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INITIAL SEGMENTS AND ISOMORPHIC IMAGES OF NONSTANDARD MODELS OF ARITHMETIC*

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ABSTRACT. Let m be a countable nonstandard model of Peano arithmetic (abbreviated by P) and $A_k^m = \{n | n \subset_e m,\ n <_{\Sigma_k} m,\ n \cong m\}$. For any $a \in |m|$, let $I_a = \{x \in |m| \mid x < a\}$. In this paper we consider those segments I_a which contain elements of A_k^m (propositions 1.1 and 1.3), and get some supplements and easier proofs of Corollary 2.7.1 and Theorem 2.9 from [1] (Corollaries 1.2,1.4,1.5 and 1.7).

INTRODUCTION. The language of P we denote by L_P, N stands for the structure of natural numbers and ω for its domain. Nonstandard models of P we denote by m, n and k, and their domains by $|m|$, $|n|$, and $|k|$ respectively; if $a \in |m|$, then $\underline{a}$ denotes the name of a. A subset I of a model m is an initial segment of m if for all $a \in |m|$ and $b \in I$, $a < b$ implies $a \in I$. If m and n are models of P and $m \subset n$, then by $m \subset_e n$ ($m <_e n$) we denote that n is an end extension (elementary end extension) of m; $m \subset_{\Sigma_k} n$ means that for all Σ_k formulas ϕ and all $a_1,\dots,a_n \in |m|$, $m \models \phi(\underline{a}_1,\dots,\underline{a}_n)$ implies $n \models \phi(\underline{a}_1,\dots,\underline{a}_n)$; similarly, we write $m <_{\Sigma_k} n$ when $m \models \phi(\underline{a}_1,\dots,\underline{a}_n)$ holds iff $n \models \phi(\underline{a}_1,\dots,\underline{a}_n)$ holds.

We use the consequence of Matijasevič's theorem asserting that for any model m, n of P, $m \subset n$ implies $m <_{\Sigma_0} n$ and $m \subset_{\Sigma_1} n$. Thus, $A_0^m = \{n | n \subset_e m,\ n \cong m\}$. Also, if I is an initial segment of m which supports a substructure I of m (i.e. I is closed for operations) then $I <_{\Sigma_0} m$ and $I \subset_{\Sigma_1} m$ holds.

Let Γ be a set of formulas of L_P and $\mathfrak{A}_1$, $\mathfrak{A}_2$ two structures for the same language, then $\mathfrak{A}_1 \equiv \mathfrak{A}_2$ means that for all sentences

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ϕ form Γ, $\mathfrak{A}_1 \models \phi$ holds iff $\mathfrak{A}_2 \models \phi$ holds. The set of sentences from Γ which are true in m we denote by $Th_\Gamma(m)$. An element $a \in |m|$ is Γ-definable in m if there is a formula $\phi(x) \in \Gamma$ such that $m \models \underline{a} = \mu x\, \phi(x)$. The set of all Γ-definable elements of m we denote by Γ^m, and $\overline{\Gamma}^m$ is the smallest initial segment of m containing Γ^m.

The standard system of m we denote by $SSy(m)$. Let $\tau(x_1, y_1, \ldots, y_n)$ be a set of formulas of L_P, and $a_1, \ldots, a_n \in |m|$. We say that $\tau(x, \underline{a}_1, \ldots, \underline{a}_n)$ is a real type over m if it is a type over m and $\{\ulcorner \phi(x, y_1, \ldots, y_n) \urcorner \mid \phi \in \tau\} \in SSy(m)$.

In this paper we use McDowell - Specker's theorem asserting that every model of P has a proper elementary end extension, and the following propositions, which can be found in [1] and [3]. Here m and n are nonstandard models of P and $a_1, \ldots, a_n \in |m|$.

PROPOSITION 0.1. Let m, n be models of P, the $m \subset_{\Sigma_{k+1}} n$ iff $m \prec_{\Sigma_k} n$.

PROPOSITION 0.2. (i) $m \subset n$ implies $SSy(m) \subset SSy(n)$;
(ii) $m \subset_e n$ implies $SSy(m) = SSy(n)$.

PROPOSITION 0.3. Let $\tau^{\Sigma_k}_{a_1 \ldots a_n}(x_1, \ldots, x_n)$ be the set of all Σ_k formulas such that $m \models \phi(\underline{a}_1, \ldots, \underline{a}_n)$. Then $\tau^{\Sigma_k}_{a_1 \ldots a_n}(x_1, \ldots, x_n)$ is real over m.

PROPOSITION 0.4. Any Σ_k-type $\tau^{\Sigma_k}(x, \underline{a}_1, \ldots, \underline{a}_n)$ of the language $L_P \cup \{\underline{a}_1, \ldots, \underline{a}_n\}$ which is real over m is realized in m.

PROPOSITION 0.5. The following are equivalent for a countable model m.

(a) (i) m has a proper initial embedding into n.
(ii) $Th_{\Sigma_1}(m) \subset Th_{\Sigma_1}(n)$ and $SSy(m) = SSy(n)$.

(b) (i) m has a proper initial Σ_k-elementary embedding into n.
(ii) $Th_{\Sigma_{k+1}}(m) \subset Th_{\Sigma_{k+1}}(n)$ and $SSy(m) = SSy(n)$.

INITIAL SEGMENTS OF m WHICH CONTAIN ELEMENTS OF A^m_k

PROPOSITION 1.1. Let m be a nonstandard countable model of P and $a \in |m|$, then the following are equivalent.

(i) There is a model $n \in A^m_o$ such that $n \subset I_a$.

(ii) There is a model $n \subset m$ such that $n \models P$, $n \equiv_{\Sigma_1} m$ and $n \subset I_a$.

(iii) There is a model $I \subset_e m$ such that $I \equiv_{\Sigma_1} m$ and $I \subset I_a$.

(iv) $\Sigma_1^m \subset I_a$.

Proof: Parts (i) → (ii) and (i) → (iii) are trivial; to prove (i) → (iv), suppose that I_a contains a model $n \in A_o^m$ and let f be an isomorphism of m onto n. Also, let $b \in \Sigma_1^m$, i.e. $m \models \underline{b} = \mu x \phi(x)$ for a Σ_1 formula ϕ, and suppose $b \notin I_a$. Then $m \models \phi(f(b))$, which is a contradiction, since $f(b) < a < b$.

Let m' be an elementary end extension of m, $c \in |m'| \setminus |m|$ and $\tau_c^{\Sigma_1}(x)$ the Σ_1-type of c in m'. According to proposition 0.3 this type is real over m'. Hence, by propositions 0.3 and 0.2 this type is real both over m' and m. Let $\tau_o^{\Sigma_1}(x)$ be an arbitrary finite subset of $\tau_c^{\Sigma_1}(x)$ and let $\mathfrak{A}$ be either model n from (ii) or model I from (iii). In both cases $\mathfrak{A} \equiv_{\Sigma_1} m$ and $\mathfrak{A} \subset_{\Sigma_1} m$ holds; thus, $m \models \exists x \wedge \tau_o^{\Sigma_1}(x)$ implies $\mathfrak{A} \models \exists x \wedge \tau_o^{\Sigma_1}(x)$. Also, if $b_o \in |\mathfrak{A}|$ and $\mathfrak{A} \models \wedge \tau_o^{\Sigma_1}(b_o)$ then $m \models \wedge \tau_o^{\Sigma_1}(b_o)$. Since $b_o \in |\mathfrak{A}| \subset I_a$, i.e. $b_o < a$, the type $\tau_c^{\Sigma_1}(x) \cup \{x < a\}$ is consistent over m. If we suppose (iv) and $m \models \underline{b}_o = \mu x \wedge \tau_o^{\Sigma_1}(x)$, then $b_o \in \Sigma_1^m \subset I_a$, so the type $\tau_c^{\Sigma_1}(x) \cup \{x<a\}$ is consistent over m. Thus, all (ii), (iii) and (iv) imply that the type $\tau_c^{\Sigma_1}(x) \cup \{x < a\}$ is consistent over m. Therefore, by proposition 0.4, it is realized in m. Let b realizes $\tau_c^{\Sigma_1}(x) \cup \{x < a\}$ in m. Define $f : |m'| \longrightarrow |m|$ such that $f(c) = b$ and then simply continue the usual Friedman's construction used in the proof of 0.5(a), to get an initial embedding of m' into m. Since $f(c) < a$, the restriction $f \upharpoonright |m|$ is an isomorphism of m onto an initial segment contained in I_a.

COROLLARY 1.2.* $\cap\{|n| \mid n \in A_o^m\} = \cap\{|I| \mid I \subset_e m,\ I \equiv_{\Sigma_1} m\} = \overline{\Sigma}_1^m$.

Proof: Immediately from (i) ↔ (ii) ↔ (iv) of Proposition 1.1.

Using Propositions 0.1 and 0.5(b) the following hierarchical variant of proposition 1.1 can be proven similarly.

* *The second equality is the answer to a question of M.Rašković.*

PROPOSITION 1.3. Let $\mathcal{M}$ be a nonstandard model of P and $a \in |\mathcal{M}|$. The following are equivalent.

(i) There is a model $\mathcal{N} \in A_k^{\mathcal{M}}$ such that $\mathcal{N} \subset I_a$.

(ii) There is a model $\mathfrak{A} \subset I_a$ such that $\mathfrak{A} \prec_{\Sigma_k} \mathcal{M}$ and $\mathfrak{A} \equiv_{\Sigma_{k+1}} \mathcal{M}$.

(iii) $\overline{\Sigma}_{k+1}^{\mathcal{M}} \subset I_a$.

COROLLARY 1.4. $\cap\{|\mathcal{N}| \mid \mathcal{N} \in A_k^{\mathcal{M}}\} = \cap\{|\overline{\mathfrak{A}}| \mid \mathfrak{A} \subset \mathcal{M},\ \mathfrak{A} \prec_{\Sigma_k} \mathcal{M},\ \mathfrak{A} \equiv_{\Sigma_{k+1}} \mathcal{M}\} = \overline{\Sigma}_{k+1}^{\mathcal{M}}$.

Proof: Immediately from propositions 1.2 and 0.1.

COROLLARY 1.5. ([1]). $\cap\{|\mathcal{N}| \mid \mathcal{N} \in A_k^{\mathcal{M}}\} = \omega$ iff $N \prec_{\Sigma_{k+1}} \mathcal{M}$.

Proof: $N \prec_{\Sigma_{k+1}} \mathcal{M}$ implies $N \in \{\overline{\mathfrak{A}} \mid \mathfrak{A} \subset \mathcal{M},\ \mathfrak{A} \prec_{\Sigma_k} \mathcal{M}, \mathfrak{A} \equiv_{\Sigma_{k+1}} \mathcal{M}\}$; thus $\cap\{|\overline{\mathfrak{A}}| \mid \mathfrak{A} \subset \mathcal{M},\ \mathfrak{A} \prec_{\Sigma_k} \mathcal{M},\ \mathfrak{A} \equiv_{\Sigma_{k+1}} \mathcal{M}\} = \omega$. Hence, by Corollary 1.4, $\cap\{|\mathcal{N}| \mid \mathcal{N} \in A_k^{\mathcal{M}}\} = \omega$. Otherwise, if $\cap\{|\mathcal{N}| \mid \mathcal{N} \in A_k^{\mathcal{M}}\} = N$ then Corollary 1.4 implies $\overline{\Sigma}_{k+1} = \omega$; thus, according to the hierarchical refinement of Tarski - Vaught's test, $N \prec_{\Sigma_{k+1}} \mathcal{M}$.

We now give a direct proof of Corollary 2.7.3 from [1] (Lemma 1.6 in this paper). This Corollary, together with Corollary 2.2.2 from [1] which asserts that $\overline{\Delta}_k^{\mathcal{M}} = \overline{\Sigma}_k^{\mathcal{M}}$, and Corollary 1.4 from this paper give us Corollary 1.7 which implies Corollary 2.7.1 from [1].

LEMMA 1.6. ([1]) For any $k \in \omega$ and any model $\mathcal{M}$ of P, $\overline{\Pi}_k = \overline{\Sigma}_{k+1}$.

Proof: Obviously $\Pi_k^{\mathcal{M}} \subset \Sigma_{k+1}^{\mathcal{M}}$; suppose $a \in \Sigma_{k+1}^{\mathcal{M}}$, then there is a Π_k formula Ψ such that $\mathcal{M} \models \underline{a} = \mu x\, \exists y\, \Psi(x,y)$ (we contract quantifiers if necessary). Let $b \in |\mathcal{M}|$ and $\mathcal{M} \models \underline{b} = \mu x \Psi((x)_o, (x)_1)$. Then $b \in \Pi_k^{\mathcal{M}}$ and $b > a$.

COROLLARY 1.7. $\cap\{|\mathcal{N}| \mid \mathcal{N} \in A_k^{\mathcal{M}}\} = \cap\{|\mathfrak{A}| \mid \mathfrak{A} \subset \mathcal{M}, \mathfrak{A} \prec_{\Sigma_k} \mathcal{M},$

$$\mathfrak{A} \equiv_{\Sigma_{k+1}} \mathcal{M}\} = \overline{\Sigma}_{k+1}^{\mathcal{M}} = \overline{\Pi}_k^{\mathcal{M}} = \overline{\Delta}_k^{\mathcal{M}}.$$

R E F E R E N C E S

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INICIJALNI SEGMENTI I IZOMORFNE SLIKE NESTANDARDNIH MODELA ARITMETIKE

SADRŽAJ. Neka je m prebrojiv nestandardan model Peanove aritmetike (skraćeno P) i $A_k^m = \{n \mid n \subset_e m,\ n <_{\Sigma_k} m,\ n \doteq m\}$. Za svako $a \in |m|$ neka je $I_a = \{x \in |m| \mid x < a\}$. U radu se razmatraju oni segmenti I_a koji sadrže elemente iz A_k^m (stavovi 1.1 i 1.3) i daju se neke dopune i jednostavniji dokazi posledice 2.7.1 i teoreme 2.9 iz [1] (posledice 1.2, 1.4, 1.5 i 1.7).

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