Delay Analysis for a Finite Buffer TDMA Model

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***Abstract*— This paper provides an efficient solution for the derivation of the delay distribution for a TDMA system with a finite buffer. Assuming successive intervals of length equal to the duration of a slot, the density is expressed as (finite) linear combinations of gamma densities and beta densities with positive coefficients. The results are verified by simulations. Comparison with a previously proposed solution demonstrates that the solution presented here is more efficient.**

I. Introduction

TDMA (Time Division Multiple Access) systems are widely used in various telecommunication applications. Given the wide applicability of TDMA, models for TDMA applications, options, and versions have been extensively studied for over a quarter of a century [1]–[13]. The finite buffer TDMA model considered in this paper and also in [14] was motivated by the application of GSM paging described in [15]. In [14], we obtained detailed analytic formulas for the density and probability distribution of the delay in such a TDMA system. A concern for numerical accuracy led us to express all the modules of the recursive computational scheme as (infinite) linear combinations *with positive coefficients only* of beta densities. Though the infinite sums are truncated when the Poisson probabilities become zero or negligible, the large amount of involved terms result in significant computational effort. In this paper, we present the delay density as (finite) linear combinations of gamma densities and beta densities with positive coefficients. We shall see that the alternative (finite-sum) solution improves the efficiency of the (infinite-sum) solution in [14] for this important problem.

The remainder of the paper is organized as follows. In Section II we describe the model. Section III deals with derivations of preliminary quantities and an outline of the mathematical derivation of the delay distribution. The details of that derivation are presented in Sections IV, and V. Finally in Section VI we present the computational results for a numerical example; we confirm these by simulation, and we compare the computation time between the finite-sum solution and the infinite-sum solution.

II. The Model

We derive the delay distribution of an arbitrary admitted customer to a finite buffer that operates under the following procedure. The customers arrive according to a homogeneous Poisson process of rate λ. If there are fewer than K customers present, an arriving item is *admitted*, otherwise it is lost. The time axis is divided equally into successive frames (slots) of length T. If at the end of a slot, there are j items in the buffer, then with conditional probability $d(i,j)$, i of the j items are removed on a first come first served (FCFS) basis. For $1 \le j \le K$, the quantities $\{d(i,j)\}$ satisfy

$$\sum_{i=0}^{j} d(i,j) = 1.$$

An item admitted during the frame $(0,T)$ may be removed at one of the epochs kT, $k = 1,2,3,\ldots$. If it is admitted at time $T-u$, $0 \le u \le T$, its delay is therefore the sum of u and $(k-1)T$.

III. An outline of the derivation

Let J_k be the number of items in the buffer at time $kT+$. $\{J_k\}$ is then a Markov chain with state space $\{0,1,\ldots,K\}$. Its probability transition matrix $P = \{P_{ij}\}$ is given by

$$\begin{aligned} P_{ij} &= P(J_{k+1} = j | J_k = i) \\ &= \sum_{\nu=max(0,j-i)}^{K-i-1} e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}\, d(i+\nu-j, i+\nu) \\ &\quad + \left[1 - \sum_{\nu=0}^{K-i-1} e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}\right] d(K-j,K), \end{aligned}$$

for $0 \le i,j \le K$. By $[\pi_0, \pi_1, \ldots, \pi_K]$, we denote the steady-state probabilities of that Markov chain and we assume that these have been computed.

It can be shown that the expected number E^* of packets admitted during a slot of length T is

$$E^* = \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{\nu=0}^{K-i-1} e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}(K-i-\nu)\right].$$

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The ratio E^*/T is the steady-state rate at which items are admitted to the buffer, so that $(E^*/T)dv$ is the elementary probability of an admission in $(v, v+dv)$.

Let $\psi(\cdot)$ be the probability density of the delay of an arbitrary admitted item. In this section, we present an outline of the derivation of $\psi(\cdot)$ with the cumbersome details to be filled in later. We choose the time origin 0 at the beginning of the slot during which the arbitrary item is admitted.

We first derive the expected number $dE^*(u)$ of items admitted during $(0, T)$ whose delay lies between u and $u+du$. That derivation is somewhat involved. When that is completed, we note that

$$[dE^*(u)/T]/[E^*/T] = \psi(u)du,$$

is the elementary probability that an arbitrary admitted item waits between u and $u+du$. Therefore, $\psi(\cdot)$ is the probability density of the delay distribution.

What requires a well-organized derivation is that the function $E^*(u)$ assumes different analytic forms on the successive intervals $(kT, kT+T)$, $k \geq 0$. To express the first density, denoted $\psi_0(\cdot)$, and to relate the form of the density $\psi_k(\cdot)$, $k \geq 1$, on a subsequent interval to the preceding one requires somewhat involved bookkeeping. These tasks are accomplished by using a convenient matrix formalism.

We must keep track of the buffer content at each epoch $kT+$ and of the position r, $K \geq r \geq 1$, of the item that we are following. While, owing to new arrivals and successive departures, the buffer content can increase and decrease, the position r is non-increasing from one slot to the next. When the tracked item leaves the buffer, we shall say that it reaches position 0.

A. Accounting for the first frame $(0, T)$

It is convenient to introduce the matrix $T_0(u)$ that serves to account for transactions during the first slot $(0, T)$. The matrix $T_0(u)$ is of dimensions $[K(K+1)/2] \times [K(K+1)/2]$. Its row indices (r, i) run from $(1, 0)$ to $(K, K-1)$, for $K \geq r \geq 1$ and $i \leq r-1$; its column indices (r', i') from $(1, 1)$ to (K, K), for $K \geq i' \geq 1$ and $r' \leq i'$. The quantity $[T_0(u)]_{(r,i),(r',i')}du$ is the elementary conditional probability that, given that the buffer contains i items at time $0+$, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$, and that at time $T+$, there are i' items present and the item we are tracking is now in the position r', with $r \geq r' \geq 1$.

For $K \geq r \geq 1$ and $i \leq r-1$, we also define the column vector $\mathbf{T}_0^0(u)$ of dimension $K(K+1)/2$. The quantity $[\mathbf{T}_0^0(u)]_{(r,i)}du$ is the elementary conditional probability that, given that the buffer content at time $0+$ is i, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$ and departs at time T.

B. Accounting for the subsequent frames $(kT, kT+T)$

To do the accounting for the subsequent slots, we define the $[K(K+1)/2] \times [K(K+1)/2]$ matrix $\widetilde{T}$. Both the row and column indices (r_1, i_1) and (r_2, i_2) of $\widetilde{T}$ range from $(1, 1)$ to (K, K), for $K \geq i_1 \geq 1$, $r_1 \leq i_1$ and $K \geq i_2 \geq 1$, $r_2 \leq i_2$. The element $\widetilde{T}_{(r_1,i_1),(r_2,i_2)}$ is the conditional probability that, given that, at the beginning of the slot, there are i_1 items in the buffer with the marked item in position r_1, by the end of the slot, there are i_2 items in the buffer and the item we are tracking has moved to position r_2, with $r_1 \geq r_2 \geq 1$.

For $K \geq i_1 \geq 1$ and $r_1 \leq i_1$, we also define the column vector $\widetilde{\mathbf{T}}^0$ of dimension $K(K+1)/2$. The quantity $[\widetilde{\mathbf{T}}^0]_{(r_1,i_1)}$ is the conditional probability that, given that, at the beginning of the slot, there are i_1 items in the buffer with the marked item in position r_1, the item we are tracking is removed at the end of that slot.

The two essential steps of the derivation of the delay distribution are the following: Let us write $\pi^*(r)$ for the vector $[\pi_{r-1}, \pi_{r-2}, \ldots \pi_0]$, and π^* for the vector $[\pi^*(K), \pi^*(K-1), \ldots \pi^*(1)]$. The expected number $dE^*(u)$ of items admitted during $(0, T)$ whose delay lies between u and $u+du$ is given by different expressions on the successive intervals $(kT, kT+T), k \geq 0$. Applying the law of total conditional expectation, we see that, for $0 \leq u < T$,

$$dE^*(u) = \pi^* \mathbf{T}_0^0(u) du, \tag{1}$$

and on the interval $(kT, kT+T)$, for $k \geq 1$,

$$dE^*(u) = \pi^* T_0(u-kT) \widetilde{T}^{k-1} \widetilde{\mathbf{T}}^0 du. \tag{2}$$

The method of computation of the function $E^*(u)$ and, therefore, of the probability density $\psi(\cdot)$, is now clear in principle. By using equation (1), we evaluate the function on $(0, T)$. Then, recursively forming the vectors $\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0$, we apply (2) to compute the function for the subsequent frames. However, by getting into the details, the analytic results and the algorithmic procedure can be made much more explicit. These matters are discussed in the next two sections.

IV. The Delay Distribution - The First Slot

The elements of the matrix $T_0(u)$ are now made explicit. We recall that the quantity $[T_0(u)]_{(r,i),(r',i')}du$ is the elementary conditional probability that, given that the buffer content at time $0+$ is i, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$, that at time $T+$, there are i' items in the buffer and the item we are tracking has moved to position r'. Clearly, we can have positive probability only when $r \geq r'$ and $i'+r-r' \leq K$. Moreover, the initial state i cannot be K, otherwise no admission during $(0, T)$ is possible. The transition during $(T-u, T)$ can occur with or without the buffer filling up. If it does not, then for $i'+r-r' < K$, there must be $r-r'$ removals at time T. That means that there must be $i'+r-r'$ items just prior to T, so that there are $r-i-1$ arrivals in $(0, T-u)$ and $i'-r'$ in $(T-u, T)$.

Therefore, the element $[T_0(u)]_{(r,i),(r',i')}$ is given by

$$\begin{aligned} &e^{-\lambda(T-u)} \frac{[\lambda(T-u)]^{r-i-1}}{(r-i-1)!} \lambda \\ &\cdot e^{-\lambda u} \frac{(\lambda u)^{i'-r'}}{(i'-r')!} d(r-r', i'+r-r'), \end{aligned} \tag{3}$$

for $i'+r-r' < K$. For future reference, let us call that the form $P_A(u)$.

If $i' + r - r' = K$, the buffer fills up during $(T-u, T)$. The corresponding expression for that case is

$$e^{-\lambda(T-u)} \frac{[\lambda(T-u)]^{r-i-1}}{(r-i-1)!} \lambda \cdot \left[1 - \sum_{\nu=0}^{K-r-1} e^{-\lambda u} \frac{(\lambda u)^\nu}{\nu!}\right] d(r-r', K). \tag{4}$$

We call that the form $P_B(u)$.

The matrix $T_0(u)$ has the structure that we show in (5, see next page) for $K = 4$. The symbols $P_A(u)$ or $P_B(u)$ indicate which of the formulas (3) or (4) that is to be used for the specific indices.

The elements of the column vector $\mathbf{T}_0^0(u)$ are similarly defined. The element $[\mathbf{T}_0^0(u)]_{(r,i)}$, the elementary conditional probability that, given that the buffer content at time $0+$ is i, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$ and *departs* at time T, is given by

$$\begin{aligned} &e^{-\lambda(T-u)} \frac{[\lambda(T-u)]^{r-i-1}}{(r-i-1)!} \lambda \\ &\cdot \left\{ \sum_{j=0}^{K-r-1} e^{-\lambda u} \frac{(\lambda u)^j}{j!} \sum_{\nu=r}^{j+r} d(\nu, j+r) \right. \\ &+ \left. \left[1 - \sum_{j=0}^{K-r-1} e^{-\lambda u} \frac{(\lambda u)^j}{j!}\right] \sum_{\nu=r}^{K} d(\nu, K) \right\}. \end{aligned} \tag{6}$$

When $r' = 0$ the item is removed at time T. It then no longer matters how many items remain in the buffer. The marked item is removed at time T if r or more items are removed at that time. The two terms in equation (6) reflect whether or not the buffer fills up in $(T-u, T)$.

The elements of the matrix $T_0(u)$ and the column vector $\mathbf{T}_0^0(u)$ are conveniently expressed as linear combinations of gamma densities

$$\gamma(t; \lambda, n) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda, \quad t > 0,$$

and beta densities

$$\beta(y; p, q) = [B(p,q)]^{-1} y^{p-1} (1-y)^{q-1}, \quad 0 < y < 1,$$

where $B(p,q)$ is the beta function

$$B(p,q) = \int_0^1 v^{p-1} (1-v)^{q-1} dv.$$

By routine manipulations, we rewrite the form $P_A(u)$ as

$$\begin{aligned} P_A(u) &= e^{-\lambda T} \frac{(\lambda T)^{r-r'+i'-i}}{(r-r'+i'-i)!} d(r-r', i'+r-r') \\ &\cdot \frac{1}{T} \beta(1 - \tfrac{u}{T}; r-i, i'-r'+1), \end{aligned}$$

and the form $P_B(u)$ as

$$\begin{aligned} P_B(u) &= d(r-r', K) \gamma(T-u; \lambda, r-i) \\ &- \sum_{\nu=0}^{K-r-1} e^{-\lambda T} \frac{(\lambda T)^{r-i+\nu}}{(r-i+\nu)!} d(r-r', K) \\ &\cdot \frac{1}{T} \beta(1 - \tfrac{u}{T}; r-i, \nu+1). \end{aligned}$$

Similarly, we rewrite the components of the vectors $\mathbf{T}_0^0(u)$. From formula (6) we obtain that

$$\begin{aligned} [\mathbf{T}_0^0(u)]_{(r,i)} &= \sum_{\nu=r}^{K} d(\nu, K) \gamma(T-u; \lambda, r-i) \\ &+ \sum_{j=0}^{K-r-1} e^{-\lambda T} \frac{(\lambda T)^{r-i+j}}{(r-i+j)!} \left[\sum_{\nu=r}^{j+r} d(\nu, j+r) - \sum_{\nu=r}^{K} d(\nu, K) \right] \\ &\cdot \frac{1}{T} \beta(1 - \tfrac{u}{T}; r-i, j+1). \end{aligned}$$

By virtue of formula (1), the probability density $\psi_0(\cdot)$ on the interval $(0, T)$ is given by

$$\psi_0(u) = [E^*]^{-1} \pi^* \mathbf{T}_0^0(u). \tag{7}$$

The delay distribution $F(x)$ at x with $0 < x < T$ is then given by

$$F(x) = \int_0^x \psi_0(u) du. \tag{8}$$

V. The Delay Distribution - The Subsequent Slots

The accounting of the transactions in the buffer content and in the position of the marked item during the subsequent slots is carried out by means of the matrix $\widetilde{T}$. We recall that the elements of $\widetilde{T}$ are the conditional probabilities that, given that at the start of the slot there are i_1 items with the marked item in position r_1, by the end of the slot there are i_2 items in the buffer and the marked item has moved to position r_2. Apparently, $\widetilde{T}$ has positive quantities only when $r_1 \geq r_2$, $i_2 \geq i_1 - r_1 + r_2$, and $i_2 \leq K - r_1 + r_2$.

The element $\widetilde{T}_{(r_1,i_1),(r_2,i_2)}$ is given by

$$\begin{aligned} &\widetilde{T}_{(r_1,i_1),(r_2,i_2)} \\ &= e^{-\lambda T} \frac{(\lambda T)^{i_2+r_1-r_2-i_1}}{(i_2+r_1-r_2-i_1)!} d(r_1 - r_2, i_2 + r_1 - r_2), \end{aligned} \tag{9}$$

for $i_2 < K - r_1 + r_2$. That is the case where, during the slot, the buffer does not fill up. Let us call that the form P_C.

The element $\widetilde{T}_{(r_1,i_1),(r_2,K-r_1+r_2)}$ is given by

$$\begin{aligned} &\widetilde{T}_{(r_1,i_1),(r_2,K-r_1+r_2)} \\ &= \left[1 - \sum_{\nu=0}^{K-i_1-1} e^{-\lambda T} \frac{(\lambda T)^\nu}{\nu!}\right] d(r_1 - r_2, K). \end{aligned} \tag{10}$$

It corresponds to $i_2 = K - r_1 + r_2$ and to the buffer filling up during the slot. We call that the form P_D.

The special structure of the matrix $\widetilde{T}$ is displayed in (11, see next page) for the representative value $K = 4$. The symbols P_C or P_D indicate which of the formulas (9) or (10) that is to be used for the specific indices.

The components of the vector $\widetilde{\mathbf{T}}^0$ are given by

$$\begin{aligned} &\sum_{j=0}^{K-i_1-1} e^{-\lambda T} \frac{(\lambda T)^j}{j!} \sum_{\nu=r_1}^{i_1+j} d(\nu, i_1+j) \\ &+ \left[1 - \sum_{\nu=0}^{K-i_1-1} e^{-\lambda T} \frac{(\lambda T)^\nu}{\nu!}\right] \sum_{l=r_1}^{K} d(l, K). \end{aligned}$$

$$
T_0(u) = \begin{array}{c||c|cc|ccc|cccc}
(r,i)\backslash(r',i') & (4,4) & (3,4) & (3,3) & (2,4) & (2,3) & (2,2) & (1,4) & (1,3) & (1,2) & (1,1) \\
\hline\hline
(4,3) & P_B(u) & 0 & P_B(u) & 0 & 0 & P_B(u) & 0 & 0 & 0 & P_B(u) \\
(4,2) & P_B(u) & 0 & P_B(u) & 0 & 0 & P_B(u) & 0 & 0 & 0 & P_B(u) \\
(4,1) & P_B(u) & 0 & P_B(u) & 0 & 0 & P_B(u) & 0 & 0 & 0 & P_B(u) \\
(4,0) & P_B(u) & 0 & P_B(u) & 0 & 0 & P_B(u) & 0 & 0 & 0 & P_B(u) \\
\hline
(3,2) & 0 & P_B(u) & P_A(u) & 0 & P_B(u) & P_A(u) & 0 & 0 & P_B(u) & P_A(u) \\
(3,1) & 0 & P_B(u) & P_A(u) & 0 & P_B(u) & P_A(u) & 0 & 0 & P_B(u) & P_A(u) \\
(3,0) & 0 & P_B(u) & P_A(u) & 0 & P_B(u) & P_A(u) & 0 & 0 & P_B(u) & P_A(u) \\
\hline
(2,1) & 0 & 0 & 0 & P_B(u) & P_A(u) & P_A(u) & 0 & P_B(u) & P_A(u) & P_A(u) \\
(2,0) & 0 & 0 & 0 & P_B(u) & P_A(u) & P_A(u) & 0 & P_B(u) & P_A(u) & P_A(u) \\
\hline
(1,0) & 0 & 0 & 0 & 0 & 0 & 0 & P_B(u) & P_A(u) & P_A(u) & P_A(u)
\end{array} \tag{5}
$$

$$
\widetilde{T} = \begin{array}{c||c|cc|ccc|cccc}
(r_1,i_1)\backslash(r_2,i_2) & (4,4) & (3,4) & (3,3) & (2,4) & (2,3) & (2,2) & (1,4) & (1,3) & (1,2) & (1,1) \\
\hline\hline
(4,4) & P_D & 0 & P_D & 0 & 0 & P_D & 0 & 0 & 0 & P_D \\
\hline
(3,4) & 0 & P_D & 0 & 0 & P_D & 0 & 0 & 0 & P_D & 0 \\
(3,3) & 0 & P_D & P_C & 0 & P_D & P_C & 0 & 0 & P_D & P_C \\
\hline
(2,4) & 0 & 0 & 0 & P_D & 0 & 0 & 0 & P_D & 0 & 0 \\
(2,3) & 0 & 0 & 0 & P_D & P_C & 0 & 0 & P_D & P_C & 0 \\
(2,2) & 0 & 0 & 0 & P_D & P_C & P_C & 0 & P_D & P_C & P_C \\
\hline
(1,4) & 0 & 0 & 0 & 0 & 0 & 0 & P_D & 0 & 0 & 0 \\
(1,3) & 0 & 0 & 0 & 0 & 0 & 0 & P_D & P_C & 0 & 0 \\
(1,2) & 0 & 0 & 0 & 0 & 0 & 0 & P_D & P_C & P_C & 0 \\
(1,1) & 0 & 0 & 0 & 0 & 0 & 0 & P_D & P_C & P_C & P_C
\end{array} \tag{11}
$$

By (2), on the interval $(kT, kT+T)$, for $k \geq 1$, the density $\psi_k(\cdot)$ is given by

$$\psi_k(u) = [E^*]^{-1}\pi^* T_0(u-kT)\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0. \tag{12}$$

The delay distribution $F(x)$ at x with $kT < x < kT+T$ is given by

$$F(x) = \sum_{j=0}^{k-1}\int_{jT}^{jT+T}\psi_j(u)du + \int_{kT}^{x}\psi_k(u)du. \tag{13}$$

VI. Numerical examples

After routine numerical computations of the vector π^* and of the quantity E^*, we obtain from formulas (7) and (12) the delay densities on successive intervals $(kT, kT+T)$, $k \geq 0$. The density $\psi(u)$ on the first slot $(0,T)$ is evaluated by implementing (7). For the subsequent slots, we recursively perform a matrix multiplication to form $\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0$. From these, the density $\psi(u)$ is readily computed on each interval by implementing (12). Inside the recursive loop, we also compute the probability distribution $F(\cdot)$ by using formula (13) and calling a library routine for the gamma distribution and the incomplete beta ratio. Particularly, since p and q are both positive integers, the incomplete beta ratio

$$I_x(y;p,q) = \int_0^x \beta(y;p,q)dy$$

can be most efficiently calculated by applying the following recurrence relations [16]:

$$\begin{cases} I_x(y;1,1) = x, \\ I_x(y;p,1) = x^p, \\ I_x(y;1,q) = 1-(1-x)^q, \\ I_x(y;p,q) = xI_x(y;p-1,q) + (1-x)I_x(y;p,q-1). \end{cases}$$

Our simulations were initiated by variates from the steady-state probability vector $[\pi_0, \pi_1, \ldots, \pi_K]$. Each run consisted of one million time slots from which a histogram and the empirical distribution function of the delay were estimated. For the histogram, the class width was set to one tenth of the slot length T.

In our numerical example, one selected from among many, we consider a buffer of size $K = 30,\ 60,\ 90$, respectively under the load $\lambda T = 10,\ 30,\ 60$. For the sake of our example, the parameters $d(i,j)$ were specified as follows: For j even, we formed the $j+1$ integers $1,\ 2, \cdots, j/2+1, j/2, \cdots, 2,\ 1$, divided by their sum and identified $d(i,j)$, for $0 \leq i \leq j$, with the corresponding ratio. For j odd, we wrote the $j+1$ integers $1,\ 2, \cdots, (j+1)/2, (j+1)/2, \cdots, 2,\ 1$, divided by their sum and similarly identified the $d(i,j)$. That avoids displaying a 71×71 matrix to complete the specification of our example.

Graphs of the corresponding delay densities and distributions, obtained by computation and simulation, are shown in Figures 1-3. We see that the results of the alternative finite-sum algorithm presented here agree satisfactorily with those of the earlier infinite-sum one, but the required computation times are much shorter. Our experience on a Sun Ultra-Enterprise-10000 machine is summarised in the following table:

	Figure 1	Figure 2	Figure 3
Finite-sum	3 sec	55 sec	547 sec
Infinite-sum	10 sec	162 sec	807 sec

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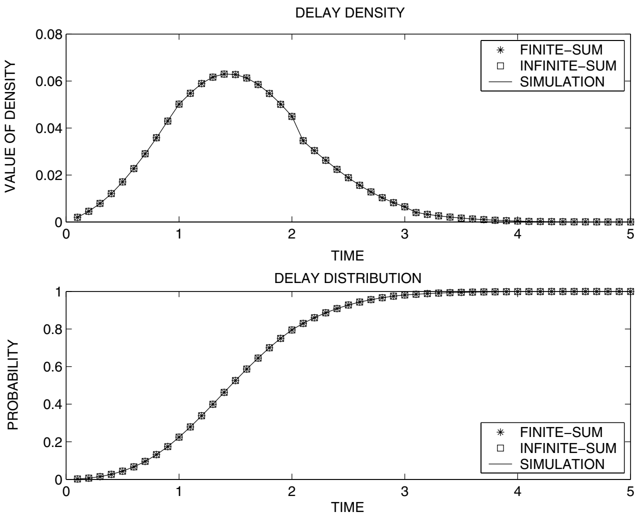

Fig. 1. Computation and simulation results for the delay density and delay distribution of the TDMA model, $K = 30$, $\lambda T = 10$.

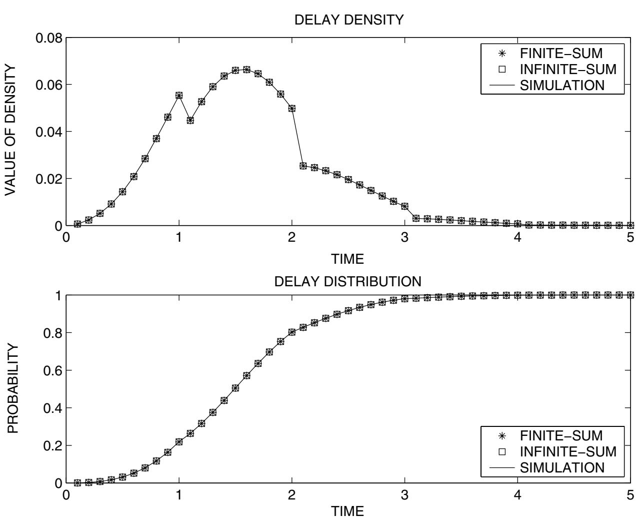

Fig. 2. Computation and simulation results for the delay density and delay distribution of the TDMA model, $K = 60$, $\lambda T = 30$.

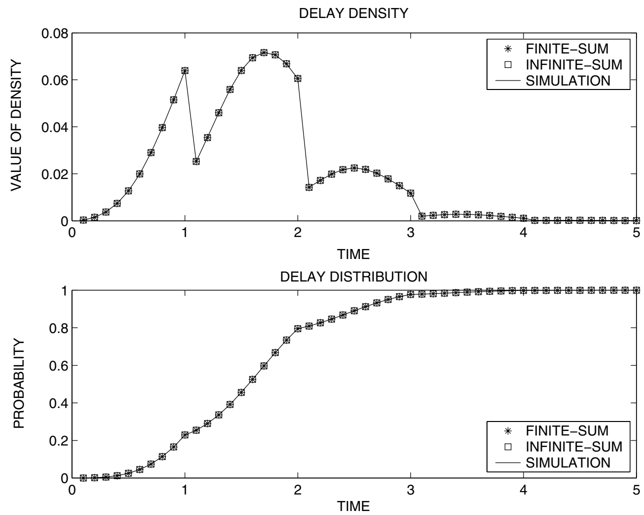

Fig. 3. Computation and simulation results for the delay density and delay distribution of the TDMA model, $K = 90$, $\lambda T = 60$.

REFERENCES

[1] T. G. Birdsall, M. P. Ristenbatt, and S. B. Weinstein, "Analysis of asynchronous time multiplexing of speech sources," *IRE Transactions on Communications Systems*, vol. 10, pp. 390–397, December 1962.

[2] W. W. Chu and A. G. Konheim, "On the analysis and modeling of a class computer communications systems," *IEEE Transactions on Communications*, vol. 20, no. 3, pp. 645–660, June 1972.

[3] L. P. Clare and I. Rubin, "Queueing analysis of TDMA with limited and unlimited buffer capacity," in *Proceedings of IEEE Infocom 83*, April 1983, pp. 229–238.

[4] K. Khan and H. Peyravi, "Delay and queue size analysis of TDMA with general traffic," in *Proceedings of the International Symposium on Modeling, Analysis and Simulation of Computer and Communication Systems*, July 1998, pp. 217–225.

[5] H. Kobayashi and A. G. Konheim, "Queueing models for computer communications system analysis," *IEEE Transactions on Communications*, vol. 25, no. 1, pp. 2–29, January 1977.

[6] S. S. Lam, "Delay analysis of a time division multiple access (TDMA) channel," *IEEE Transactions on Communications*, vol. 25, no. 12, pp. 1489–1494, December 1977.

[7] L. F. M. de Moraes, "Message delay analysis for a TDMA scheme operating under a preemptive priority discipline," *IEEE Transactions on Communications*, vol. 38, no. 1, pp. 67–73, January 1990.

[8] L. F. M. de Moraes and I. Rubin, "Message delays for a TDMA scheme under a nonpreemptive priority discipline," *IEEE Transactions on Communications*, vol. 32, pp. 583–588, May 1984.

[9] I. Rubin, "Message delays in FDMA and TDMA communication channels," *IEEE Transactions on Communications*, vol. 27, pp. 769–777, 1979.

[10] I. Rubin, "Access-control disciplines for multi-access communication channels: reservation and TDMA schemes," *IEEE Transactions on Information Theory*, vol. 25, no. 5, pp. 516–536, 1979.

[11] I. Rubin and Z.-H. Tsai, "Message delay analysis of multiclass priority TDMA, FDMA, and discrete-time queueing systems," *IEEE Transactions on Information Theory*, vol. 35, no. 3, pp. 637–647, May 1989.

[12] I. Rubin and Z. Zhang, "Message delay and queue-size analysis for circuit-switched TDMA systems," *IEEE Transactions on Communications*, vol. 39, pp. 905–914, June 1991.

[13] I. Rubin and Z. Zhang, "Message delay analysis for TDMA schemes using contiguous-slot assignments," *IEEE Transactions on Communications*, vol. 40, pp. 730–737, April 1992.

[14] M. F. Neuts, J. Guo, M. Zukerman, and H. L. Vu, "The waiting time distribution for a TDMA model with a finite buffer," in *Proceedings of IEEE Infocom 03*, 2003, in press.

[15] M. Ivanovich, P. Fitzpatrick, and M. Hesse, "GSM paging analysis," Telstra Research Laboratories, Radio Networks Report 02-3, February 2002.

[16] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. New York: Dover, 1972.