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AN ORDERING OF THE SET OF SENTENCES OF PEANO ARITHMETIC

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Abstract. We consider a partial ordering of the set of sentences of Peano arithmetic P induced by a theory T extending P, which orders sentences according to the complexity of their "proofs". Using some properties of the ordering induced by the theory $P + \neg\mathrm{Con}_p$ we prove that P doesn't have the Joint Embedding Property. We also describe models for P which do not enrich the ordering induced by P, i.e. models satisfying $<_{\mathrm{Th}(\mathfrak{M})} = <_P$, and we prove that for every consistent theory T, $T \supset P$, there is a theory $T' \supset P$ such that the ordering induced by the theory T' is a linear extension of the ordering induced by the theory T.

By L_P we denote the language of P and by $S(P)$ the set of sentences of P. Any consistent extension of P we denote by T, and N stands for the structure of natural numbers. By $\mathfrak{M}, \mathfrak{N}, \ldots$ we denote nonstandard models for P, and by $|\mathfrak{M}|, |\mathfrak{N}|, \ldots$ their domains respectively. If $\mathfrak{M}$ and $\mathfrak{N}$ are models for P, then $\mathfrak{M} \subset_{\sum_1} \mathfrak{N}$ means that for all $\sum_1$-formulas φ and all $a_1, \ldots, a_n \in |\mathfrak{M}|$, $\mathfrak{M} \models \varphi[a_1, \ldots, a_n]$ implies $\mathfrak{N} \models \varphi[a_1, \ldots, a_n]$; similarly, we write $\mathfrak{M} \prec_{\sum_1} \mathfrak{N}$ when $\mathfrak{M} \models \varphi[a_1, \ldots, a_n]$ holds iff $\mathfrak{N} \models \varphi[a_1, \ldots, a_n]$ holds.

We use the following model-theoretical consequence of Matijasevič's theorem.

LEMMA 0. *Let $\mathfrak{M}$, $\mathfrak{N}$ be models for P; then $\mathfrak{M} \subset \mathfrak{N}$ implies $\mathfrak{M} \prec_{\sum_0} \mathfrak{N}$ and $\mathfrak{M} \subset_{\sum_1} \mathfrak{N}$.* □

For any sentences φ and ψ from $S(P)$, by $\varphi < \psi$ we denote the $\sum_1$-sentence

$$\exists x(\mathrm{Prf}_p(x, \ulcorner\varphi\urcorner) \wedge (\forall y \leq x)\neg\mathrm{Prf}_p(y, \ulcorner\psi\urcorner)).$$

The following lemma enables us to introduce the ordering in $S(P)$. msk
LEMMA 1. *Let T be a consistent extension of P; then the relation $<_T$ defined by $\varphi <_T \psi$ iff $T \vdash \varphi < \psi$ is transitive and irreflexive.* □

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Since the sentence $\varphi < \psi$ is $\sum_1$, it is obvious that $\varphi <_P \psi$ holds iff $\varphi < \mathrm{Th}(N)\psi$ holds, i.e. $<_P = <_{Th(N)}$. The order type of $<_P$ is $\omega + \Lambda$; the set of theorems of P has order type ω, and Λ is the type of the empty ordering of the countable set of sentences from $S(P)$ not provable in P.

The ordering induced by the theory of a model $\mathfrak{M} \models P$ is linear iff $\mathfrak{M} \models \neg\mathrm{Con}_P$. If $\mathfrak{M} \models \mathrm{Con}_P$, then the ordering consists of a linearly ordered set of sentences having "proofs" in $\mathfrak{M}$, i.e. sentences satisfying $\mathfrak{M} \models \mathrm{Th}mp(\ulcorner\varphi\urcorner))$, and the countable remainder of $S(P)$ with empty ordering; it is obvious that if φ belongs to the first set and ψ to the second, then $\varphi <_{\mathrm{Th}(\mathfrak{M})} \psi$ holds.

Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are models for P and $\mathfrak{M} \subset \mathfrak{N}$. Since $\varphi < \psi$ is a $\sum_1$-sentence, $\varphi <_{\mathrm{Th}(\mathfrak{M})} \psi$ implies $\varphi <_{\mathrm{Th}(\mathfrak{N})} \psi$, i.e. $<_{\mathrm{Th}(\mathfrak{M})} \subset <_{\mathrm{Th}(\mathfrak{N})}$. Also, $<_T \subset <_{\mathrm{Th}(\mathfrak{M})}$ holds for any $T \supset P$ and $\mathfrak{M} \models T$. The next proposition describes models for P satisfying $<_P = <_{\mathrm{Th}(\mathfrak{M})}$.

PROPOSITION 1. *Let $\mathfrak{M}$ be a model for P; then $<_P = <_{\mathrm{Th}(\mathfrak{M})}$ iff $N \prec_{\sum_1} \mathfrak{M}$.*

Proof. Since $<_P = <_{\mathrm{Th}(N)}$, from $N \prec_{\sum_1} \mathfrak{M}$, it immediately follows $\varphi <_P \psi$ iff $\varphi <_{\mathrm{Th}(N)} \psi$ iff $\varphi <_{\mathrm{Th}(\mathfrak{M})} \psi$ i.e. $<_P = <_{\mathrm{Th}(\mathfrak{M})}$. Conversely, suppose that $<_p = <_{\mathrm{Th}(\mathfrak{M})}$ holds, and let φ be a $\sum_1$-sentence true in $\mathfrak{M}$. Since for $\sum_1$-sentences φ, $P \vdash \varphi \to \mathrm{Th}m_P(\ulcorner\varphi\urcorner)$ holds (see 5.3.4 in [SM]), we get that $\mathfrak{M} \models \mathrm{Th}m_P(\ulcorner\varphi\urcorner)$. Then, for every other sentence $\psi \in S(P)$, either $\varphi <_{\mathrm{Th}(\mathfrak{M})} \psi$ or $\psi <_{\mathrm{Th}(\mathfrak{M})} \varphi$ holds. This and the assumption $<_P = <\mathrm{Th}(\mathfrak{M})$ imply that φ belongs to the linearly ordered part of $<_P$. Thus, φ is a theorem of P and consequently $\mathfrak{M} \models \varphi$. Since for all $\sum_1$-sentences φ, $N \models \varphi$ implies $\mathfrak{M} \models \varphi$, we get $N \prec_{\sum_1} \mathfrak{M}$. *square*

Using Corollary 2.9.1 from [MI], asserting that $N \prec_{\sum_1} \mathfrak{M}$, iff $\bigcap\{\mathfrak{N} | \mathfrak{N} \subset_e \mathfrak{M}, \mathfrak{N} \cong \mathfrak{M}\} = n$, we get the following algebraic characterization of models satisfying $<_P = <_{\mathrm{Th}(\mathfrak{M})}$.

COROLLARY 1. *$<_P = <_{\mathrm{Th}(\mathfrak{M})}$ iff $\bigcap\{\mathfrak{N} | \mathfrak{N} \subset_e \mathfrak{M},\ \mathfrak{N} \cong \mathfrak{M}\} = N$.* □

We use the following lemma to prove that, although the ordering $<_{\mathrm{Th}(\mathfrak{M})}$ is linear for any model $\mathfrak{M}$ of the theory $P + \neg\mathrm{Con}_P$, the ordering $<_{P+\neg\mathrm{Con}_P}$ itself is not linear.

LEMMA 2. *There is a sentence varphi, independent of the theory $P + \neg\mathrm{Con}_P$, such that $P \vdash \varphi \leftrightarrow ((\varphi \to \mathrm{Con}_P) < (\neg\mathrm{Con}_P \to \varphi))$.*

Proof. Gödel's diagonalization technique and usual arguments for sentences of the Rosser type.

PROPOSITION 2. *There are two sentences σ, $\psi \in S(P)$, such that neither $\sigma <_{P+\neg\mathrm{Con}_P} \psi$, nor $\psi <_{P+\neg\mathrm{Con}_P} \sigma$ holds.*

Proof. Let σ be the sentence $\varphi \to \mathrm{Con}_P$ and ψ the sentence $\neg\mathrm{Con}_P \to \varphi$, where φ is the sentence from Lemma 2. From the same lemma we get that there are two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of the theory $P + \neg\mathrm{Con}_P$ such that $\mathfrak{M}_1 \models \varphi$ and $\mathfrak{M}_1 \models \neg\varphi$. It is obvious that $\sigma <_{\mathrm{Th}(\mathfrak{M}_1)} \psi$ and $\neg(\sigma <_{\mathrm{Th}(\mathfrak{M}_2)} \psi)$. But $\mathfrak{M}_2 \models \neg\mathrm{Con}_P$ implies that the ordering $<_{\mathrm{Th}(\mathfrak{M}_2)}$ is linear; so we get $\psi <_{\mathrm{Th}(\mathfrak{M}_2)} \sigma$. Thus, neither $\sigma < P + \neg\mathrm{Con}_p\psi$ nor $\psi < P + \neg\mathrm{Con}_P\sigma$ holds.

Models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ described in the proof of Proposition 2 don't have a common extension which is a model for P: $\mathfrak{M}_1 \subset \mathfrak{N}$, $\mathfrak{M}_2 \subset \mathfrak{N}$ and $\mathfrak{N} \models P$ would imply $\sigma <_{\mathrm{Th}(\mathfrak{N})\psi}$ and $\psi <_{\mathrm{Th}(\mathfrak{N})\sigma}$, which is a contradiction. Thus, we get the following corollary.

COROLLARY 2. *Peano arithmetic does not have the Joint Embedding Property, i.e. there are two models of P which cannot be embedded in any common extension which is a model for P.* □

Let T be a consistent extension of P; if the theory $T' = T + \neg\mathrm{Con}_P$ is consistent and $\mathfrak{M} \models T'$, then the ordering $<_{\mathrm{Th}(\mathfrak{M})}$ is a linear extension of the ordering $<_T$. Using the following lemma [**KR**] we prove that the ordering $<_T$ induced by any consistent extension T of P can be extended in a similar way.

LEMMA 3 (Kreisel). *The theory $P + \neg\mathrm{Con}_P$ is a Π_1-conservative extension of P, i.e. for any Π_1-sentence φ, $P + \neg\mathrm{Con}_P \vdash \varphi$ iff $P \vdash \varphi$.*

PROPOSITION 3. *For any consistent theory $T \supset P$ there is a theory $T' \supset P$ such that the ordering $<_{T'}$ is a linear extension of $<_T$.*

Proof. Let $S = \{\varphi < \psi | \, \varphi <_T \psi; \varphi, \psi \in S(P)\}$. The theory $P + \neg\mathrm{Con}_P + S$ is consistent: for any finite $S_0 \subset S$, Lemma 3 and $P + \neg\mathrm{Con}_P \vdash \neg\wedge_{\varphi\in S_0}$ would imply $P \vdash \neg\wedge_{\varphi\in S0}$; this is a contradiction, since $T \supset P$ is consistent and $T \vdash \wedge_{\varphi\in S_0}$. Let $\mathfrak{M}$ be a model for the theory $P + \neg\mathrm{Con}_P + S$; the ordering $<_{\mathrm{Th}(\mathfrak{M})}$ is a linear extension of $<_T$ induced by the consistent extension $T' = \mathrm{Th}(\mathfrak{M})$ of P.

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