Signal interpolation using numerically robust differential operators

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Abstract: In our work on frequency estimation based on local signal behavior [15] for testing purposes we needed a signal $\phi(t)$ which over some disjoint intervals of (continuous) time I_n is equal to a corresponding linear combination $f_n(t)$ of up to N sine waves, possibly damped and phase shifted, of (normalized) frequencies smaller than π. The signal should also satisfy the following constraints: $\phi(t)$ should contain a minimal amount of out-of-band energy, i.e., the energy of its Fourier transform $\widehat{\phi}(\omega)$ outside interval $[-\pi, \pi]$ should be as small as possible; $\phi(t)$ should fit within an as narrow envelope as possible; $\phi(t)$ should also have a finite support in the time domain, which is as short as possible. Clearly, these are mutually conflicting requirements and we want to look for a compromise solution which is nevertheless good in all of these respects. A computationally efficient method for producing such a signal can be useful for designing novel digital modulation schemas which satisfy stringent conditions on out of band leakage and envelope properties of the generated signal. The method we propose in this paper employs some special, numerically robust linear differential operators, called the chromatic derivatives, which were introduced relatively recently, and which we believe hold yet unexplored promise in signal and image processing.

Key–Words: numerical differentiation, chromatic derivatives, local signal representation, signal interpolation, digital modulation

1 Introduction

Assume that $F_1(t), F_2(t), \ldots, F_n(t)$ are π-band limited signals; thus, their corresponding Nyquist rate sampling interval is of unit length.[1] Let also I_1, I_2, ..., I_n be disjoint intervals of time and let $\chi_{I_j}(t)$ denote the characteristic function of the interval I_j, $1 \leq j \leq n$. Thus, $\chi_{I_j}(t) = 1$ for $t \in I_j$ and $\chi_{I_j}(t) = 0$ outside I_j.

We denote by $f_j(t)$ the section of $F_j(t)$ on the corresponding interval of time I_j, i.e., for all $1 \leq j \leq n$ let $f_j(t) = F_j(t)$ on I_j and $f_j(t) = 0$ outside I_j. Thus, $f_j(t) = F_j(t)\chi_{I_j}(t)$ and the support $\mathrm{supp}(f_j)$ of f_j satisfies $\mathrm{supp}(f_j) \subseteq I_j$.

For simplicity of our presentation we will assume that intervals $I_1, I_2, \ldots, I_n$ are of equal duration of T Nyquist rate intervals and that they are equally spaced N Nyquist rate intervals apart; it is straightforward to adapt our method to more general cases. In particular,

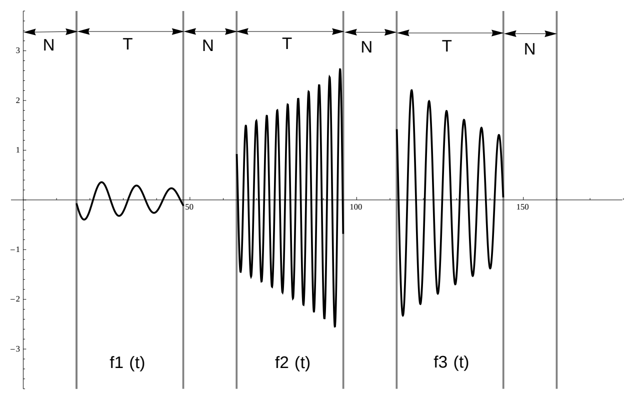

Figure 1: *Pieces of three exponentially modified sine waves of duration T spaced N unit intervals apart.*

let us also assume that $I_j = [(j-1)T + jN, j(T+N)]$ for all $1 \leq j \leq n$; see Figure 1.

Let $c = nT + (n+1)N$; our goal is to produce an interpolation signal $\phi(t)$ with finite support $\mathrm{supp}(\phi) = [0, c]$, which has the following properties: (1) for all $1 \leq j \leq n$ we have $\phi(t) = f_j(t)$ for all $t \in \mathrm{supp}(f_j)$; (2) $\phi(t)$ has minimal out of band leakage, i.e., the Fourier transform $\widehat{\phi}(\omega)$ of $\phi(t)$ is such that $\int_{|\omega|>\pi} |\widehat{\phi}(\omega)|^2 d\omega$ is as small fraction of the total energy of $\widehat{\phi}(\omega)$ as possible; (3) $\phi(t)$ fits within

[1] These signals might belong to the space $\mathbf{BL}(\pi)$ of π-band limited signals of finite energy, i.e., the space of continuous L_2 functions whose Fourier transform is supported within $[-\pi, \pi]$. However, they can also be of infinite energy and with a Fourier transform which exists only as a generalized function. As it will be observed later, our method applies to an even broader subset of analytic functions.

an as narrow envelope as possible; thus, between disjoint intervals I_j the signal should not have transients of large amplitude. Clearly, computationally efficient methods which produce such signals can be useful for designing digital modulation schemes which satisfy stringent requirements on out of band leakage and tightness of the envelope of the generated signal.[2] Interpolation produced by our algorithm for fragments shown on Figure 1 is shown on Figure 2.

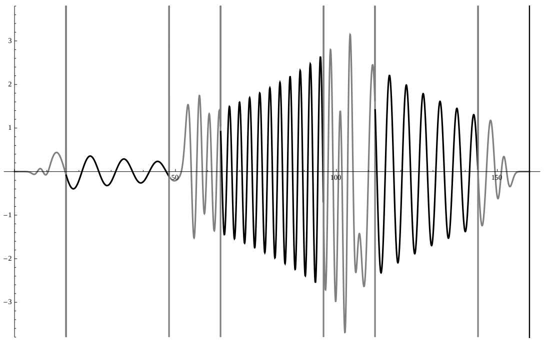

Figure 2: *Interpolation for fragments on Figure 1.*

We now sketch the main idea which we will refine to obtain our method. We do not provide the details, because they will be provided for the actual construction described in Section 3.

We will define a signal $\phi(t)$ supported on $[0, c]$ extending all of $f_j(t)$, ($1 \leq j \leq n$), such that $\phi(t)$ has continuous derivatives of orders up to $N-1$ and such that $\phi^{(k)}(0) = \phi^{(k)}(c) = 0$ for all $k < N$. Thus, if we extend ϕ to the entire $\mathbb{R}$ by setting it equal to zero outside $[0, c]$, the resulting function is $N-1$ times continuously differentiable on $\mathbb{R}$ and, since it is also finitely supported, $\phi, \phi', \ldots, \phi^{(N-1)}$ all belong to L_1. Low out of band leakage of $\phi(t)$ can now be shown to follow from the following well known theorem (see, for example [2]):

Theorem 1 *Assume that ϕ is n times continuously differentiable and that $\phi, \phi', \ldots, \phi^{(n)} \in L_1$. Then the Fourier transform $(\phi^{(n)})\hat{}(\omega)$ of $\phi^{(n)}(t)$ satisfies $|(\phi^{(n)})\hat{}(\omega)| \leq M$ for some $M \geq 0$, and $\lim_{|\omega|\to\infty}(\phi^{(n)})\hat{}(\omega) = 0$. Moreover, the Fourier transform $\widehat{\phi}(\omega)$ of $\phi(t)$ satisfies*

$$|\widehat{\phi}(\omega)| \leq \frac{|(\phi^{(n)})\hat{}(\omega)|}{|\omega|^n} \leq \frac{M}{|\omega|^n}$$

and consequently decreases rapidly as $|\omega| \to \infty$.

To obtain such interpolation $\phi(t)$ we will first prove the following "pulse shaping" extrapolation theorem.

[2] An early version of this method was used at author's startup *Kromos Technology Inc.*, to design a digital transceiver.

Theorem 2 *Let $a \in \mathbb{R}$ be arbitrary and let $f(t) = F(t)\chi_{[a+N,a+N+T)}(t)$ be a fragment of a π band limited signal $F(t)$ on an interval $[a+N, a+N+T]$. Then there exist corresponding $G_L, G_R \in \mathbf{BL}(\pi)$ such that for $k < N$: $G_L^{(k)}(a) = 0$; $G_L^{(k)}(a+N) = F^{(k)}(a+N)$; $G_R^{(k)}(a+N+T) = F^{(k)}(a+N+T)$; $G_R^{(k)}(a+2N+T) = 0$. Let $g_L(t) = G_L(t)\chi_{[a,a+N)}(t)$ and $g_R(t) = G_R(t)\chi_{[a+N+T,a+2N+T)}(t)$; then $\phi_f(t) = g_L(t) + f(t) + g_R(t)$ has support contained in $[a, a+2N+T]$, coincides with $f(t)$ on $[a+N, a+N+T]$ and has $N-1$ continuous derivatives on $\mathbb{R}$.*

Let $\mathrm{supp}(f_j) = [(j-1)T + jN, j(T+N)]$ be the supports of fragments $f_j(t)$, ($1 \leq j \leq n$) and let functions $g_L^{f_j}, g_R^{f_j}$ be as provided by the above theorem, with $a_j = (j-1)(T+N)$. Then, setting $f_0(t) = f_{n+1}(t) = 0$ for all $t \in \mathbb{R}$, we can define $\phi(t) = \sum_{j=0}^{n+1} g_L^{f_j}(t) + f_j(t) + g_R^{f_j}(t)$ to obtain an $N-1$ times continuously differentiable function with support $\mathrm{supp}(\phi) = [0, c]$ which extends each $f_j(t)$, $1 \leq j \leq n$.

Functions G_L^f, G_R^f satisfying the conditions of Theorem 2 exist and are supplied in the course of the proof of a special case of Papoulis' Sampling Theorem, see [3, 4, 11], which extends Shannon's Sampling Theorem,[3] and which states that a π band limited signal $f(t)$ of finite energy is uniquely determined by the samples of the derivatives $f^{(k)}(t_j)$, $0 \leq k \leq N-1$, taken every N Nyquist rate intervals, i.e., such that $t_{j+1} - t_j = N$. Such special case of Papoulis' theorem represents a band limited signal in the form

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} f^{(k)}(Nn)\phi_k(t - nN) \qquad (1)$$

where interpolation functions $\phi_k(t)$ satisfy $\phi_k^{(m)}(0) = \delta(m-k)$ and $\phi_k^{(m)}(nN) = 0$ for all $0 \leq m, k < N$ and all $n \neq 0$. Functions $\phi_k(t)$ can be obtained by suitably choosing polynomials $P_k(t)$ of degree at most $N-1$ in the expression $\phi_k(t) = P_k(t)\,[\mathrm{sinc}\,(t/N)]^N$. Given a π band limited fragment $f(t)$ supported within $[a+N, a+N+T]$, functions $G_L(t)$ and $G_R(t)$ whose existence is asserted by Theorem 2 can now be obtained by setting $G_L(t) = \sum_{k=0}^{N-1} f^{(k)}(a+N)\phi_k(t-(a+N))$ and $G_R(t) = \sum_{k=0}^{N-1} f^{(k)}(a+N+T)\phi_k(t-(a+N+T))$.

Unfortunately, such functions $\phi_k(t)$ obtained from Papoulis' Sampling Theorem do not produce functions $G_L(t)$ and $G_R(t)$ which can be used to define interpolation $\phi(t)$ which also satisfies condition

[3] Shannon stated the theorem for $N = 2$ (i.e., for f and f') in his seminal paper [1], without a proof.

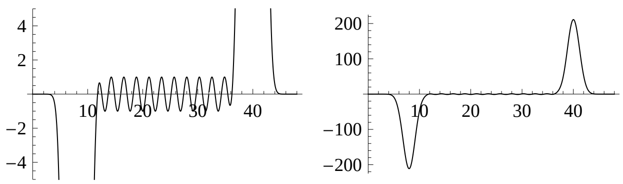

Figure 3: *Interpolation based on the Papoulis Sampling Theorem shown on two scales for the y-axis.*

(3) mentioned at the beginning of the paper, i.e., an interpolation which fits in an as narrow envelope as possible. A closer look reveals that some of $\phi_k(t)$ attain very large amplitudes in the interval $[-N, N]$. For example, for $N = 16$, the corresponding function $\phi_5(t)$ attains values larger than 69. Thus, interpolation functions G_L and G_R defined from such ϕ_k might have transients of very large amplitudes in intervals $[a, a+N]$ and $(a+N+T, a+2N+T]$. Figure 3 shows interpolation function $\phi_f(t)$ for $f(t) = \sin(7/8\,\pi\, t)\chi_{[N,2N]}(t)$ with $N = T = 16$ and $a = 0$; such interpolation attains values larger than 200.

This problem can be solved by observing that we do not need the "global" features of $\phi_k(t)$, namely that $\phi_k^{(m)}(nN) = 0$ for all $0 \leq k, m < N$ and all $n \neq 0$; we only need

$$\phi_k^{(m)}(0) = \delta(m-k); \quad \phi_k^{(m)}(N) = 0; \tag{2}$$

for all $0 \leq k, m < N$, without any constraints on values at $t = nN$ for integers $n \neq 0, 1$. Then for every particular $f(t) = F(t)\chi_{[a+N,a+N+T]}(t)$ the corresponding functions $G_L^f(t), G_R^f(t)$ can be defined by $G_L^f(t) = \sum_{k=0}^{N-1} f^{(k)}(a+N)(-1)^k\phi_k(a+N-t)$ and $G_R^f(t) = \sum_{k=0}^{N-1} f^{(k)}(a+N+T)\phi_k(t-(a+N+T))$.

With so relaxed constraints, we can now look for interpolation functions $\phi_k(t)$ with much smaller maximal amplitudes.

To obtain such functions $\phi_k(t)$ we need a "generic" representation of $\phi_k(t)$ which is in a form convenient for numerical evaluation of its derivatives, and which involves parameters that can be chosen to meet the requirements (1) - (3) in the best possible way. Preferably, the method should be applicable to signals $F(t)$ given by their sampled values, rather than by their corresponding analytic expressions.

This is a non trivial task because of the problems associated with numerical evaluation of derivatives of high order; it can be elegantly achieved using some special, numerically robust differential operators called *the chromatic derivatives*, introduced in [6], and their corresponding local signal expansions akin to the Taylor expansion, called *the chromatic expansions*, introduced in [7] and first published in [8, 9, 10]. A comprehensive presentation can be found in [12, 13, 14].

2 Chromatic derivatives and chromatic expansions

As is well known, truncations of the expansion of band limited signals provided by the Whittaker–Kotelnikov–Nyquist–Shannon Sampling Theorem $f(t) = \sum_{n=-\infty}^{\infty} f(n)\ \mathrm{sinc}\,(t-n)$ do not provide good local signal approximations, because, for a fixed t, the values of interpolation functions $\mathrm{sinc}\,(t-n)$ decay slowly as $|n|$ grows. Thus, to achieve a good local approximation a very large number of samples $f(n)$ are necessary.

On the other hand, signals $f \in \mathbf{BL}(\pi)$ are analytic functions; thus they can be locally represented by the truncations of the Taylor expansion $f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\ t^n/n!$. Taylor's expansion is local in nature, because the values of the derivatives $f^{(n)}(0)$ are determined by the values of the signal in an arbitrarily small neighborhood of zero.

However, Taylor's formula has an extremely limited use in signal processing because accurate evaluation of derivatives of higher orders from discrete noisy samples of a signal is essentially impossible. Moreover, the functions used in the expansion, i.e., the monomials $t^n/n!$, do not correspond to band limited signals; the approximation is unbounded, it converges non-uniformly and its error increases rapidly when moving away from the center of expansion.

Chromatic derivatives and chromatic expansions were introduced to provide a framework for local approximations of band limited signals which do not suffer from any of the above problems.

2.1 Chromatic derivatives

Chromatic derivatives are linear differential operators with constant coefficients obtained from suitably chosen families of orthogonal polynomials.[4] Thus, let polynomials $P_n(\omega)$ satisfy

$$\int_{-\pi}^{\pi} P_n(\omega)P_m(\omega)w(\omega)d\omega = \delta(m-n), \tag{3}$$

where $w(\omega)$ is a non-negative symmetric weight function and δ the Kronecker delta function. It can be shown that, if the weight function $w(\omega)$ is symmetric, then each polynomial $P_n(\omega)$ contains only powers of the same parity as n, and that such polynomials satisfy the recurrence relation

$$P_{n+1}(\omega) = \frac{1}{\gamma_n}\,\omega\, P_n(\omega) - \frac{\gamma_{n-1}}{\gamma_n}\, P_{n-1}(\omega) \tag{4}$$

for some positive constants γ_n. We define linear differential operators associated with such family of

[4] See [12, 14] for details regarding which families of orthogonal polynomials produce satisfactory families of chromatic derivatives.

ortho-*normal* polynomials by the operator polynomials

$$\mathcal{K}^n = (-j)^n P_n\left(j\,\frac{\mathrm{d}}{\mathrm{d}t}\right). \qquad (5)$$

Thus, $\mathcal{K}^n$ is obtained by replacing ω^k in $P_n(\omega)$ by $j^k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t)$.

For example, let $P^L_n(\omega)$ be obtained by re-scaling and re-normalizing the Legendre polynomials so that $1/(2\pi)\int_{-\pi}^{\pi} P^L_n(\omega)P^L_m(\omega)d\omega = \delta(m-n)$, i.e., such that $P^L_n(\omega)$ are orthonormal with respect to the constant weight function $w(\omega) = (2\pi)^{-1}$. Then it is easily computed that $P^L_0(\omega) = 1$, $P^L_1(\omega) = \sqrt{3}\omega/\pi$, $P^L_2(\omega) = \sqrt{5}(3\omega^2-\pi^2)/2\pi^2$ and $P^L_3(\omega) = \sqrt{7}(5\omega^3 - 3\omega\pi^2)/2\pi^3$. Consequently, the substitution given by (5) yields $\mathcal{K}^0[f](t) = f(t)$, $\mathcal{K}^1[f](t) = \sqrt{3}f'(t)/\pi$, $\mathcal{K}^2[f](t) = \sqrt{5}(3f''(t)+\pi^2 f(t))/2\pi^2$ and $\mathcal{K}^3[f](t) = \sqrt{7}(5f'''(t) - 3\pi^2 f'(t))/2\pi^2$.

For symmetric weight functions $w(\omega)$ in (3) the corresponding operators $\mathcal{K}^n$ have real coefficients, each $\mathcal{K}^n$ contains only derivatives of the same parity as n and $\mathcal{K}^n$ satisfy the three term recurrence relation

$$\mathcal{K}^{n+1} = \frac{1}{\gamma_n}\,(\mathrm{d}\circ\mathcal{K}^n) + \frac{\gamma_{n-1}}{\gamma_n}\,\mathcal{K}^{n-1}, \qquad (6)$$

with the same coefficients γ_n as in (4). Using (4) and (6), it is easy to verify that

$$\mathcal{K}^n_t[\mathrm{e}^{j\,\omega t}] = j^n P_n(\omega)\,\mathrm{e}^{j\,\omega t}. \qquad (7)$$

Thus, if $f \in \mathbf{BL}(\pi)$ and $\widehat{f}(\omega)$ is its Fourier transform, then

$$\mathcal{K}^n[f](t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n P_n(\omega)\widehat{f}(\omega)e^{j\,\omega t}d\omega. \qquad (8)$$

In comparison, if we normalize the "standard" derivatives so that the magnitude of their frequency response is bounded uniformly in n within $[-\pi,\pi]$, we get

$$\frac{f^{(n)}(t)}{\pi^n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} j^n\left(\frac{\omega}{\pi}\right)^n \widehat{f}(\omega)\mathrm{e}^{j\,\omega t}d\omega. \qquad (9)$$

Figure 4 compares the plots of $(\omega/\pi)^n$ (left), which are the transfer functions of the normalized derivatives $1/\pi^n\ \mathrm{d}^n/\mathrm{d}t^n$ (modulo a factor of j^n) with the plots of the transfer functions $P^L_n(\omega)$ of the chromatic derivatives $\mathcal{K}^n$ associated with the Legendre polynomials (right). Plots on the left reveal why numerical evaluation of higher order derivatives from signal samples makes no practical sense. Multiplication of the Fourier transform of a signal by the transfer function of a derivative of higher order essentially obliterates the spectrum of the signal, leaving only its edges, which in practice contain mostly noise. Figure 4 (left) also shows that the graphs of the transfer functions of the normalized derivatives of high orders and of the

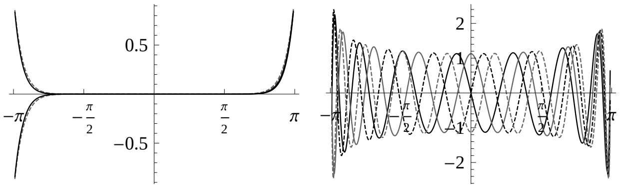

Figure 4: *Graphs of* $(\omega/\pi)^n$ *(left) and* $P^L_n(\omega)$ *(right) for* $n = 15$ *to* $n = 18$.

same parity cluster so tightly together that are essentially indistinguishable. Thus, from a numerical perspective, the set of the derivatives $\{f, f', f'', \ldots\}$ is a very poor base of the vector space of linear differential operators with real coefficients. On the other hand, the right plot on Figure 4 shows that the transfer functions of the chromatic derivatives $\mathcal{K}^n$ form a family of well separated, interleaved and increasingly refined comb-like filters. Instead of obliterating, such operators encode the spectral features of the signal. For this reason, we call operators $\mathcal{K}^n$ *the chromatic derivatives*.

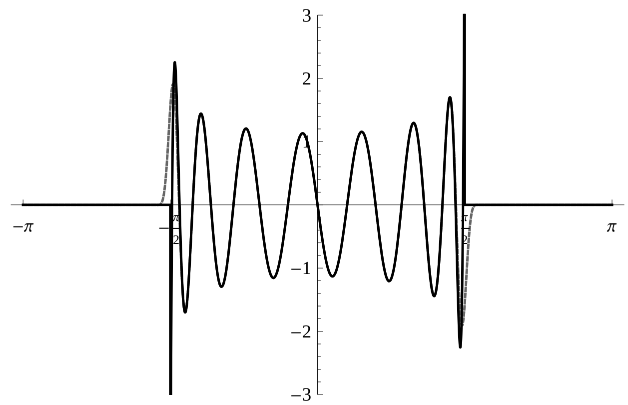

Figure 5: *The transfer functions of the operator* $\mathcal{K}^{15}$ *(black) and of its transversal filter (gray).*

While introducing chromatic derivatives amounts to simply replacing the usual base $\{f, f', f'', \ldots\}$ of the vector space of linear differential operators with an orthonormal base, it turns out that such a base has some remarkable properties.[5]

First of all, evaluation of chromatic derivatives from samples of the signal taken at twice the usual Nyquist rate is very accurate and noise robust. Figure 5 shows the transfer function of a digital transversal filter $A[f](t) = \sum_{i=-64}^{64} c_i\, f(t - i/2)$ which approximates the chromatic derivative $\mathcal{K}^{15}$ (gray) associated with the Legendre polynomials, and the transfer function of the ideal filter corresponding to $\mathcal{K}^{15}$

[5] It was shown in [12] that one can introduce a scalar product on the vector space of linear differential operators with constant coefficients which makes operators $\{\mathcal{K}^n\}_{n\in\mathbb{N}}$ orthonormal.

(black). The filter was designed using the Remez exchange method [5], and has 129 taps, spaced two taps per Nyquist rate interval. Thus, the transfer function of the corresponding ideal filter $\mathcal{K}^{15}$ is $P^L_{15}(2\omega)$ for $|\omega| \leq \pi/2$ and zero outside this interval. The passband of the filter is 90% of the interval $[-\pi/2, \pi/2]$, and the transition region extends 10% of the bandwidth $\pi/2$ on each side of the boundaries $-\pi/2$ and $\pi/2$. Outside the transition region the error of approximation is less than 1.3×10^{-4}. Implementations of filters for operators $\mathcal{K}^n$ of orders $0 \leq n \leq 30$ have been tested in practice and proved to be both accurate and noise robust, as expected from the above considerations.

Secondly, chromatic derivatives can be used to produce local approximations of band limited signals which do not suffer from the mentioned shortcomings of the Taylor expansion.

Proposition 3 *Let $\mathcal{K}^n$ be the chromatic derivatives associated with the Legendre polynomials, and let $f(t)$ be any function analytic on $\mathbb{R}$; then for all $t \in \mathbb{R}$,*

$$f(t) = \sum_{n=0}^{\infty} (-1)^n \; K^n[f](u) \; K^n[\mathrm{sinc}\,](t-u) \quad (10)$$

If in addition $f \in \mathbf{BL}(\pi)$, then the series converges uniformly on $\mathbb{R}$ and in the space $\mathbf{BL}(\pi)$.

The series in (10) is called *the chromatic expansion of f associated with the Legendre polynomials*; a truncation of this series is called a *chromatic approximation* of f. Just like a Taylor approximation, a chromatic approximation is also a local approximation: its coefficients are the values of differential operators $\mathcal{K}^m[f](u)$ at a single instant u, and for all $k \leq n$,

$$f^{(k)}(u) = \frac{\mathrm{d}^k}{\mathrm{d}t^n} \left[\sum_{m=0}^{n} (-1)^m \; \mathcal{K}^m[f](u) \; \mathcal{K}^m[\mathrm{sinc}\,](t-u) \right]_{t=u}.$$

Figure 6 compares the behavior of the chromatic approximation (black) of a signal $f \in \mathbf{BL}(\pi)$ (gray) with the behavior of the Taylor approximation of $f(t)$ (dashed). Both approximations are of order sixteen, and the signal $f(t)$ is defined using the Nyquist expansion, with randomly generated samples $\{f(n) \; : \; |f(n)| < 1, \; -32 \leq n \leq 32\}$.

By Proposition 3, the chromatic expansion of such a signal converges uniformly on $\mathbb{R}$, and the plot reveals that, when approximating a signal $f \in \mathbf{BL}(\pi)$, a chromatic approximation has a much gentler error accumulation when moving away from the point of expansion than the Taylor approximation of the same order.

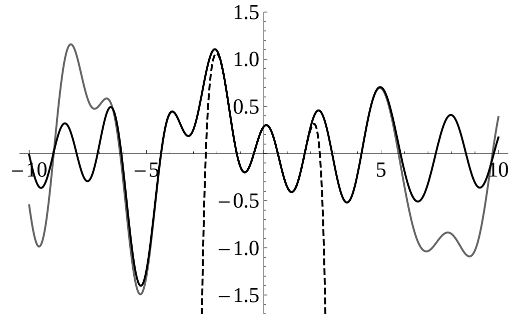

Figure 6: *A signal $f \in \mathbf{BL}(\pi)$ (gray) and its chromatic and Taylor approximations (black, dashed).*

The error of chromatic approximation of a function $f(t) \in \mathbf{BL}(\pi)$ is given by

$$\left| f(t) - \sum_{n=0}^{N-1} (-1)^n \; K^n[f](u) \; K^n[\mathrm{sinc}\,](t-u) \right|^2 \leq \|f\|_2 \left(1 - \sum_{k=0}^{N-1} K^n[\mathrm{sinc}\,](t-u)^2 \right) \quad (11)$$

The error bound term in the brackets on the right-hand side of (11) for the case of the chromatic expansion associated with the Legendre polynomials for $N = 16$ is shown on Figure 7; note that in this case the approximation is highly accurate on the interval $[-4, 4]$.

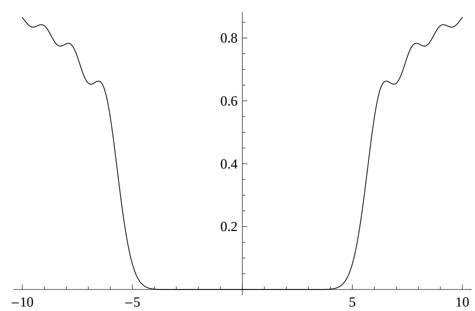

Figure 7: *Behavior of the error term of chromatic approximation for $N = 16$.*

Functions $\mathcal{K}^n[\mathrm{sinc}\,](t)$ in the chromatic expansion associated with the Legendre polynomials are given by $\mathcal{K}^n[\mathrm{sinc}\,](t) = (-1)^n \sqrt{2n+1} \;\; j_n(\pi t)$, where j_n is the spherical Bessel function of the first kind of order n.

Unlike the monomials that appear in the Taylor formula, functions $\mathcal{K}^n[\mathrm{sinc}\,](t)$ belong to $\mathbf{BL}(\pi)$ and satisfy $|\mathcal{K}^n[\mathrm{sinc}\,](t)| \leq 1$ for all $t \in \mathbb{R}$. Consequently, the chromatic approximations are bounded on $\mathbb{R}$ and belong to $\mathbf{BL}(\pi)$. Since by Proposition 3 the chromatic approximation of a signal $f \in \mathbf{BL}(\pi)$ converges to f in $\mathbf{BL}(\pi)$, if A is a filter, then A commutes with the differential operators $\mathcal{K}^n$ and thus

$$A[f](t) = \sum_{n=0}^{\infty}(-1)^n\ \mathcal{K}^n[f](0)\ \mathcal{K}^n[A[\operatorname{sinc}]](t)$$

for every $f \in \mathbf{BL}(\pi)$. Note that this is fully analogous to the representation of the actions of such operators via the Nyquist expansion:

$$A[f](t) = \sum_{n=0}^{\infty}(-1)^n\ f(n)\ A[\operatorname{sinc}](t-n).$$

Thus, while local, chromatic expansions possess the features which make the Nyquist expansion so useful in signal processing. This, together with numerical robustness of chromatic derivatives, makes chromatic approximations applicable in fields involving empirically sampled data, such as digital signal and image processing.

The next proposition demonstrates another remarkable property of chromatic derivatives relevant to signal processing.

Proposition 4 *Let $\mathcal{K}^n$ be the chromatic derivatives associated with the (rescaled and normalized) Legendre polynomials, and $f, g \in \mathbf{BL}(\pi)$. Then for all $t \in \mathbb{R}$,*[6]

$$\sum_{n=0}^{\infty} K^n[f](t)^2 = \int_{-\infty}^{\infty} f(x)^2 dx;$$

$$\sum_{n=0}^{\infty} K^n[f](t)K^n[g](t) = \int_{-\infty}^{\infty} f(x)g(x)dx;$$

$$\sum_{n=0}^{\infty} K^n[f](t)K^n_t[g(u-t)] = \int_{-\infty}^{\infty} f(x)g(u-x)dx.$$

Moreover, if a function f is analytic on $\mathbb{R}$ and satisfies $\sum_{n=0}^{\infty} K^n[f](0)^2 < \infty$, then such f must belong to $\mathbf{BL}(\pi)$.

Note that the sums in the above theorem provide local representations of the usual norm, the scalar product and the convolution, respectively, which are defined in $\mathbf{BL}(\pi)$ globally, as improper integrals. We now return to the main aim of this paper.

3 Band Limited Signal Interpolation

We now refine the ideas presented in the Introduction. We first show that fragments $g_L(t)$ and $g_R(t)$, whose existence is claimed in Theorem 2 (for simplicity we set $a = 0$), can be chosen so that the resulting signal $\phi_f(t) = g_L(t) + f(t) + g_R(t)$ has both very small out of band energy and fits in a narrow envelope.

Let $\mathcal{K}^n$ denote the chromatic derivatives associated with the Legendre polynomials. We first produce functions $\phi_k(t)$ which for all $0 \le m, k < N$ instead of conditions (2) satisfy[7]

$$\mathcal{K}^m[\phi_k](0) = \delta(m-k); \quad \mathcal{K}^m[\phi_k](N) = 0. \qquad (12)$$

We can then define $G_L(t)$ and $G_R(t)$ as

$$G_L(t) = \sum_{j=0}^{N-1} \mathcal{K}^j[f](N)(-1)^j \phi_j(N-t); \qquad (13)$$

$$G_R(t) = \sum_{j=0}^{N-1} \mathcal{K}^j[f](N+T)\phi_j(t-(N+T)).$$

To explain why an approach using the chromatic derivatives associated with the Legendre polynomials is better than the one based on the "standard" derivatives, we note that (8) implies that for $f \in \mathbf{BL}(\pi)$,

$$|\mathcal{K}^n[f](t)|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P^L_n(\omega)^2 d\omega \int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2 d\omega.$$

Since polynomials $P^L_n(\omega)$ are orthonormal with the weight function $(2\pi)^{-1}$, we get that $|\mathcal{K}^n[f](t)| \le \|f\|_2$ for all $t \in \mathbb{R}$, i.e., chromatic derivatives of signals $f \in \mathbf{BL}(\pi)$ are bounded uniformly in n and t.

Similar is true for a large class of signals f which do not have a finite L_2 norm. For example, (7) implies that $|\mathcal{K}^n_t[e^{j\,\omega t}]| = |P^L_n(\omega)|$, and polynomials $P^L_n(\omega)$ satisfy that for every $0 \le b < \pi$ there exists $M > 0$ such that $|P^L_n(\omega)| < M$ for all n and all $|\omega| \le b$. For example, for $|\omega| < .99\pi$ we have $|P^L_n(\omega)| < 3.01$.

Thus, unlike the standard derivatives of $e^{j\,\omega t}$ whose absolute values grow rapidly for frequencies larger than 1 and vanish rapidly for frequencies smaller than 1, the chromatic derivatives always attain values in a range which insures that our constrained optimizations described below will be numerically stable.[8] More over, the values of chromatic derivatives of $f(t) = F(t)\chi_{[N,N+T]}(t)$ at the end points of the support of $f(t)$ and which appear in (13) can be obtained in a noise robust way from sampled values of $F(t)$, thus eliminating the need for any analytic differentiation.

We want now to chose $\phi_k(t)$ so that $G_L(t)$ and $G_R(t)$ given by (13) are such that the of out of band energy of the corresponding extrapolation $\phi(t) = g_L(t) + f(t) + g_R(t)$ whose existence is asserted in Theorem 2 is as small fraction of the total energy

[6]Thus, the sums on the left hand side of the above equations do not depend on the choice of the instant t; see [12, 14] for details.

[7]Note that, since chromatic derivatives are linear combinations of the "standard" derivatives, $\mathcal{K}^n[f](t_0) = \mathcal{K}^n[g](t_0)$ holds for all $n < N$ if and only if $f^{(n)}(t_0) = g^{(n)}(t_0)$ holds for all $n < N$.

[8]We note that we will use the fact that fragments f_i come from π band limited signals only to conclude that their chromatic derivatives associated with the Legendre polynomials do not have values much larger than their amplitude. Thus, or method applies to all analytic functions having such property

of ϕ as possible, as well as that $g_L(t)$ and $g_R(t)$ on their corresponding supports have as small amplitude as possible.

To achieve the first objective we note that (8) implies that

$$|\widehat{\phi}(\omega)| = \frac{|(\mathcal{K}^{N-1}[\phi])^{\wedge}(\omega)|}{|P^L_{N-1}(\omega)|}$$

Since ϕ is finitely supported and $N-1$ times continuously differentiable, $|(\mathcal{K}^{N-1}[\phi])^{\wedge}(\omega)|$ is bounded and converges to zero as $|\omega| \to \infty$. More over, since all the zeros of $P^L_{N-1}(\omega)$ are within interval $[-\pi, \pi]$, outside this interval the value of $|P^L_{N-1}(\omega)|$ grows very rapidly. Thus, if we ensure that $|(\mathcal{K}^{N-1}[\phi])^{\wedge}(\omega)|$ is bounded with a reasonably small bound, $\widehat{\phi}(\omega)$ will have very low energy outside $[-\pi, \pi]$. Since $|(\mathcal{K}^{N-1}[\phi])^{\wedge}(\omega)| = \left|\int_0^{2N+T} \mathcal{K}^{N-1}[\phi](t) e^{j\,\omega t} dt\right| \leq \int_0^{2N+T} |\mathcal{K}^{N-1}[\phi](t)| dt$, to make $|(\mathcal{K}^{N-1}[\phi])^{\wedge}(\omega)|$ uniformly small, it is enough to keep $|\mathcal{K}^{N-1}[\phi](t)|$ as small as possible for all t. Since the value of $|\mathcal{K}^{N-1}[\phi](t)|$ is equal to $|\mathcal{K}^{N-1}[f](t)|$ in the interval $[N, N+T]$, we have to make sure that the values of $|\mathcal{K}^{N-1}[g_L](t)|$ on $[0, N]$ and the values of $|\mathcal{K}^{N-1}[g_R](t)|$ on $[N+T, 2N+T]$ are not much larger. This will be accomplished by keeping the values of $\mathcal{K}^{N-1}[\phi_k](t)$ small on $[0, N]$ for all $0 \leq k < N$.

Similarly, we will limit the maximal value of $|g_L(t)|$ and $|g_R(t)|$ by limiting the maximal values of all $|\phi_k(t)|$.

We note that controlling the out of band energy as well as the maximal amplitude of ϕ on $[0, 2N+T]$ by controlling the same features of all ϕ_k is suboptimal; applying our minimization techniques to g^f_L and g^f_R directly gives better results. However, this requires running optimization algorithms for each f separately, while the above approach uses optimization only to obtain the family $\{\phi_k(t)\}_{0\leq k<N}$; the corresponding g^f_L and g^f_R can then be obtained for every f as sections of linear combinations (13); this results in a very efficient interpolation algorithm.

To obtain an appropriate family $\phi_m(t)$ we consider a "generic" chromatic expansion associated with the Legendre polynomials, centered at $t = N/2$, of the form

$$\begin{aligned} \psi(t) &= \sum_{n=0}^{3N} X_k\, \mathcal{K}^n[\mathrm{sinc}\,](t - N/2)) \\ &= \sum_{n=0}^{3N} X_k\, j_n(\pi(t - N/2) \qquad (14) \end{aligned}$$

where X_k are variables. The chromatic expansion associated with the Legendre polynomials is a good choice, because it provides a rapidly converging uniform approximation of exactly all $\mathbf{BL}(\pi)$ signals. The upper limit of summation was chosen to allow sufficient "degrees of freedom" of the resulting generic approximation, based on the type of constraints we will impose and the properties of the error function of the chromatic approximation given by (11), which, due to lack of space, we cannot detail here.

Chromatic derivatives of $\psi(t)$ are of the form $\mathcal{K}^m[\psi](t) = \sum_{n=0}^{3N+m} (-1)^n\, Y_n\, \mathcal{K}^n[\mathrm{sinc}\,](t)$ where Y_n are linear combinations of X_k with reasonable coefficients. This is due to the fact that (8) implies

$$(\mathcal{K}^m \circ \mathcal{K}^n)[\mathrm{sinc}\,](t) = \frac{j^{n+m}}{2\pi} \int_{-\pi}^{\pi} P^L_n(\omega) P^L_m(\omega) e^{j\,\omega t} d\omega,$$

and $P^L_n(\omega)P^L_m(\omega) = \sum_{k=|m-n|}^{m+n} c_k P^L_k(\omega)$, with $c_k = (2\pi)^{-1} \int_{-\pi}^{\pi} P^L_n(\omega) P^L_m(\omega) P^L_k(\omega) d\omega$. Note also that for $n \leq 3N$,

$$\mathcal{K}^n[\psi](N/2) = (-1)^n X_n, \qquad (15)$$

and $\mathcal{K}^n[\psi](N/2) = 0$ for $n > 3N$.

Let us now fix an $m < N$; to obtain $\phi_m(t)$ we start by imposing the following $2N$ constraints on such generic expansion ψ: for all $0 \leq k < N$,

$$\mathcal{K}^k[\psi](0) = \delta(m-k); \quad \mathcal{K}^k[\psi](N) = 0. \qquad (16)$$

As we have just explained, we need to find values of X_k for which the maximal values of both $|\mathcal{K}^{N-1}[\psi](t)|$ and $|\psi(t)|$ on the interval $[0, N]$ are as small as possible.

To ensure that the maximal value of $|\mathcal{K}^{N-1}[\psi](t)|$ is small we use the fact that the first equality of Theorem 4 implies that for every $\psi \in \mathbf{BL}(\pi)$ the sum $\sum_{k=0}^{\infty} \mathcal{K}^n[\psi](t)^2$ does not depend on t. Thus, it is enough to make this sum small for one value of t to obtain a small upper bound for the absolute values of all chromatic derivatives of $\psi(t)$ for all values of t. Using (15), this will be accomplished by minimizing[9] $S = \sum_{k=0}^{3N} X_n^2$.

To ensure that the maximal value of $|\psi(t)|$ is small, we first note that the behavior of ψ around the end points of the interval $[0, N]$ is already determined by (16); to control the maximal amplitude of ψ in the interior of this interval we chose a sufficiently dense set of equally spaced points s_j such that $N/4 \leq s_j \leq 3N/4$ (we used $s_{j+1} - s_j = 1/8$) and impose for each such s_j linear conditions $\psi(s_j) < B_m$ and $\psi(s_j) > -B_m$. Here B_m is a positive bound that has to be obtained first; the values of $\psi(s_j)$ are expressed in terms of the variables X_k, $0 \leq k < 3N$ using (14).

[9] One can use more explicit bound on $|\mathcal{K}^{N-1}[\psi](t)|$ on $[0, N]$, along the lines below used to keep $|\psi(t)|$ small. However, this does not produce significantly better results, mainly because the above minimization already produces excellent results.

To obtain B_m we solve the following linear programming optimization problem.[10] We introduce an additional variable X, and minimize the objective $F(X, X_0, \ldots, X_{3N}) = X$, subject to linear constraints on variables $X_0, \ldots, X_{3N}$ which follow from (16), plus linear constraints $\psi(s_j) \leq X$ and $\psi(s_j) \geq -X$, with one such pair for each s_j, and finally, $X > 0$. For each $0 \leq m < N$ the corresponding bound B_m is now obtained by slightly relaxing (increasing) the minimal value of the objective X.[11]

Having obtained B_m, we can now solve the following quadratic minimization problem: Minimize the objective $F(X_0, \ldots, X_{3N}) = \sum_{k=0}^{3N} X_k^2$ subject to $2N$ linear constraints (2) plus linear constraints $\psi(s_j) < B_m$ and $\psi(s_j) > -B_m$ for each s_j.

Figure 3 shows extrapolation for the fragment $f(t) = \sin(7/8\,\pi t)\chi_{[16,32]}(t)$, with $N = T = 16$ and $a = 0$, obtained using functions ϕ_k produced by the above algorithm; compare it with the interpolation of the same fragment produced with the functions ϕ_k provided by the Papoulis Theorem, shown on Figure 3.

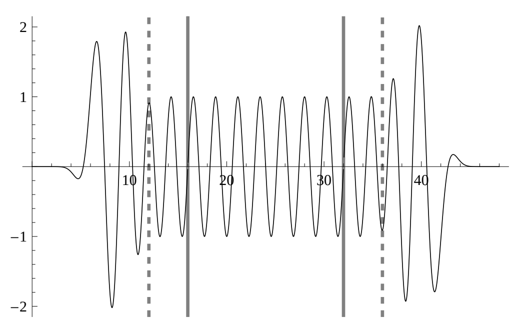

Figure 8: *Interpolation for the same fragment as shown on Figure 3.*

Note that fragment $f(t)$ is supported only on $[16, 32]$ (between the solid grid lines); however, the extrapolation accurately approximates $f[t]$ on the interval $[12, 36]$. This is due to the fact that, by (16), 15 derivatives of the extrapolation at end points of the support of f are equal to the corresponding 15 derivatives of f. Thus, as it can be seen on Figure 7, the approximation is accurate over about four Nyquist rate intervals on both sides of the support of f.

If the supports of the fragments $f_i(t)$ degenerate to a single point t_i, interpolations $G_L^{f_i}(t), G_R^{f_i}(t)$ can still be chosen such that $(G_L^{f_i})^{(n)}(t_i) = (G_R^{f_i})^{(n)}(t_i) = f_i^{(n)}(t_i)$. The resulting interpolation $\phi(t)$ will be a good approximation of each $f_i(t)$ over intervals centered at t_i of approximate length $N/2$. Figure 9 shows interpolation $\phi(t)$ (black) whose derivatives of orders up to 15 at $t_1 = 16$ are equal to the corresponding derivatives of $f_1(t) = \sin(3\pi/4\,t)$ and whose derivatives of the same orders at $t_2 = 32$ are equal to the corresponding derivatives of $f_2(t) = \sin(\pi/4\,(t-1))$ (f_1 and f_2 are shown in gray). While the supports of f_1, f_2 are only the single points $t_1 = 16$ and $t_2 = 32$ (at the solid grid lines), the corresponding interpolation $G_R^{f_1}(t) + G_L^{f_2}(t)$ is accurate within intervals between the dashed grid lines, each of length of about 8 Nyquist rate intervals.

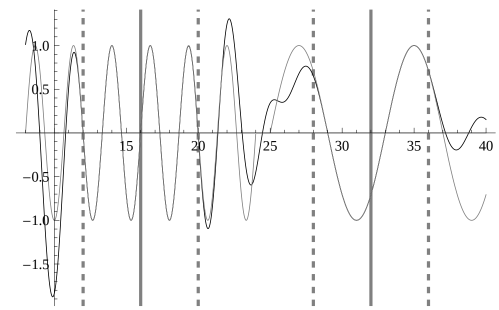

Figure 9: *Interpolation (black) for two single point fragments (gray).*

Thus, about half of the length of the spacing between the supports of the fragments is actually used for approximation of these fragments. Note that such approximation is accurate on the continuous time domain; thus, there is absolutely no "inter-symbol interference" between the fragments. This fact can be used for modulation schemas in which detection of the fragments (symbols) is achieved using local signal behavior in the continuous time domain, as encoded by the (discrete) values of the chromatic derivatives of the signal, obtained by sampling a corresponding filter bank; see [10] and [15].[12]

For evaluation we ran our interpolation method on sequences of 1000 fragments of signals which were sums of two exponentially damped or expanding sine waves, with randomly generated frequencies $0 < \omega < 3$ and damping factors $-0.03 < d < 0.03$. Both the duration of these fragments and their spacing was 16 Nyquist rate intervals. The maximal amplitude of these fragments was about 2.1. The resulting interpolation had maximal amplitude of about 6.5, achieving such value very infrequently and mostly staying within the maximal amplitude of the signal, see Figure 8 (left). Such interpolation was sampled twice per Nyquist rate interval and the corresponding discrete Fourier transform was computed. We found that the

[10] All our optimization problems are easily solved using any of the standard numerical software packages.

[11] We found that increasing the minimal value of X for only about 20% allows excellent simultaneous optimization for both condition (i) and (ii).

[12] Some of the references as well as our *Mathematica* simulations are available at http://www.cse.unsw.edu.au/˜ignjat/diff.

ratio of out of band energy, contained in frequencies above π and the total energy of the signal was about 10^{-5}, see Figure 8 (right).

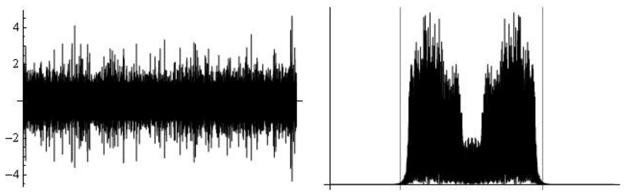

Figure 10: *The envelope and the Fourier transform of an interpolation.*

This demonstrates that our method produces an interpolation signal with both a tight envelope and an extremely low out of band leakage. We believe that, when combined with a detection of symbols based on local signal behavior as captured by the sampled values of several chromatic derivatives of the signal, our method could be used for novel digital modulation techniques satisfying very stringent requirements for out of band leakage and envelope properties of the generated signal.

4 Conclusion

We have shown that chromatic derivatives and chromatic expansions allow control of local signal behavior, otherwise poorly captured by Nyquist rate samples. They also allow control of global signal features, by either controlling the smoothness of the signal or by using the fact that the values of the chromatic derivatives at $t = 0$ provide coefficients of the expansion of the Fourier transform of the signal in series of orthogonal polynomials (see [14]):

$$\widehat{f}(\omega) = \sum_{n=0}^{\infty} (-j)^n \mathcal{K}^n[f](0) P_n^L(\omega).$$

Finally, we would like to stress that, while the interpolation method presented in this paper is interesting in its own right, our main aim has been to hint at the practical relevance and the potential of chromatic derivatives and chromatic expansions in signal processing. Since this is an entirely unexplored area, we hope that this paper as well as [15] might motivate the signal processing community to investigate further these notions and their potential applications.

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