The Waiting Time Distribution for a TDMA Model With a Finite Buffer and State-Dependent Service

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***Abstract*—We obtain detailed analytic formulas for the density and probability distribution of the waiting time in a time-division multiple-access (TDMA) model with a finite buffer and state-dependent service. On successive intervals of length equal to the duration of a slot, the density is expressed as a linear combination of beta densities with positive coefficients. A recursive scheme, obtained by a matrix-analytic derivation, allows for the highly efficient computations of the coefficient sequences. An expression for the mean waiting time is derived using the classical queueing formula $L = \lambda W$. We also demonstrate that our methodology provides a concise treatment of various special cases that have been studied over the past half century.**

***Index Terms*—Finite buffer, queueing model, state-dependent service, time division multiple access (TDMA), waiting time distribution.**

I. INTRODUCTION

TIME-DIVISION multiple-access (TDMA) systems are widely used in various telecommunication applications. Given the wide applicability of TDMA, models for TDMA applications, options, and versions were extensively studied for over four decades [1]–[17]. Due to the complexity of the analysis of exact waiting time (or delay) distribution in finite buffer TDMA models, only a few results for very special cases are given in the literature [1], [8], [15]–[17]. While in [1], [8], and [17] models with constant unit service are studied, [15] and [16] treat cases with constant batch service.

The more general state-dependent service process in this paper is motivated by the modeling of GSM paging in [18] and of an optical burst switching (OBS) edge router in [19]. As will be clear from this paper, a finite buffer TDMA model with that service process gives rise to the interesting, complex problem of deriving the probability distribution for the waiting time of an arbitrary admitted customer.

In TDMA models with constant service, the amount of service on each successive time slot is deterministic. The waiting time of a customer finding a certain number of items ahead of it, under the first-come-first-served (FCFS) queueing discipline, is therefore bounded. In Section X, we see that the treatment of the delay distribution in such cases is straightforward. However, with state-dependent service, the exact number of customers that can be served is random on each successive time slot and depends on the number of waiting customers. If there is a positive probability that the waiting customers do not receive service within a time slot, the delay of a customer can therefore be arbitrarily large with positive probability despite the fact that the customer arrives at a buffer with a finite number of customers ahead of it.

The methodological interest of this work lies in the way we handle the complex accounting of the queueing process. The main structural results are derived by a transparent matrix formalism that is later unpacked to reveal the analytic form of the density of the waiting time distribution. We shall see that, on each of the successive TDMA fixed intervals, the density is given by a finite, positive linear combination of beta densities. The coefficients of those linear combinations are computed by a recursive scheme. We thus arrive at a nearly explicit analytic characterization of the waiting time density and at an algorithm for its numerical computation.

The remainder of the paper is organized as follows. In Section II, we describe the model. Sections III and IV deal with derivations of preliminary quantities. An outline of the mathematical derivation of the delay distribution is given in Section V. The details of that derivation are presented in Sections VI and VII. The results for the mean and variance of the waiting time are discussed in Section VIII. In Section IX we present the computational results for a numerical example, we confirm these by simulation, and we give an intuitive explanation for the behavior of the waiting time density function. Finally, in Section X, we provide straightforward derivations of the waiting time densities in two special cases of the TDMA model with constant service.

II. MODEL

We derive the waiting time distribution of an arbitrary admitted customer to a finite buffer that operates under the following procedure. The customers arrive according to a homogeneous Poisson process of rate λ. The probability $\alpha(t; \lambda, n)$ of n arrivals during time t is

$$\alpha(t; \lambda, n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

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If there are fewer than K customers present, an arriving item is *admitted*; otherwise it is lost. The time axis is divided equally into successive frames (slots) of length T. An item admitted during the frame (0, T) may be removed at one of the epochs kT, $k \geq 1$. If it is admitted at time $T-u$, $0 < u < T$, its waiting time is therefore the sum of u and an integer multiple $(k-1)T$ of T. If at the end of a slot there are j items in the buffer, then with conditional probability $d(i,j)$, i of the j items are removed on an FCFS basis. For $1 \leq j \leq K$, the quantities $\{d(i,j)\}$ satisfy

$$\sum_{i=0}^{j} d(i,j) = 1. \tag{1}$$

That service process has considerable versatility. It may cater for a wide range of applications such as demand assigned and quality of service (QoS) based classification. In the case of demand assigned TDMA, more TDMA "slots" are allocated to a station where the demand is higher, so that the longer the queue in terms of the number of packets, the more packets are served. In the case of QoS-based classification, separate logical buffers are allocated to different QoS classes and each may be treated in accordance with its specific QoS requirements. The service process also generalizes the cases of constant service. If the quantities $\{d(i,j)\}$ are set so that

$$\begin{cases} d(j,j) = 1, & \text{for } 0 \leq j \leq N \\ d(N,j) = 1, & \text{for } N < j \leq K, \\ d(i,j) = 0, & \text{elsewhere} \end{cases}$$

then our TDMA model (motivated by the applications in [18] and [19]) generalizes the one with constant batch service in [15] and [16]. More specifically, if $N = 1$, it further reduces to the one with constant unit service in [1].

III. EMBEDDED MARKOV CHAIN

Let J_k be the number of items in the buffer at time kT^+ immediately after removal. $\{J_k\}$ is then a Markov chain with state space $\{0, 1, \ldots, K\}$. Its probability transition matrix $P = \{P_{ij}\}$ is given by

$$\begin{aligned} P_{ij} &= P(J_{k+1} = j | J_k = i) \\ &= \sum_{\nu=\max(0,j-i)}^{K-i-1} \alpha(T;\lambda,\nu)\, d(i+\nu-j, i+\nu) \\ &\quad + \left[1 - \sum_{\nu=0}^{K-i-1} \alpha(T;\lambda,\nu)\right] d(K-j, K) \end{aligned}$$

for $0 \leq i, j \leq K$. By $[\pi_0, \pi_1, \ldots, \pi_K]$, we denote the steady-state probabilities of that Markov chain and we assume that these have been computed.

IV. EXPECTED NUMBER OF ITEMS ADMITTED PER TIME FRAME

Theorem 1: The expected number E^* of customers admitted during a slot of length T is

$$E^* = \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{\nu=0}^{K-i-1} \alpha(T;\lambda,\nu)(K-i-\nu)\right]. \tag{2}$$

Proof: Given that there are i, $0 \leq i \leq K-1$, items in the buffer at the beginning of a slot, the item must arrive so that it can occupy one of the positions r with $i+1 \leq r \leq K$. Using the indicator random variables of the corresponding events, we readily see that

$$E^* = \sum_{i=0}^{K-1} \pi_i \sum_{r=i+1}^{K} \int_0^T \alpha(u;\lambda, r-i-1)\, \lambda du \tag{3}$$

but $\alpha(u;\lambda,r-i-1)\,\lambda$ is the Erlang density function of order $r-i$, and

$$\int_0^T \alpha(u;\lambda,r-i-1)\,\lambda du = 1 - \sum_{\nu=0}^{r-i-1} \alpha(T;\lambda,\nu).$$

Thus, (3) becomes

$$\begin{aligned} E^* &= \sum_{i=0}^{K-1} \pi_i \sum_{r=i+1}^{K} \left[1 - \sum_{\nu=0}^{r-i-1} \alpha(T;\lambda,\nu)\right] \\ &= \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{l=0}^{K-i-1} \sum_{\nu=0}^{l} \alpha(T;\lambda,\nu)\right] \\ &= \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{\nu=0}^{K-i-1} \alpha(T;\lambda,\nu)(K-i-\nu)\right]. \end{aligned}$$

■

The ratio E^*/T is the steady-state rate at which items are admitted to the buffer, so that $(E^*/T)dv$ is the elementary probability of an admission in $(v, v+dv)$, and $1 - E^*/(\lambda T)$ is the probability that an arriving item is blocked.

V. OUTLINE OF THE DERIVATION

Let $\psi(\cdot)$ be the probability density of the delay of an arbitrary admitted item. In this section, we present an outline of the derivation of $\psi(\cdot)$ with the cumbersome details to be filled in later. We choose the time origin 0 at the beginning of the slot during which the arbitrary item is admitted.

We first derive the expected number $dE^*(u)$ of items admitted during (0, T) whose waiting time lies between u and $u+du$. That derivation is somewhat involved. When that is completed, we note that

$$\frac{\left[\frac{dE^*(u)}{T}\right]}{\left[\frac{E^*}{T}\right]} = \psi(u)du$$

is the elementary probability that an arbitrary admitted item waits between u and $u+du$. Therefore, $\psi(\cdot)$ is the probability density of the waiting time distribution.

What requires a well-organized derivation is that the function $dE^*(u)$ assumes different analytic forms on the successive intervals (kT, $kT+T$), i.e., for $kT < u < kT+T$, $k \geq 0$. To express the first density and to relate the form of the density on a subsequent interval to the preceding one requires somewhat involved bookkeeping. That second task is accomplished by using a convenient matrix formalism.

We must keep track of the buffer content at each epoch kT^+ and of the position r, $K \geq r \geq 1$, of the item that we are following. While, owing to new arrivals and successive departures, the buffer content can increase and decrease, the position r is nonincreasing from one slot to the next. When the tracked item leaves the buffer, we shall say that it reaches position 0.

A. Accounting for the First Frame (0, T)

It is useful to introduce vectors $\mathbf{g}^*(r';u)$, for $K \geq r' \geq 1$ and $0 < u < T$. The vector $\mathbf{g}^*(r';u)$ is of dimension $K - r' + 1$. The infinitesimal quantity $g^*_{i'}(r';u)du$, $K \geq i' \geq r'$, is the expected number of items admitted between $T-u$ and $T-u+du$, evaluated over the event where the buffer content at time T^+ equals i' and the position of that item at time T^+ is r'. As there can be at most one such arrival, $g^*_{i'}(r';u)du$ is also the elementary probability that between $T-u$ and $T-u+du$ an item is admitted, that its position at time T^+ is r', and that the buffer content then equals i'. Expressions for the vectors $\mathbf{g}^*(r';u)$ are derived in Section VI-B.

Next, we define some convenient matrices that serve to account for transactions during the first slot $(0, T)$. The matrices $T_0(r,r';u)$ are defined for $K \geq r \geq 1$ and for $r' \leq r$. The matrix $T_0(r,r';u)$ is of dimensions $r \times (K-r'+1)$. Its row indexes i run from 0 to $r-1$; its column indexes i' from r' to K. The quantity $[T_0(r,r';u)]_{i,i'}du$ is the elementary conditional probability that, given that the buffer contains i items at time 0^+, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$, and that at time T^+, there are i' items present and the item we are tracking is now in the position r'.

For $K \geq r \geq 1$, we also define column vectors $\mathbf{T}^0_0(r;u)$ of dimension r. The quantity $[\mathbf{T}^0_0(r;u)]_i du$ is the elementary conditional probability that, given that the buffer content at time 0^+ is i, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$ and departs at time T.

B. Accounting for the Subsequent Frames (kT, $kT+T$)

To do the accounting for the subsequent slots, we define the $(K-r_1+1) \times (K-r_2+1)$ matrices $T(r_1,r_2)$, for $K \geq r_1 \geq 1$ and for $r_2 \leq r_1$. The row and column indexes i_1 and i_2 of $T(r_1,r_2)$ range from K down to r_1 and from K down to r_2, respectively. The element $[T(r_1,r_2)]_{i_1,i_2}$ is the conditional probability that, given that, at the beginning of the slot, there are i_1 items in the buffer with the marked item in position r_1, by the end of the slot, there are i_2 items in the buffer and the item we are tracking has moved to position r_2, with $r_1 \geq r_2 \geq 1$.

For $K \geq r_1 \geq 1$, we also define column vectors $\mathbf{T}^0(r_1)$ of dimension $K-r_1+1$. The quantity $[\mathbf{T}^0(r_1)]_{i_1}$ is the conditional probability that, given that, at the beginning of the slot, there are i_1 items in the buffer with the marked item in position r_1, the item we are tracking is removed at the end of that slot.

It will be convenient to consider the matrix T^* whose structure is displayed in (4), shown at the bottom of the page, for the representative value $K = 6$.

We see that this matrix is of the form

$$T^* = \begin{vmatrix} \widetilde{T} & \widetilde{\mathbf{T}}^0 \\ \mathbf{0} & 1 \end{vmatrix}$$

and we shall later verify that T^* is stochastic. The derivation of the delay distribution is based on rudiments of the theory of discrete *phase-type* distributions.

The two essential steps of the derivation of that distribution are the following. Let us write $\boldsymbol{\pi}^*(r)$ for the vector $[\pi_{r-1},\ \pi_{r-2},\ \ldots \pi_0]$. The expected number $dE^*(u)$ of items admitted during $(0, T)$ whose waiting time lies between u and $u+du$ is given by different expressions on the successive intervals $(kT,\ kT+T)$, $k \geq 0$. Applying the law of total conditional expectation, we see that, for $0 < u < T$,

$$dE^*(u) = \sum_{r=1}^{K} \boldsymbol{\pi}^*(r)\mathbf{T}^0_0(r;u)du. \tag{5}$$

The direct sum of a finite ordered set of (row) vectors is the vector obtained by concatenating these vectors into a single row vector. For convenience, we form the direct sum $\boldsymbol{\gamma}^*(u)$ of the vectors $\mathbf{g}^*(r';u)$ with r' running from K down to 1. Then, a further application of the law of total conditional expectation yields that, on the interval $(kT,\ kT+T)$, for $k \geq 1$, we have

$$dE^*(u) = \boldsymbol{\gamma}^*(u-kT)\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0 du. \tag{6}$$

The method of computation of the function $E^*(u)$ and, therefore, of the probability density of the delay, is now clear in principle. By using (5), we evaluate the function on $(0, T)$. Then, recursively forming the vectors $\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0$, we apply (6) to compute the function for the subsequent frames. However, by getting into the details, the analytic results and the algorithmic procedure can be made much more explicit. These matters are discussed in Sections VI and VII.

VI. DELAY DISTRIBUTION–THE FIRST SLOT

The elements of the matrices $T_0(r,r';u)$, $K \geq r \geq 1$, for $r' \leq r$ are now made explicit. We recall that the quan-

$$T^* = \begin{array}{c||cccccc|c}
 & \underline{6} & \underline{5} & \underline{4} & \underline{3} & \underline{2} & \underline{1} & 0 \\
\hline\hline
\underline{6} & T(6,6) & T(6,5) & T(6,4) & T(6,3) & T(6,2) & T(6,1) & \mathbf{T}^0(6) \\
\underline{5} & 0 & T(5,5) & T(5,4) & T(5,3) & T(5,2) & T(5,1) & \mathbf{T}^0(5) \\
\underline{4} & 0 & 0 & T(4,4) & T(4,3) & T(4,2) & T(4,1) & \mathbf{T}^0(4) \\
\underline{3} & 0 & 0 & 0 & T(3,3) & T(3,2) & T(3,1) & \mathbf{T}^0(3) \\
\underline{2} & 0 & 0 & 0 & 0 & T(2,2) & T(2,1) & \mathbf{T}^0(2) \\
\underline{1} & 0 & 0 & 0 & 0 & 0 & T(1,1) & \mathbf{T}^0(1) \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array} \tag{4}$$

tity $[T_0(r, r'; u)]_{i,i'}du$ is the elementary conditional probability that, given that the buffer content at time 0^+ is i, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$, that at time T^+, there are i' items in the buffer, and the item we are tracking has moved to position r'. Clearly, we can have positive probability only when $r \geq i+1, r \geq r' \geq 0$. Moreover, the initial state i cannot be K, otherwise no admission during (0, T) is possible. The transition during ($T-u$, T) can occur with or without the buffer filling up. If it does not, then, for $r \geq i+1$, $i' \geq r' \geq 1$, and $i'+r-r' < K$, there must be $r-r'$ removals at time T. That means that there must be $i'+r-r'$ items just prior to T, so that there are $r-i-1$ arrivals in (0, $T-u$) and $i'-r'$ in ($T-u$, T).

Therefore, for $r \geq i+1, i' \geq r' \geq 1$, and $i'+r-r' < K$, the quantity $[T_0(r, r'; u)]_{i,i'}du$ is given by

$$\alpha(T-u; \lambda, r-i-1)\lambda du \cdot \alpha(u; \lambda, i'-r')d(r-r', i'+r-r').$$

If $i'+r-r' = K$, the buffer fills up during ($T-u$, T). The corresponding expression for that case is

$$\alpha(T-u; \lambda, r-i-1)\lambda du\left[1-\sum_{j=0}^{K-r-1}\alpha(u; \lambda, j)\right]d(r-r', K)$$

for $r \geq i+1, i' = K+r'-r$, and $r' \geq 1$.

The elements of the column vectors $\mathbf{T}_0^0(r; u)$, $K \geq r \geq 1$, are similarly defined. The quantity $[\mathbf{T}_0^0(r; u)]_i du$, the elementary conditional probability that, given that the buffer content at time 0^+ is i, an item is admitted into the rth buffer position between $T-u$ and $T-u+du$ and *departs* at time T, is given by

$$\alpha(T-u; \lambda, r-i-1)\lambda du\left\{\sum_{j=0}^{K-r-1}\alpha(u; \lambda, j)\sum_{\nu=r}^{j+r}d(\nu, j+r) + \left[1-\sum_{j=0}^{K-r-1}\alpha(u; \lambda, j)\right]\sum_{\nu=r}^{K}d(\nu, K)\right\} \quad (7)$$

for $r \geq i+1$. When $r' = 0$, the item is removed at time T. It then no longer matters how many items remain in the buffer. The marked item is removed at time T if r or more items are removed at that time. The two terms in (7) reflect whether or not the buffer fills up in ($T-u$, T).

The elements of the matrix $T_0(r, r'; u)$ are conveniently expressed as linear combinations of beta densities

$$\beta(y; p, q) = [B(p, q)]^{-1}y^{p-1}(1-y)^{q-1}, \quad 0 < y < 1$$

where $B(p, q)$ is the beta function

$$B(p, q) = \int_0^1 v^{p-1}(1-v)^{q-1}dv.$$

By routine manipulations, we rewrite the elements of $T_0(r, r'; u)$ as

$$[T_0(r, r'; u)]_{i,i'} = \alpha(T; \lambda, r-r'+i'-i) \cdot d(r-r', i'+r-r')\frac{1}{T}\beta\left(1-\frac{u}{T}; r-i; i'-r'+1\right) \quad (8)$$

for $r \geq i+1, i' \geq r' \geq 1, i'+r-r' < K$, and for $r \geq i+1$, $1 \leq r' \leq r$,

$$\begin{aligned}&[T_0(r, r'; u)]_{i,K+r'-r}\\ &= d(r-r', K)\alpha(T-u; \lambda, r-i-1)\lambda\\ &\quad -\sum_{j=0}^{K-r-1}\alpha(T; \lambda, r-i+j)d(r-r', K)\frac{1}{T}\beta\Big(1-\frac{u}{T}; r-i, j+1\Big).\end{aligned}$$

Similarly, we rewrite the components of the vectors $\mathbf{T}_0^0(r; u)$. From (7), we obtain, for $i \leq r-1$, the equation shown at the bottom of the page.

A. Density on (0, T)

The probability density $\psi(\cdot)$ on the interval (0, T) can now be written explicitly in terms of beta densities. By virtue of formula (5), the density $\psi(\cdot)$ is given by

$$\psi(u) = (E^*)^{-1}\sum_{r=1}^{K}\sum_{i=0}^{r-1}\pi_i[\mathbf{T}_0^0(r; u)]_i.$$

We set $i = r-h-1$ and carefully interchange the summations in the resulting formulas. That successively yields (9), shown at the bottom of the next page, for $0 < u < T$.

We see that $\psi(u)$ is expressed as a finite, positive linear combination of beta densities on (0, 1). Correspondingly, the distribution of the delay is given by the same positive linear combination of beta distributions.

Moreover, $\psi(u)$ is of the form

$$\psi(u) = (E^*)^{-1}\sum_{h=0}^{K-1}b(h)\alpha(T-u; \lambda, h)\lambda + (E^*)^{-1}\sum_{h=0}^{K-2}\sum_{j=0}^{K-h-2}c(h, j)\frac{1}{T}\beta\left(1-\frac{u}{T}; h+1, j+1\right) \quad (10)$$

$$\begin{aligned}[\mathbf{T}_0^0(r; u)]_i &= \sum_{\nu=r}^{K}d(\nu, K)\alpha(T-u; \lambda, r-i-1)\lambda\\ &\quad+\sum_{j=0}^{K-r-1}\alpha(T; \lambda, r-i+j)\left[\sum_{\nu=r}^{j+r}d(\nu, j+r)-\sum_{\nu=r}^{K}d(\nu, K)\right]\frac{1}{T}\beta\left(1-\frac{u}{T}; r-i, j+1\right)\end{aligned}$$

where the coefficients $b(h)$ and $c(h,j)$ are given by

$$b(h) = \sum_{r=h+1}^{K} \pi_{r-h-1} \sum_{\nu=r}^{K} d(\nu, K)$$

and

$$\begin{aligned} c(h,j) = {} & \alpha(T; \lambda, h+j+1) \\ & \cdot \sum_{r=h+1}^{K-j-1} \pi_{r-h-1} \left[\sum_{\nu=r}^{j+r} d(\nu, j+r) - \sum_{\nu=r}^{K} d(\nu, K) \right]. \end{aligned}$$

B. Auxiliary Vectors $\mathbf{g}^*(r'; u)$

Next, we give explicit expressions for the vectors $\mathbf{g}^*(r';u)$, for $K \geq r' \geq 1$, for $0 < u < T$. We recall that $\boldsymbol{\pi}^*(r) = [\pi_{r-1}, \pi_{r-2}, \ldots \pi_0]$. If we premultiply the matrix $T_0(r, r'; u)$ by $\boldsymbol{\pi}^*(r)$, we obtain a row vector of dimension $K - r' + 1$ and with component indexes running from K down to r'. The explicit computation of the component with index $K + r' - r$ requires the second formula in (8); that, of all other components, utilizes the first formula.

As $g^*_{i'}(r';u)du$ is the elementary probability that, between $T - u$ and $T - u + du$, an item is admitted, that its position at time T^+ is r', and that the buffer content then equals i', we see that

$$g^*_{i'}(r';u) = \sum_{r=r'}^{K} [\boldsymbol{\pi}^*(r) T_0(r, r'; u)]_{i'}$$

for i' running from K down to r'. We now do a careful accounting of the terms that contribute to each component of $\mathbf{g}^*(r';u)$ and we find that

$$g^*_K(r';u) = \sum_{i=0}^{r'-1} \pi_i [T_0(r', r'; u)]^*_{i,K}$$

and, for $i' = K-1, \ldots, r'$,

$$\begin{aligned} g^*_{i'}(r';u) = {} & [\boldsymbol{\pi}^*(K + r' - i') T_0(K + r' - i', r'; u)]^*_{i'} \\ & + \sum_{r=r'}^{K+r'-i'-1} [\boldsymbol{\pi}^*(r) T_0(r, r'; u)]_{i'} \end{aligned}$$

where the asterisks remind us of which terms are given by the second rather than the first formula in (8). We now do those substitutions and we simplify the resulting formulas. It is convenient to evaluate the vectors and their components with decreasing indices, so, in the analytic expressions that follow, we define the indexes accordingly in (11), shown at the bottom of the page.

$$\begin{aligned} \psi(u) = {} & (E^*)^{-1} \sum_{r=1}^{K} \sum_{h=0}^{r-1} \pi_{r-h-1} \sum_{\nu=r}^{K} d(\nu, K) \alpha(T-u; \lambda, h) \lambda \\ & + (E^*)^{-1} \sum_{r=1}^{K-1} \sum_{h=0}^{r-1} \sum_{j=0}^{K-r-1} \pi_{r-h-1} \alpha(T; \lambda, h+j+1) \left[\sum_{\nu=r}^{j+r} d(\nu, j+r) - \sum_{\nu=r}^{K} d(\nu, K) \right] \frac{1}{T} \beta \left(1 - \frac{u}{T}; h+1, j+1 \right) \\ = {} & (E^*)^{-1} \sum_{h=0}^{K-1} \alpha(T-u; \lambda, h) \lambda \sum_{r=h+1}^{K} \pi_{r-h-1} \sum_{\nu=r}^{K} d(\nu, K) \\ & + (E^*)^{-1} \sum_{h=0}^{K-2} \sum_{j=0}^{K-h-2} \frac{1}{T} \beta \left(1 - \frac{u}{T}; h+1, j+1 \right) \alpha(T; \lambda, h+j+1) \sum_{r=h+1}^{K-j-1} \pi_{r-h-1} \left[\sum_{\nu=r}^{j+r} d(\nu, j+r) - \sum_{\nu=r}^{K} d(\nu, K) \right] \end{aligned} \tag{9}$$

$$\begin{aligned} g^*_K(r';u) = {} & \sum_{h=0}^{r'-1} \pi_{r'-h-1} d(0,K) \alpha(T-u; \lambda, h) \lambda - \sum_{h=0}^{r'-1} \sum_{j=0}^{K-r'-1} \pi_{r'-h-1} d(0,K) \alpha(T; \lambda, h+j+1) \frac{1}{T} \beta \left(1 - \frac{u}{T}; h+1, j+1 \right). \\ g^*_{K-v}(r';u) = {} & \sum_{h=0}^{r'+v-1} \pi_{r'+v-h-1} d(v,K) \alpha(T-u; \lambda, h) \lambda \\ & - \sum_{h=0}^{r'+v-1} \pi_{r'+v-h-1} d(v,K) \sum_{j=0}^{K-v-r'-1} \alpha(T; \lambda, h+j+1) \frac{1}{T} \beta \left(1 - \frac{u}{T}; h+1, j+1 \right) \\ & + \sum_{i=0}^{r'+v-2} \sum_{r=\max(r', i+1)}^{r'+v-1} \pi_i \alpha(T; \lambda, r-i+K-v-r') d(r-r', K-v+r-r') \frac{1}{T} \beta \left(1 - \frac{u}{T}; r-i, K-v-r'+1 \right), \\ & \text{for } 1 \leq v \leq K - r' \end{aligned} \tag{11}$$

In this last sum, we make the change of indexes $r-i-1=h$. The result is most conveniently written as the sum of two terms, as shown in (12) at the bottom of the page. So, finally, for $1 \le v \le K-r'$, $g^*_{K-v}(r';u)$ is given by (13), shown at the bottom of the page.

We see that, for $1 \le r' \le K$, the vector $\mathbf{g}^*(r';u)$ of dimension $K-r'+1$, is of the form

$$\mathbf{g}^*(r';u) = \sum_{h=0}^{K+r'-1} \mathbf{b}^*(r';h)\alpha(T-u;\lambda,h)\lambda + \sum_{h=0}^{K+r'-1}\sum_{j=0}^{K-r'-1} \mathbf{c}^*(r';h,j)\frac{1}{T}\beta\Big(1-\frac{u}{T};h+1,j+1\Big) \tag{14}$$

where the coefficient vectors $\mathbf{b}^*(r';h)$ and $\mathbf{c}^*(r';h,j)$ are given by

$$\begin{aligned}
b^*_K(r';h) &= \pi_{r'-h-1}d(0,K), \text{ for } 0 \le h \le r'-1,\\
b^*_K(r';h) &= 0, \text{ for } r' \le h \le K+r'-1,\\
c^*_K(r';h,j) &= -\pi_{r'-h-1}\alpha(T;\lambda,h+j+1)d(0,K),\\
&\qquad \text{for } 0 \le h \le r'-1,\\
c^*_K(r';h,j) &= 0, \text{ for } r' \le h \le K+r'-1
\end{aligned}$$

and, for $1 \le v \le K-r'$, we have the equations shown at the bottom of the page. These coefficient vectors are computed once and stored for use in the recursive computation of the density $\psi(\cdot)$ on the subsequent slots.

VII. Delay Distribution—The Subsequent Slots

The accounting of the transactions in the buffer content and in the position of the marked item during the subsequent slots is carried out by means of the partitioned matrix T^* defined in (4). We recall that the elements of $T(r_1,r_2)$ are the conditional probabilities that, given that at the start of the slot there are i_1 items in the buffer with the marked item in position r_1, by the end of the slot there are i_2 items in the buffer and the marked item has moved to position r_2, with $r_1 \ge r_2 \ge 1$. The components of $\mathbf{T}^0(r_1)$ are the conditional probabilities that, given that at the start of the slot there are i_1 items in the buffer with the marked item in position r_1, the marked item is removed at the end of that slot.

The element $[T(r_1,r_2)]_{i_1,i_2}$ is given by

$$\begin{aligned}
&[T(r_1,r_2)]_{i_1,i_2}\\
&\quad = \alpha(T;\lambda,i_2+r_1-r_2-i_1)d(r_1-r_2,i_2+r_1-r_2)
\end{aligned} \tag{15}$$

$$\left\{\sum_{h=0}^{r'-1}\sum_{i=r'-h-1}^{r'+v-h-2} + \sum_{h=r'}^{r'+v-2}\sum_{i=0}^{r'+v-h-2}\right\}\pi_i\alpha(T;\lambda,h+K-v-r'+1)d(i+h-r'+1,K-v+i+h-r'+1) \cdot\frac{1}{T}\beta\left(1-\frac{u}{T};h+1,K-v-r'+1\right) \tag{12}$$

$$\begin{aligned}
&g^*_{K-v}(r';u)\\
&= \sum_{h=0}^{r'+v-1}\pi_{r'+v-h-1}d(v,K)\alpha(T-u;\lambda,h)\lambda - \sum_{h=0}^{r'+v-1}\pi_{r'+v-h-1}d(v,K)\sum_{j=0}^{K-v-r'-1}\alpha(T;\lambda,h+j+1)\frac{1}{T}\beta\Big(1-\frac{u}{T};h+1,j+1\Big)\\
&\quad + \left\{\sum_{h=0}^{r'-1}\sum_{i=r'-h-1}^{r'+v-h-2} + \sum_{h=r'}^{r'+v-2}\sum_{i=0}^{r'+v-h-2}\right\}\pi_i\alpha(T;\lambda,h+K-v-r'+1)\\
&\quad \cdot d(i+h-r'+1,K-v+i+h-r'+1)\frac{1}{T}\beta\left(1-\frac{u}{T};h+1,K-v-r'+1\right)
\end{aligned} \tag{13}$$

$$\begin{aligned}
c^*_{K-v}(r';h,j) &= -\pi_{r'+v-h-1}\alpha(T;\lambda,h+j+1)d(v,K), \quad \text{for } 0 \le h \le r'+v-1,\ 0 \le j \le K-v-r'-1\\
c^*_{K-v}(r';h,j) &= \sum_{i=r'-h-1}^{r'+v-h-2}\pi_i\alpha(T;\lambda,h+K-v-r'+1)d(i+h-r'+1,K-v+i+h-r'+1)\\
&\qquad \text{for } 0 \le h \le r'-1,\ j=K-v-r'\\
c^*_{K-v}(r';h,j) &= \sum_{i=0}^{r'+v-h-2}\pi_i\alpha(T;\lambda,h+K-v-r'+1)d(i+h-r'+1,K-v+i+h-r'+1)\\
&\qquad \text{for } r' \le h \le r'+v-2,\ j=K-v-r'\\
c^*_{K-v}(r';h,j) &= 0, \quad \text{for } h=r'+v-1,\ j=K-v-r', \text{ or for } r'+v \le h \le K+r'-1,\\
&\qquad \text{or for } K-v-r' < j \le K-r'-1
\end{aligned}$$

for $i_2 \geq i_1 - r_1 + r_2$, $i_2 < K - r_1 + r_2$. That is the case where, during the slot, the buffer does not fill up. For future reference, let us call that the form P_C.

The element $[T(r_1, r_2)]_{i_1, K-r_1+r_2}$ is given by s

$$[T(r_1, r_2)]_{i_1, K-r_1+r_2} = \left[1 - \sum_{j=0}^{K-i_1-1} \alpha(T; \lambda, j)\right] d(r_1 - r_2, K). \tag{16}$$

It corresponds to $i_2 = K - r_1 + r_2$ and to the buffer filling up during the slot. We call that the form P_D.

For $1 \leq i_1 \leq K$, the components of the vector $\mathbf{T}^0(r_1)$ are given by

$$\sum_{j=0}^{K-i_1-1} \alpha(T; \lambda, j) \sum_{\nu=r_1}^{i_1+j} d(\nu, i_1 + j) + \left[1 - \sum_{j=0}^{K-i_1-1} \alpha(T; \lambda, j)\right] \sum_{\nu=r_1}^{K} d(\nu, K). \tag{17}$$

The matrices $T(r_1, r_2)$ have further special structure that we display for the representative values $K = 6, r_1 = 4$, and $r_2 = 4$, 3, 2, 1. The symbols P_C or P_D indicate which of the formulas (14) or (15) are to be used for the specific indices

$$T(4,4) = \begin{array}{c|ccc} i_1 \backslash i_2 & 6 & 5 & 4 \\ \hline 6 & P_D & 0 & 0 \\ 5 & P_D & P_C & 0 \\ 4 & P_D & P_C & P_C \end{array}$$

$$T(4,3) = \begin{array}{c|cccc} i_1 \backslash i_2 & 6 & 5 & 4 & 3 \\ \hline 6 & 0 & P_D & 0 & 0 \\ 5 & 0 & P_D & P_C & 0 \\ 4 & 0 & P_D & P_C & P_C \end{array}$$

$$T(4,2) = \begin{array}{c|ccccc} i_1 \backslash i_2 & 6 & 5 & 4 & 3 & 2 \\ \hline 6 & 0 & 0 & P_D & 0 & 0 \\ 5 & 0 & 0 & P_D & P_C & 0 \\ 4 & 0 & 0 & P_D & P_C & P_C \end{array}$$

$$T(4,1) = \begin{array}{c|cccccc} i_1 \backslash i_2 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 6 & 0 & 0 & 0 & P_D & 0 & 0 \\ 5 & 0 & 0 & 0 & P_D & P_C & 0 \\ 4 & 0 & 0 & 0 & P_D & P_C & P_C \end{array}.$$

Lemma 1: T^* is a stochastic matrix.

Proof: For $K \geq i_1 \geq r_1$, the components of the row sum vector $T(r_1, r_2)\mathbf{e}$ are given by

$$\begin{aligned} [T1, r_2)\mathbf{e}]_{i_1} &= \sum_{i_2=i_1-r_1+r_2}^{K-r_1+r_2-1} \alpha(T; \lambda, i_2+r_1-r_2-i_1) \\ &\quad \times d(r_1-r_2, i_2+r_1-r_2) \\ &\quad + \sum_{j=K-i_1}^{\infty} \alpha(T; \lambda, j) d(r_1 - r_2, K) \\ &= \sum_{j=0}^{K-i_1-1} \alpha(T; \lambda, j) d(r_1 - r_2, j + i_1) \\ &\quad + \sum_{j=K-i_1}^{\infty} \alpha(T; \lambda, j) d(r_1 - r_2, K). \end{aligned}$$

Now we sum these quantities over r_2 from one to r_1. Finally, we add the term in (17) and, by (1), we obtain an expression that is clearly equal to one. ■

To complete the argument, we verify that $\psi(\cdot)$ is a valid probability density. We integrate the expressions in (5) and (6) and sum over k and check that we thus obtain the quantity E^*. We specifically have that

$$\begin{aligned} \sum_{k=0}^{\infty} \int_{kT}^{kT+T} \psi(u) du &= \sum_{k=0}^{\infty} \int_0^T \psi(u + kT) du \\ &= (E^*)^{-1} \left[\sum_{r=1}^{K} \boldsymbol{\pi}^*(r) \int_0^T \mathbf{T}_0^0(r; u) du \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \int_0^T \boldsymbol{\gamma}^*(u) du \, \widetilde{T}^{k-1} \widetilde{\mathbf{T}}^0 \right]. \end{aligned}$$

However, since T^* is stochastic, it follows that

$$(I - \widetilde{T})^{-1} \widetilde{\mathbf{T}}^0 = \mathbf{e}.$$

It remains to show that

$$(E^*)^{-1} \int_0^T \left[\sum_{r=1}^{K} \boldsymbol{\pi}^*(r) \mathbf{T}_0^0(r; u) + \boldsymbol{\gamma}^*(u) \mathbf{e}\right] du = 1.$$

That can be easily shown without using the fairly involved expressions for $\boldsymbol{\gamma}^*(u)$. From the definition of the matrices $T_0(r, r'; u)$ and the vectors $\mathbf{T}_0^0(r; u)$, we readily see that, for $0 \leq i \leq r - 1$,

$$\sum_{r'=1}^{r} \sum_{i'=r'}^{K} [T_0(r, r'; u)]_{i,i'} + [\mathbf{T}_0^0(r; u)]_i = \alpha(u; \lambda, r - i - 1)\lambda.$$

These quantities need to be multiplied by π_i, summed over i with $0 \leq i \leq r - 1$, and then summed over r with $1 \leq r \leq K$. We remember that, for $0 \leq i \leq r - 1$,

$$\int_0^T \alpha(u; \lambda, r - i - 1)\lambda du = 1 - \sum_{\nu=0}^{r-i-1} \alpha(T; \lambda, \nu)$$

so that

$$\begin{aligned} (E^*)^{-1} \sum_{r=1}^{K} \sum_{i=0}^{r-1} \pi_i \left[1 - \sum_{\nu=0}^{r-i-1} \alpha(T; \lambda, \nu)\right] &= (E^*)^{-1} \sum_{i=0}^{K-1} \pi_i \sum_{h=0}^{K-i-1} \left[1 - \sum_{\nu=0}^{h} \alpha(T; \lambda, \nu)\right] \\ &= (E^*)^{-1} \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{h=0}^{K-i-1} \sum_{\nu=0}^{h} \alpha(T; \lambda, \nu)\right] \\ &= (E^*)^{-1} \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{\nu=0}^{K-i-1} \alpha(T; \lambda, \nu)(K - i - \nu)\right] \\ &= 1. \end{aligned}$$

A. *Final Recursive Algorithm*

To describe the final recursive algorithm concisely, it is convenient to partition the vector $\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0 = \mathbf{w}^*(k)$, $k \geq 1$, into vectors $\mathbf{w}(r', k)$, of dimension $K - r' + 1$, for $K \geq r' \geq 1$. The recursive computation of the vectors $\mathbf{w}^*(k)$ is obvious.

We then readily see that, on the interval $(kT, kT + T)$, the density $\psi(\cdot)$ is given by (18), shown at the bottom of the page, which is again a finite, positive linear combination of beta densities on (0, 1). As the coefficients in (14) have been computed, those in (18) are readily evaluated for successive k. Once those coefficients are stored, the density is readily computed at a set of equidistant points in $(kT, kT + T)$ and can be plotted.

Upon integration in (10) and (18), we find that the values $F(kT)$ at the points kT of the probability distribution of the delay are given by

$$F(kT) = \sum_{\nu=1}^{k} \phi_\nu,$$

where

$$\begin{aligned}\phi_1 &= (E^*)^{-1} \sum_{h=0}^{K-1} b(h) \left[1 - \sum_{v=0}^{h} \alpha(T; \lambda, v)\right] \\ &\quad + (E^*)^{-1} \sum_{h=0}^{K-2} \sum_{j=0}^{K-h-2} c(h, j)\end{aligned}$$

and for $\nu \geq 2$

$$\begin{aligned}\phi_\nu &= (E^*)^{-1} \\ &\quad \cdot \sum_{h=0}^{K+r'-1} \left\{\sum_{r'=1}^{K} \mathbf{b}^*(r'; h)\mathbf{w}(r', \nu-1)\right\} \left[1 - \sum_{v=0}^{h} \alpha(T; \lambda, v)\right] \\ &\quad + (E^*)^{-1} \sum_{h=0}^{K+r'-1} \sum_{j=0}^{K-r'-1} \left\{\sum_{r'=1}^{K} \mathbf{c}^*(r'; h, j)\mathbf{w}(r', \nu-1)\right\}.\end{aligned}$$

The distribution $F(x)$ at x with $kT < x < kT + T$ is given by

$$F(x) = F(kT) + \int_{kT}^{x} \psi(u)du \qquad (19)$$

and can be evaluated along with the density $\psi(\cdot)$ on that interval simply by substituting the corresponding beta distributions for the beta densities.

We note that, for $\nu \geq 2$, ϕ_ν is the probability that the marked item departs at the end of $(\nu - 1)$ after the interval in which it arrives. ϕ_1 is the probability that it departs at the earliest possible time T.

Since $\psi(\cdot)$ is a finite, positive linear combination of beta densities, the computation of the distribution $F(\cdot)$ is of polynomial complexity.

VIII. MEAN WAITING TIME

Since Stidham's rigorous proof in [20], we know that the classical Little's formula $L = \lambda W$ holds generally for various queueing systems in equilibrium. We recall in such a deterministic relation that λ is the steady-state arrival rate, W is the mean waiting time, and L is the time-averaged number of customers in the system. In our context, it follows that, given E^*/T, which is the steady-state rate at which customers are admitted, and letting Q be the time-averaged number of customers in the buffer, Little's formula gives the mean delay of an arbitrary admitted customer as $W = QT/E^*$.

Theorem 2: The time-averaged number Q of customers in the buffer is shown in (20) at the bottom of the page.

Proof: Let $\pi_j(t)$, $0 \leq j \leq K$, $0 < t < T$, be the steady-state probability that the number of items in the buffer at a random time t within a time slot is j. Using π_i, $0 \leq i \leq K$, we can express $\pi_j(t)$ as

$$\pi_j(t) = \sum_{i=0}^{j} \pi_i \alpha(t; \lambda, j - i), \quad \text{for } 0 \leq j < K,$$

and

$$\pi_K(t) = \sum_{i=0}^{K} \pi_i \left[1 - \sum_{l=0}^{K-i-1} \alpha(t; \lambda, l)\right].$$

$$\begin{aligned}\psi(u) &= (E^*)^{-1} \sum_{r'=1}^{K} \mathbf{g}^*(r'; u - kT)\mathbf{w}(r', k) \\ &= (E^*)^{-1} \sum_{h=0}^{K+r'-1} \left\{\sum_{r'=1}^{K} \mathbf{b}^*(r'; h)\mathbf{w}(r', k)\right\} \alpha(kT + T - u; \lambda, h)\lambda \\ &\quad + (E^*)^{-1} \sum_{h=0}^{K+r'-1} \sum_{j=0}^{K-r'-1} \left\{\sum_{r'=1}^{K} \mathbf{c}^*(r'; h, j)\mathbf{w}(r', k)\right\} \frac{1}{T}\beta\left(k + 1 - \frac{u}{T}; h + 1, j + 1\right)\end{aligned} \qquad (18)$$

$$Q = K - \frac{1}{2\lambda T} \sum_{i=0}^{K-1} \pi_i \left[(K - i)(K - i + 1) - \sum_{\nu=0}^{K-i-1} \alpha(T; \lambda, \nu)(K - i - \nu)(K - i - \nu + 1)\right] \qquad (20)$$

The average number of items in the buffer at time t is then given by $\sum_{j=0}^{K} j\pi_j(t)$. After some algebraic manipulations, the expected value equals

$$K - \sum_{i=0}^{K-1} \pi_i \sum_{l=0}^{K-i-1} (K-i-l)\alpha(t;\lambda,l). \tag{21}$$

To calculate the time-averaged number Q of items in the buffer, we integrate (21) with respect to t over the time slot and divide the deduced term by T, thus giving (22), shown at the bottom of the page. ■

There is no simple analytic expression for the variance of the delay, but the quantity of the second moment is a byproduct of the computation of the distribution $F(\cdot)$.

IX. NUMERICAL EXAMPLES

After routine numerical computations of the steady-state probability vector $[\pi_0, \pi_1, \dots, \pi_K]$ and of the quantity E^*, we implement the recursive scheme for the computation of the coefficient sequences in (10) and (18). The recurrence is initiated by the coefficients for the first slot (0, T). The density $\psi(u)$ on that interval is evaluated by implementing (9). For the subsequent slots, we perform a matrix multiplication to form the required vector $\mathbf{w}^*(k) = \widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0$, and we evaluate the required coefficient series. From these, the density $\psi(u)$ is readily computed on each interval. Inside the recursive loop, we also compute the probability distribution $F(\cdot)$ by using (19) and calling a library routine for the incomplete beta ratio. Particularly, since p and q are both positive integers, the incomplete beta ratio

$$I_x(y;p,q) = \int_0^x \beta(y;p,q)dy$$

can be most efficiently calculated by applying the following recurrence relations [21]:

$$\begin{cases} I_x(y;1,1) = x \\ I_x(y;p,1) = x^p \\ I_x(y;1,q) = 1-(1-x)^q \\ I_x(y;p,q) = xI_x(y;p-1,q) + (1-x)I_x(y;p,q-1). \end{cases}$$

We remember that $\alpha(u;\lambda,n)\lambda$ is the Erlang density function of order n, and $\int_0^x \alpha(u;\lambda,n)\lambda du$ is evaluated by a simple summation of the corresponding Poisson probabilities. Thus, getting $F(\cdot)$ along with the density only requires a modicum of additional computation.

Our simulations were initiated by variates from the probability vector $[\pi_0, \pi_1, \dots, \pi_K]$. Each run consisted of one million time slots from which a histogram, the empirical distribution function, and the mean of the delay were estimated. For the histogram, the class width was set to one tenth of the slot length T.

In our numerical example, one selected from among many, we consider a buffer of size $K = 70$, respectively, under light, medium, and high loads ($\lambda T = 1$, 60, 600). For the sake of our example, the parameters $d(i,j)$ were specified as follows. For j even, we formed the $j+1$ integers $1,\ 2, \dots, j/2+1, j/2, \dots, 2,\ 1$, divided by their sum and identified $d(i,j)$, for $0 \le i \le j$, with the corresponding ratio. For j odd, we wrote the $j+1$ integers $1,\ 2, \dots, (j+1)/2, (j+1)/2, \dots, 2,\ 1$, divided by their sum and similarly identified the $d(i,j)$. That avoids displaying a 71×71 matrix to complete the specification of our example.

Graphs of the corresponding delay densities and distributions, obtained by computation and simulation, are shown in Fig. 1. CPU times for computation on a SunFire 280R machine are presented. Results of the mean of the waiting time are reported in Table I.

The apparent approach to a discrete density under the high load ($\lambda T = 600$) is to be expected. When the arrival rate is very high, the occasional job that is admitted will arrive very early in the slot and will wait (approximately) for a duration that is a multiple of T.

X. SPECIAL CASES

Two special cases of our TDMA model with constant service were studied in [1], [15], and [16]. We shall see that the methodology we used to derive the waiting time density of an arbitrary admitted customer readily gives rise to straightforward solutions for the constant service models.

A. Constant Unit Service

Let $dE^*(u)$ again be the expected number of customers admitted during (0, T) whose waiting time lies between u and

$$\begin{aligned} Q &= K - \frac{1}{\lambda T}\sum_{i=0}^{K-1}\pi_i \sum_{l=0}^{K-i-1}(K-i-l)\int_0^T \alpha(t;\lambda,l)\,\lambda dt \\ &= K - \frac{1}{\lambda T}\sum_{i=0}^{K-1}\pi_i \sum_{l=0}^{K-i-1}(K-i-l)\left[1-\sum_{\nu=0}^{l}\alpha(T;\lambda,\nu)\right] \\ &= K - \frac{1}{\lambda T}\sum_{i=0}^{K-1}\pi_i \left[\frac{(K-i)(K-i+1)}{2} - \sum_{l=0}^{K-i-1}(K-i-l)\sum_{\nu=0}^{l}\alpha(T;\lambda,\nu)\right] \\ &= K - \frac{1}{2\lambda T}\sum_{i=0}^{K-1}\pi_i \left[(K-i)(K-i+1) - \sum_{\nu=0}^{K-i-1}\alpha(T;\lambda,\nu)(K-i-\nu)(K-i-\nu+1)\right] \end{aligned} \tag{22}$$

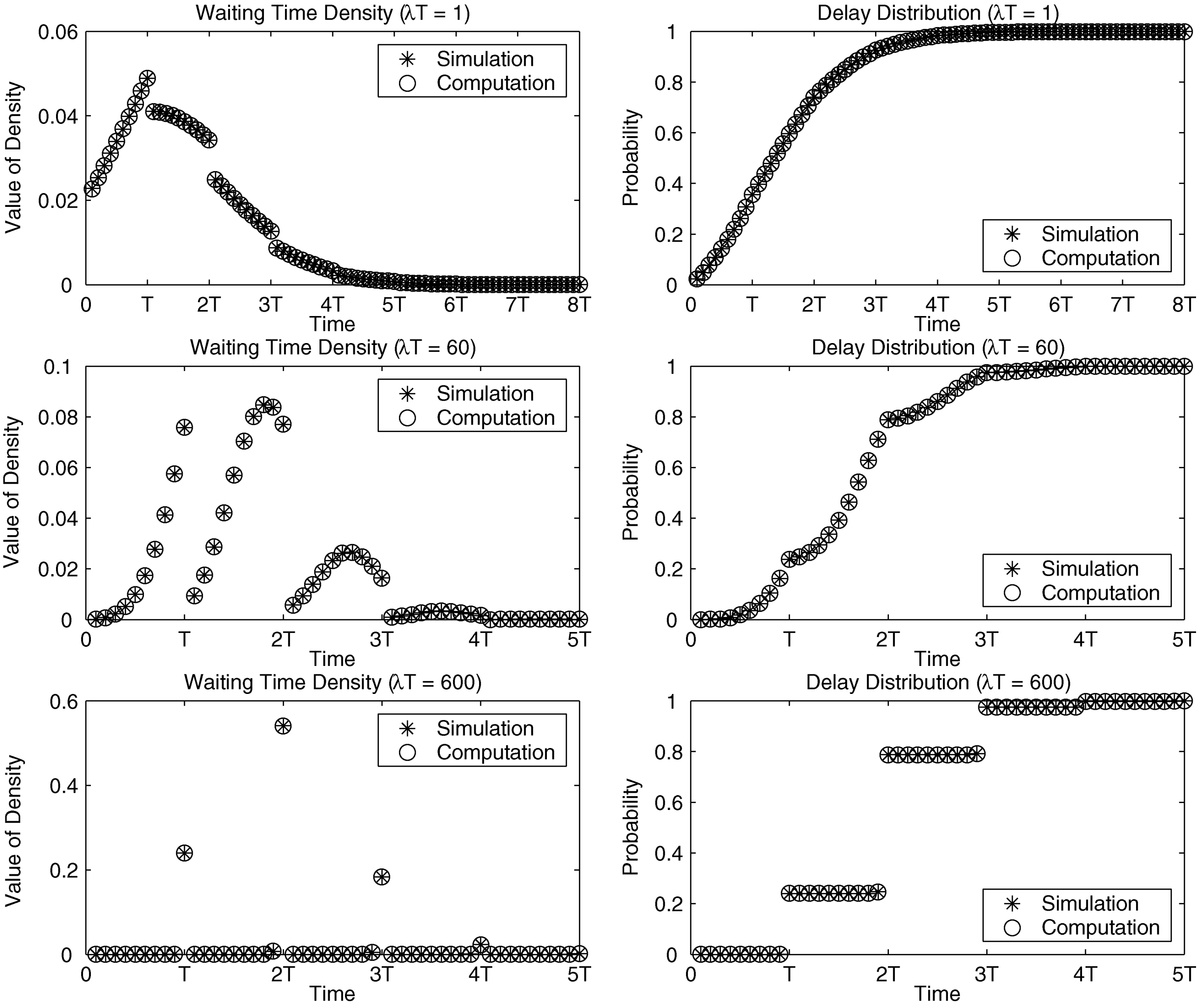

Fig. 1. Computation and simulation results for the waiting time density and delay distribution of the TDMA model. The computation time is 116 s for $\lambda T = 1$, 70 s for $\lambda T = 60$, and 72 s for $\lambda T = 600$ on a SunFire 280R machine.

TABLE I
VALUES OF THE MEAN WAITING TIME (IN T)

	$\lambda T = 1$	$\lambda T = 60$	$\lambda T = 600$
Simulation	1.4998	1.6564	1.9648
Computation	1.5000	1.6565	1.9649

$u + du$. Since the buffer is of finite capacity K, we must have $0 < u < KT$. Then, on the interval $(kT, kT + T)$, $0 \leq k \leq K - 1$, we have

$$dE^*(u) = \sum_{i=0}^{k} \pi_i \alpha(kT + T - u; \lambda, k - i)\lambda du \tag{23}$$

so that, on the interval $(kT, kT + T)$, the density $\psi(\cdot)$ is given by

$$\psi(u) = (E^*)^{-1} \sum_{i=0}^{k} \pi_i \alpha(kT + T - u; \lambda, k - i)\lambda du.$$

The iterative procedure to find the steady-state probabilities π_i, $0 \leq i \leq K - 1$, in this special case was discussed in [1] and [2]. We compute the expected number E^* of items admitted during $(0, T)$ by using (2). To verify that $\psi(\cdot)$ is a valid probability density, we integrate (23) and sum over all allowable values of k and so obtain the quantity E^*.

B. Constant Batch Service

Following the methodology we used in this paper, we readily see that the general expression for $dE^*(u)$ can be derived as follows. Let $M = \lceil K/N \rceil$, where $\lceil x \rceil$ is the smallest integer larger than or equal to x. We recall that, with that service process, if there are N or more customers in the buffer at the end of a time slot, N items are removed; otherwise, all items in the buffer will be removed. Thus, the waiting time of an arbitrary admitted customer must be bounded by MT. We then have, on the interval $(kT, kT + T)$, $0 \leq k \leq M - 3$

$$dE^*(u) = \sum_{r=Nk+1}^{Nk+N} \sum_{i=0}^{r-1} \pi_i \alpha(kT + T - u; \lambda, r - i - 1)\lambda du$$

$$dE^*(u) = \left\{ \sum_{r=NM-2N+1}^{K-N} \sum_{i=0}^{r-1} + \sum_{r=K-N+1}^{NM-N} \sum_{i=0}^{K-N} \right\} \pi_i \alpha(MT - T - u; \lambda, r - i - 1)\lambda du$$

and on the interval $(MT - 2T, MT - T)$

$$dE^*(u) = \sum_{r=NM-2N+1}^{NM-N} \sum_{i=0}^{\min(r-1,K-N)} \pi_i \alpha(MT - T - u; \lambda, r - i - 1)\lambda du \tag{24}$$

and on the interval $(MT - T, MT)$

$$dE^*(u) = \sum_{r=NM-N+1}^{K} \sum_{i=0}^{K-N} \pi_i \alpha(MT - u; \lambda, r - i - 1)\lambda du.$$

The $\min$ operator in (24) can be further removed, since, if $K = NM$, we have

$$\begin{aligned} NM - N - 1 &= K - N - 1 \\ &< K - N. \end{aligned}$$

Thus, (24) is simplified as

$$dE^*(u) = \sum_{r=NM-2N+1}^{NM-N} \sum_{i=0}^{r-1} \pi_i \alpha(MT - T - u; \lambda, r - i - 1)\lambda du.$$

However, if $K < NM$, we have

$$\begin{aligned} NM - N - 1 &> K - N - 1 \\ &\geq K - N. \end{aligned}$$

Then, (24) is conveniently rewritten as shown in the equation at the top of the page.

The probability transition matrix $\{P_{ij}\}$ of the Markov chain $\{J_k\}$ in this special case is given by

$$P_{ij} = \begin{cases} \sum_{\nu=0}^{N-i} \alpha(T; \lambda, \nu), & \text{for } j = 0, i \leq N \\ \alpha(T; \lambda, N + j - i), & \text{for } 0 < j < K - N, i \leq N + j \\ 1 - \sum_{\nu=0}^{K-i-1} \alpha(T; \lambda, \nu), & \text{for } j = K - N \\ 0, & \text{elsewhere} \end{cases}$$

for $0 \leq i, j \leq K - N$. We compute the steady-state probabilities π_i, $0 \leq i \leq K - N$, of that Markov chain and the quantity E^* by summing π_i over i with $0 \leq i \leq K - N$ in (2). The validity of the waiting time density obtained from $dE^*(u)/E^*$ is again easily verified.

XI. CONCLUSION AND FUTURE WORK

We have analyzed a TDMA model with a finite buffer and state-dependent service. We have derived exact analytical results which have been confirmed by simulation. Extensions of the model to more general arrival processes, e.g., the Markovian arrival process [22], are currently under study.

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