circ$. The Frobenius norm of such μK° is $\mu \sqrt{N+1}$; thus the RMS value of its entries is equal to $\mu / \sqrt{Q+1}$. Consequently, the perturbation μK° can be accounted for as the effect of the noise if $\mu / \sqrt{Q+1}$ is equal to the RMS value of the noise, i.e., if $\mu \approx \rho \sqrt{Q+1}$.

Given $K_{f+\nu}$, to find such $\mu > 0$, K^* and K° , we consider

the singular value decomposition $K_{f+\nu} = U\Sigma V^T$, where Σ is a diagonal matrix with the singular values of $K_{f+\nu}$ on the diagonal and U and V are orthogonal matrices comprising of left and right singular vectors, of sizes $(Q+1) \times (Q+1)$ and $(N+1) \times (N+1)$ respectively. Generalizing an argument from [9], it is easy to prove that such μ must be equal to the smallest singular value of $K_{f+\nu}$ and that $K^\circ = U\Sigma_1 V^T$, where Σ_1 is obtained from Σ by replacing all singular values with 1. We now set $K^* = K_{f+\nu} - \mu K^\circ$; then the rank of K^* is at most N ; the vector $\vec{X} = (X_0, \dots, X_N)$ such that $K^* \vec{X} = 0$ is the right singular vector of $K_{f+\nu}$, corresponding to the smallest singular value μ of $K_{f+\nu}$. Taking \vec{X} for the coefficients of B , we get $B[f](x) = 0$ for $x = t, \dots, t+Q$.

We now note that $K_{f+\nu}^T K_{f+\nu} = (Q+1)C$, where C is the matrix from our first heuristics. As it is well known, the singular values of $K_{f+\mu}$ are equal to the square roots of the eigenvalues of $K_{f+\mu}^T K_{f+\mu}$; thus, $\mu = \sqrt{(Q+1)\beta}$, where β is the smallest eigenvalue of C , and the right singular vectors of $K_{f+\mu}$ are the corresponding eigenvectors of $K_{f+\mu}^T K_{f+\mu}$. This implies that the condition $\mu \approx \rho\sqrt{Q+1}$ is equivalent to the condition $\beta \approx \rho^2$. Consequently, both heuristics result in the same criterion for determining if $f(t)$ satisfies a differential equation of order N over $[t, t+Q]$ and in the same coefficients \vec{X} of such a differential equation. This gives an indication that our heuristics are natural, and we believe it should be possible to extend the above arguments into proofs that the frequency estimators derived from our heuristics are consistent, unbiased and efficient; our initial simulations have strongly suggested that this indeed is the case.

3) *Test results:* We now summarize the results of our preliminary testing of an initial implementation of our detection and frequency estimation method, which leaves much room for improvement. To tests our algorithms, we have used a method described in [10] to join fragments of linear combinations of up to 4 damped sinusoids of frequencies less than π into a signal with extremely low frequency content out of the bandwidth $[-\pi, \pi]$; see [10] for the details. Such signal is then sampled at half integers, thus at twice the Nyquist rate, and the samples are corrupted with white Gaussian noise.

Figure 2 shows the plots of the values of the square root of the smallest eigenvalue of $(C[m, n])_{m,n=0}^4$ (top), and of $(C[m, n])_{m,n=0}^2$ (bottom), for the case where each fragment was either a single damped sinusoid or sum of two such sinusoids, with the corresponding time supports represented by the gray rectangles of height 1 or 2, respectively. The horizontal line corresponds to the threshold 1.1ρ for the SNR=25 db. The above plots show that the algorithm correctly identifies fragments I_n as well as the number of sinusoids over each fragment. More over, it also indicates that we do not need to know in advance the signal to noise ratio, because the regions where the corresponding eigenvalues are consistently small over an interval can be identified and used to determine the signal to noise ratio.

For signals consisting of fragments of single sinusoids of length of 8 Nyquist rate intervals and with SNR=10db, despite the fact that we did not attempt to (recursively) compensate for

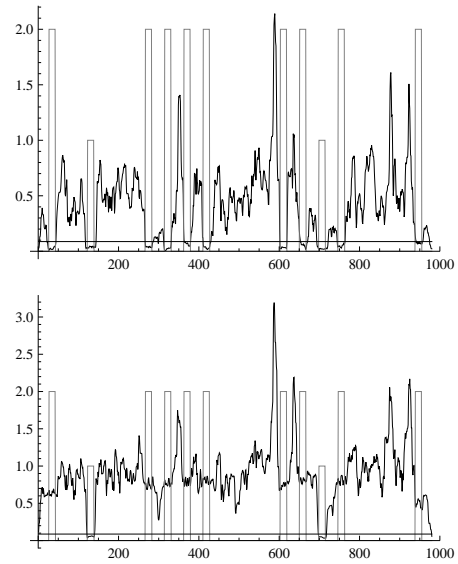


Fig. 2: The smallest eigenvalue of $(C[m, n])_{m,n=0}^4$ (top, black) and of $(C[m, n])_{m,n=0}^2$ (bottom, black). Markers showing the position and the number of components of fragments in gray.

the ripple of our FIR chromatic derivative filters, the standard deviation of our method was less than 2.5 times the theoretical minimum given by the Cramér - Rao bound. To test the resolution of our methods we generated signals of the form $f(t) = \cos(\omega t + \theta) + \cos((\omega + 0.05)t)$ and corrupted them with noise (SNR = 35db). Their 128 point sampling distributions at half integers have a symmetric Fourier transform with a single peak. However, our method correctly identified that there are two sinusoids, and estimated their frequencies with an RMS errors of less than 0.008 radians. The programs used in simulations, a detailed interactive tutorial on the subject and the references mentioned in this paper can be found at <http://www.cse.unsw.edu.au/~ignjat/diff>.

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