COMP4161
Advanced Topics in Software Verification

Gerwin Klein, Miki Tanaka, Johannes Åman Pohjola, Robert Sison

T3/2023

Binary Search (`java.util.Arrays`)

```
1:     public static int binarySearch(int[] a, int key) {
2:         int low = 0;
3:         int high = a.length - 1;
4:
5:         while (low <= high) {
6:             int mid = (low + high) / 2;
7:             int midVal = a[mid];
8:
9:             if (midVal < key)
10:                 low = mid + 1
11:             else if (midVal > key)
12:                 high = mid - 1;
13:             else
14:                 return mid; // key found
15:         }
16:         return -(low + 1);  // key not found.
17:     }
```

```
6:             int mid = (low + high) / 2;
```

http://googleresearch.blogspot.com/2006/06/extra-extra-read-all-about-it-nearly.html

Organisatorials

When	Wed	16:00 – 18:00
	Fri	15:00 – 17:00

`http://www.cse.unsw.edu.au/~cs4161/`

About us: Proofcraft and Trustworthy Systems

- **TS** (Trustworthy Systems) is a research group at UNSW
  - track record of research and real world impact in verified software
  - biggest achievement: formal verification of **seL4**
- **Proofcraft** is a new company
  - from former leaders of TS
  - providing services in software verification
- **seL4** is an operating microkernel used around the world in critical systems
  - with a proof of functional correctness and security: Security $\leftrightarrow$ Isabelle/HOL model $\leftrightarrow$ Haskell model $\leftrightarrow$ C code $\leftrightarrow$ Binary
  - 10 000 LOC / more than 1 million lines of proof
  - Open source, `http://sel4.systems`

We are always embarking on exciting new projects. Talk to us!

What you will learn

- ➜ how to use a theorem prover
- ➜ background, how it works
- ➜ how to prove and specify
- ➜ how to reason about programs

Health Warning

Theorem Proving is addictive

Prerequisites

This is an advanced course. It assumes knowledge in

- ➜ Functional programming
- ➜ First-order formal logic

The following program should make sense to you:

```
map f []       =  []
map f (x:xs)   =  f x : map f xs
```

You should be able to read and understand this formula:

$$\exists x.\ (P(x) \longrightarrow \forall x.\ P(x))$$

Content — Using Theorem Provers

	Rough timeline
➜ Foundations & Principles	
• Intro, Lambda calculus, natural deduction	[1,2]
• Higher Order Logic, Isar (part 1)	[2,3[a]]
• Term rewriting	[3,4]
➜ Proof & Specification Techniques	
• Inductively defined sets, rule induction	[4,5]
• Datatype induction, primitive recursion	[5,7]
• General recursive functions, termination proofs	[7[b]]
• Proof automation, Isar (part 2)	[8]
• Hoare logic, proofs about programs, invariants	[8,9]
• C verification	[9,10]
• Practice, questions, exam prep	[10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

To have a chance at succeeding

you should:

- ➜ attend lectures
- ➜ try Isabelle early
- ➜ redo all the demos alone
- ➜ try the exercises/homework we give, when we do give some
- ➜ **DO NOT CHEAT**
  - Assignments and exams are take-home. This does NOT mean you can work in groups. Each submission is personal.
  - For more info, see Plagiarism Policy[a]

[a] `https://student.unsw.edu.au/plagiarism`

Credits

some material (in using-theorem-provers part) shamelessly stolen from



Tobias Nipkow, Larry Paulson, Markus Wenzel


David Basin, Burkhardt Wolff

Don't blame them, errors are ours

What is a formal proof?

A derivation in a formal calculus

Example: $A \wedge B \longrightarrow B \wedge A$ derivable in the following system

Rules: $\frac{X \in S}{S \vdash X}$ (assumption) $\frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}$ (impI)

$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}$ (conjI) $\frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}$ (conjE)

Proof:

1.	$\{A, B\} \vdash B$	(by assumption)
2.	$\{A, B\} \vdash A$	(by assumption)
3.	$\{A, B\} \vdash B \wedge A$	(by conjI with 1 and 2)
4.	$\{A \wedge B\} \vdash B \wedge A$	(by conjE with 3)
5.	$\{\} \vdash A \wedge B \longrightarrow B \wedge A$	(by impI with 4)

What is a theorem prover?

Implementation of a formal logic on a computer.

- ➜ fully automated (propositional logic)
- ➜ automated, but not necessarily terminating (first order logic)
- ➜ with automation, but mainly interactive (higher order logic)

There are other (algorithmic) verification tools:

- ➜ model checking, static analysis, ...
- ➜ See COMP3153: Algorithmic Verification

Why theorem proving?

- ➜ Analyse systems/programs thoroughly
- ➜ Findi design and specification errors early
- ➜ High assurance: mathematical, machine checked proofs
- ➜ It's not always easy
- ➜ It's fun!

Main theorem proving system for this course

Isabelle

- ➜ used at TS for research, teaching and proof engineering

What is Isabelle?

A generic interactive proof assistant

- **generic:**
  not specialised to one particular logic
  (two large developments: HOL and ZF, will mainly use HOL)
- **interactive:**
  more than just yes/no, you can interactively guide the system
- **proof assistant:**
  helps to explore, find, and maintain proofs

If I prove it on the computer, it is correct, right?

If I prove it on the computer, it is correct, right?

No, because:

① hardware could be faulty
② operating system could be faulty
③ implementation runtime system could be faulty
④ compiler could be faulty
⑤ implementation could be
⑥ logic could be inconsistent
⑦ theorem could mean something else

If I prove it on the computer, it is correct, right?

No, but:
probability for

- ➜ OS and H/W issues reduced by using different systems
- ➜ runtime/compiler bugs reduced by using different compilers
- ➜ faulty implementation reduced by having the right prover architecture
- ➜ inconsistent logic reduced by implementing and analysing it
- ➜ wrong theorem reduced by expressive/intuitive logics

No guarantees, but assurance immensly higher than manual proof

If I prove it on the computer, it is correct, right?

Soundness architectures

careful implementation	PVS ACL2
LCF approach, small proof kernel	HOL4 Isabelle HOL-light
explicit proofs + proof checker	Coq Lean Twelf Isabelle HOL4 Agda

Meta Logic

Meta language:
The language used to talk about another language.

Examples:
English in a Spanish class, English in an English class

Meta logic:
The logic used to formalize another logic

Example:
Mathematics used to formalize derivations in formal logic

Meta Logic – Example

Syntax:

Formulae: $F ::= V \mid F \longrightarrow F \mid F \wedge F \mid False$

$V ::= [A - Z]$

Judgement: $S \vdash X$ X a formula, S a set of formulae

logic / meta logic

$$\frac{X \in S}{S \vdash X} \qquad \frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}$$

$$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y} \qquad \frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}$$

Isabelle's Meta Logic

$$\bigwedge \qquad \Longrightarrow \qquad \lambda$$

$\bigwedge$

Syntax: $\bigwedge x.\ F$ (F another meta logic formula)
in ASCII: `!!x. F`

➜ this is the meta-logic universal quantifier
➜ example and more later

$\Longrightarrow$

Syntax: $A \Longrightarrow B$ (A, B other meta logic formulae)
in ASCII: `A ==> B`

Binds to the right:

$$A \Longrightarrow B \Longrightarrow C \quad = \quad A \Longrightarrow (B \Longrightarrow C)$$

Abbreviation:

$$[\![A; B]\!] \Longrightarrow C \quad = \quad A \Longrightarrow B \Longrightarrow C$$

- ➜ read: A and B implies C
- ➜ used to write down rules, theorems, and proof states

Example: a theorem

mathematics: if $x < 0$ and $y < 0$, then $x + y < 0$

formal logic: $\vdash \; x < 0 \wedge y < 0 \longrightarrow x + y < 0$

variation: $x < 0; y < 0 \;\vdash\; x + y < 0$

Isabelle: **lemma** "$x < 0 \wedge y < 0 \longrightarrow x + y < 0$"

variation: **lemma** "$[\![x < 0; y < 0]\!] \Longrightarrow x + y < 0$"

variation: **lemma**

assumes "$x < 0$" and "$y < 0$" shows "$x + y < 0$"

Example: a rule

logic: $$\frac{X \quad Y}{X \wedge Y}$$

variation: $$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}$$

Isabelle: $[\![X; Y]\!] \Longrightarrow X \wedge Y$

Example: a rule with nested implication

logic:

$$\frac{X \vee Y \quad \begin{array}{c} X \\ \vdots \\ Z \end{array} \quad \begin{array}{c} Y \\ \vdots \\ Z \end{array}}{Z}$$

variation:

$$\frac{S \cup \{X\} \vdash Z \quad S \cup \{Y\} \vdash Z}{S \cup \{X \vee Y\} \vdash Z}$$

Isabelle: $[\![X \vee Y; X \Longrightarrow Z; Y \Longrightarrow Z]\!] \Longrightarrow Z$

λ

Syntax: $\lambda x.\ F$ (F another meta logic formula)
in ASCII: `%x. F`

- ➜ lambda abstraction
- ➜ used to represent functions
- ➜ used to encode bound variables
- ➜ more about this soon

Enough Theory!

Getting started with Isabelle

System Architecture

Prover IDE (jEdit) – user interface

HOL, ZF – object-logics

Isabelle – generic, interactive theorem prover

Standard ML – logic implemented as ADT

User can access all layers!

System Requirements

- ➜ **Linux**, **Windows**, or **MacOS X (10.8 +)**
- ➜ **Standard ML** (PolyML implementation)
- ➜ **Java** (for jEdit)

Premade packages for Linux, Mac, and Windows + info on:
`http://mirror.cse.unsw.edu.au/pub/isabelle/`

Documentation

Available from `http://isabelle.in.tum.de`

➜ Learning Isabelle

- Concrete Semantics Book
- Tutorial on Isabelle/HOL (LNCS 2283)
- Tutorial on Isar

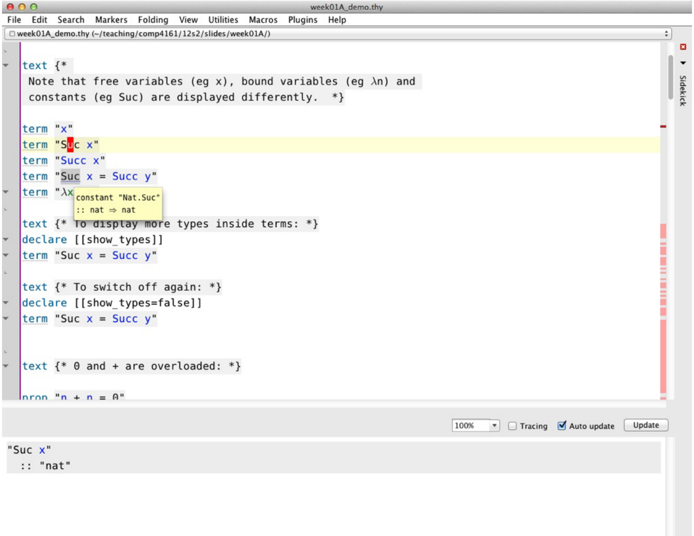
➜ Reference Manuals

- Isabelle/Isar Reference Manual
- Isabelle Reference Manual
- Isabelle System Manual

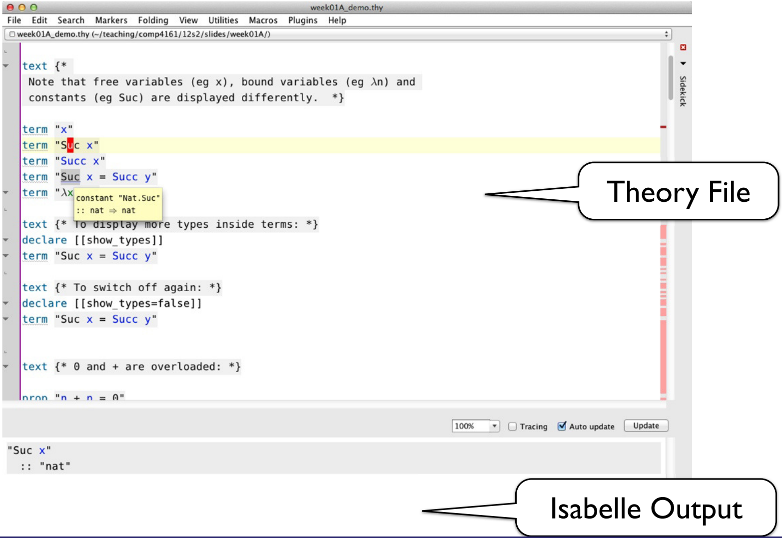
➜ Reference Manuals for Object Logics

Demo

jEdit/PIDE

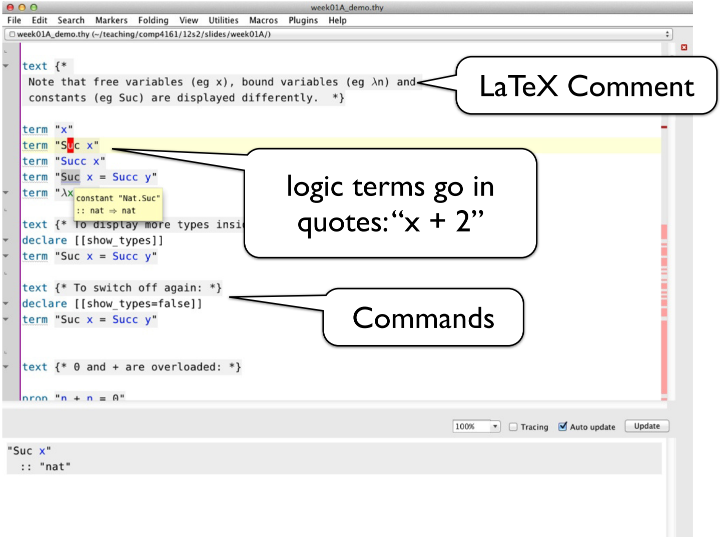

week01A_demo.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
week01A_demo.thy (~/teaching/comp4161/12s2/slides/week01A/)
text {*
Note that free variables (eg x), bound variables (eg λn) and
constants (eg Suc) are displayed differently. *}
term "x"
term "Suc x"
term "Succ x"
term "Suc x = Succ y"
term "λx
constant "Nat.Suc"
:: nat ⇒ nat
text {* To display more types inside terms: *}
declare [[show_types]]
term "Suc x = Succ y"
text {* To switch off again: *}
declare [[show_types=false]]
term "Suc x = Succ y"
text {* 0 and + are overloaded: *}
prop "n + n = 0"
100%
Tracing
Auto update
Update
"Suc x"
:: "nat"
Sidekick

jEdit/PIDE

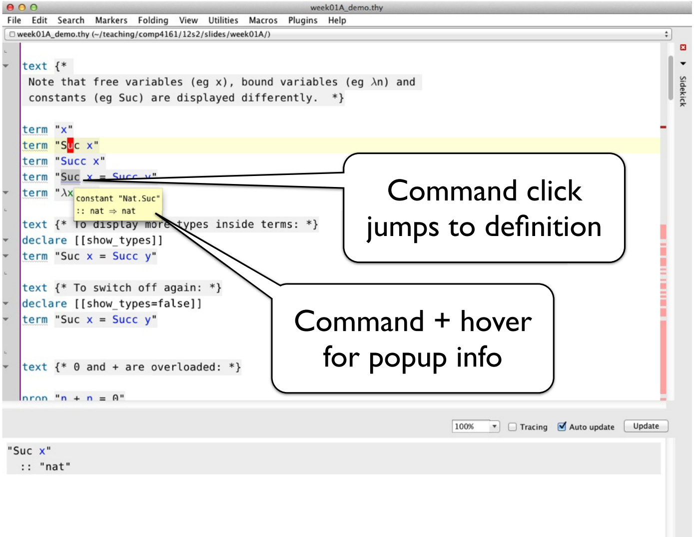


week01A_demo.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
week01A_demo.thy (~/teaching/comp4161/12s2/slides/week01A/)
text {*
Note that free variables (eg x), bound variables (eg λn) and
constants (eg Suc) are displayed differently. *}
term "x"
term "Suc x"
term "Succ x"
term "Suc x = Succ y"
term "λx
constant "Nat.Suc"
:: nat ⇒ nat
text {* To display more types inside terms: *}
declare [[show_types]]
term "Suc x = Succ y"
text {* To switch off again: *}
declare [[show_types=false]]
term "Suc x = Succ y"
text {* 0 and + are overloaded: *}
prop "n + n = 0"
Sidekick
100%
Tracing
Auto update
Update
"Suc x"
:: "nat"
Theory File
Isabelle Output

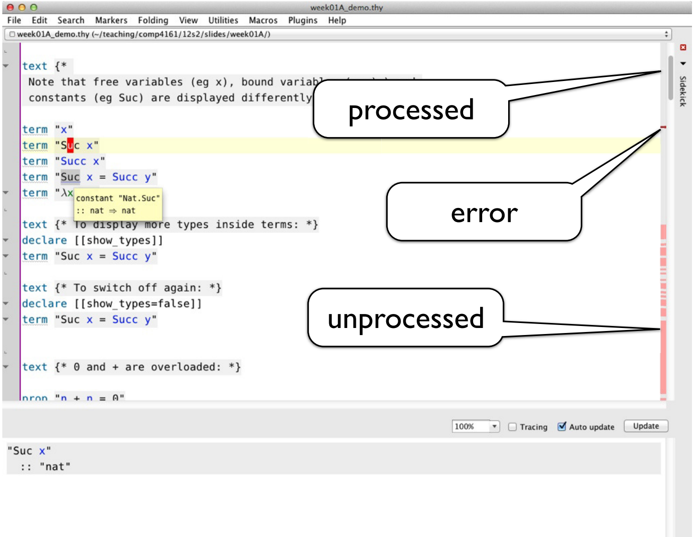
jEdit/PIDE


week01A_demo.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
week01A_demo.thy (~/teaching/comp4161/12s2/slides/week01A/)
text {*
Note that free variables (eg x), bound variables (eg λn) and
constants (eg Suc) are displayed differently. *}
term "x"
term "Suc x"
term "Succ x"
term "Suc x = Succ y"
term "λx
constant "Nat.Suc"
:: nat ⇒ nat
text {* To display more types insi
declare [[show_types]]
term "Suc x = Succ y"
text {* To switch off again: *}
declare [[show_types=false]]
term "Suc x = Succ y"
text {* 0 and + are overloaded: *}
prop "n + n = 0"
100%
Tracing
Auto update
Update
"Suc x"
:: "nat"
LaTeX Comment
logic terms go in quotes: "x + 2"
Commands

jEdit/PIDE


week01A_demo.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
week01A_demo.thy (~/teaching/comp4161/12s2/slides/week01A/)
text {*
Note that free variables (eg x), bound variables (eg λn) and
constants (eg Suc) are displayed differently. *}
term "x"
term "Suc x"
term "Succ x"
term "Suc x = Succ y"
constant "Nat.Suc"
:: nat ⇒ nat
text {* To display more types inside terms: *}
declare [[show_types]]
term "Suc x = Succ y"
text {* To switch off again: *}
declare [[show_types=false]]
term "Suc x = Succ y"
text {* 0 and + are overloaded: *}
Command click jumps to definition
Command + hover for popup info
100%
Tracing
Auto update
Update
"Suc x"
:: "nat"
Sidekick

jEdit/PIDE


week01A_demo.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
week01A_demo.thy (~/teaching/comp4161/12s2/slides/week01A/)
text {*
Note that free variables (eg x), bound variab
constants (eg Suc) are displayed differently
term "x"
term "Suc x"
term "Succ x"
term "Suc x = Succ y"
term "λx
constant "Nat.Suc"
:: nat ⇒ nat
text {* To display more types inside terms: *}
declare [[show_types]]
term "Suc x = Succ y"
text {* To switch off again: *}
declare [[show_types=false]]
term "Suc x = Succ y"
text {* 0 and + are overloaded: *}
processed
error
unprocessed
Sidekick
100%
Tracing
Auto update
Update
"Suc x"
:: "nat"

Exercises

- ➜ Download and install Isabelle from `http://mirror.cse.unsw.edu.au/pub/isabelle/`
- ➜ Step through the demo files from the lecture web page
- ➜ Write your own theory file, look at some theorems in the library, try 'find_theorems'
- ➜ How many theorems can help you if you need to prove something containing the term "Suc(Suc x)"?
- ➜ What is the name of the theorem for associativity of addition of natural numbers in the library?

λ-Calculus

Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

λ-calculus

Alonzo Church

- lived 1903–1995
- supervised people like Alan Turing, Stephen Kleene
- famous for Church-Turing thesis, lambda calculus, first undecidability results
- invented λ calculus in 1930's

λ-calculus

- originally meant as foundation of mathematics
- important applications in theoretical computer science
- foundation of computability and functional programming

untyped λ-calculus

- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

Basic intuition:

instead of	$f(x) = x + 5$
write	$f = \lambda x.\ x + 5$

$\lambda x.\ x + 5$

- ➜ a term
- ➜ a nameless function
- ➜ that adds 5 to its parameter

Function Application

For applying arguments to functions

instead of	$f(a)$
write	$f\ a$

Example: $(\lambda x.\ x + 5)\ a$

Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t
(computation!)

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

That’s it!

Now Formal

Syntax

Terms: $t \quad ::= \quad v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- ➜ v, x variables
- ➜ c constants
- ➜ $(t\ t)$ application
- ➜ $(\lambda x.\ t)$ abstraction

Conventions

- leave out parentheses where possible
- list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

Rules:

- list variables: $\lambda x.\ (\lambda y.\ t) = \lambda x\ y.\ t$
- application binds to the left: $x\ y\ z = (x\ y)\ z \neq x\ (y\ z)$
- abstraction binds to the right: $\lambda x.\ x\ y = \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$
- leave out outermost parentheses

Getting used to the Syntax

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$

$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

Computation

Intuition: replace parameter by argument
this is called β-reduction

Remember: $(\lambda x.\ t)\ a$ is evaluated (noted $\longrightarrow_\beta$) to t where x is replaced by a

Example

$$(\lambda x\ y.\ Suc\ x\ =\ y)\ 3 \longrightarrow_\beta$$
$$(\lambda x.\ (\lambda y.\ Suc\ x\ =\ y))\ 3 \longrightarrow_\beta$$
$$(\lambda y.\ Suc\ 3\ =\ y)$$

$$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$$
$$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$$
$$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$$
$$f\ 5$$

Defining Computation

β **reduction:**

$$
\begin{array}{rcccrcl}
 & & & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\beta & (\lambda x.\ s')
\end{array}
$$

Still to do: define $s[x \leftarrow t]$

Defining Substitution

Easy concept. Small problem: variable capture.
Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables

Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

Free variables FV of a term:

$$\begin{array}{ll} FV\ (x) & = \{x\} \\ FV\ (c) & = \{\} \\ FV\ (s\ t) & = FV(s) \cup FV(t) \\ FV\ (\lambda x.\ t) & = FV(t) \setminus \{x\} \end{array}$$

Example: $FV(\ \ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ \ ) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

The substitution example, $(\lambda x.\ x\ z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".

Substitution

$$
\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
c\ [x \leftarrow t] & = c & \\
\\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
\\
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda y.\ s[x \leftarrow t]) & \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) & \text{if } x \neq y \\
 & & \text{and } z \notin FV(t) \cup FV(s)
\end{array}
$$

Substitution Example

$$
\begin{array}{ll}
 & (x \;\; (\lambda x.\; x) \;\; (\lambda y.\; z\; x))[x \leftarrow y] \\
= & (x[x \leftarrow y]) \;\; ((\lambda x.\; x)[x \leftarrow y]) \;\; ((\lambda y.\; z\; x)[x \leftarrow y]) \\
= & y\; (\lambda x.\; x)\; (\lambda y'.\; z\; y)
\end{array}
$$

α Conversion

Bound names are irrelevant:
$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

α **conversion:**
$s =_\alpha t$ means $s = t$ up to renaming of bound variables.

Formally:

$$
\begin{array}{llllrll}
 & & & & (\lambda x.\ t) & \longrightarrow_\alpha & (\lambda y.\ t[x \leftarrow y]) \ \text{ if } y \notin FV(t) \\
s & \longrightarrow_\alpha & s' & \Longrightarrow & (s\ t) & \longrightarrow_\alpha & (s'\ t) \\
t & \longrightarrow_\alpha & t' & \Longrightarrow & (s\ t) & \longrightarrow_\alpha & (s\ t') \\
s & \longrightarrow_\alpha & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\alpha & (\lambda x.\ s')
\end{array}
$$

$$s =_\alpha t \quad \text{iff} \quad s \longrightarrow_\alpha^* t$$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

α Conversion

Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

$$
\begin{array}{ll}
 & x\ (\lambda x\ y.\ x\ y) \\
=_\alpha & x\ (\lambda y\ x.\ y\ x) \\
=_\alpha & x\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & z\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & x\ (\lambda x\ x.\ x\ x)
\end{array}
$$

Back to β

We have defined β reduction: $\longrightarrow_\beta$
Some notation and concepts:

- β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- t is reducible iff it contains a redex
- if it is not reducible, t is in **normal form**

Does every λ term have a normal form?

No!

Example:

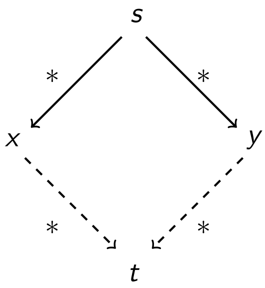
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \ldots$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \longrightarrow_\beta \lambda y.\ y$)

λ calculus is not terminating

β reduction is confluent

Confluence: $s \longrightarrow_\beta^* x \wedge s \longrightarrow_\beta^* y \Longrightarrow \exists t.\ x \longrightarrow_\beta^* t \wedge y \longrightarrow_\beta^* t$



Order of reduction does not matter for result
Normal forms in λ calculus are unique

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Definition:

$$
\begin{array}{lllllll}
 & & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t \qquad \text{if } x \notin FV(t) \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s'\ t) \\
t & \longrightarrow_\eta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s\ t') \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\eta & (\lambda x.\ s')
\end{array}
$$

$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

In fact ...

Equality in Isabelle is modulo α, β, and η conversion.

We will see later why that is possible.

Isabelle Demo

So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

Examples:

$$\begin{aligned} \texttt{true} &\equiv \lambda x\ y.\ x \\ \texttt{false} &\equiv \lambda x\ y.\ y \\ \texttt{if} &\equiv \lambda z\ x\ y.\ z\ x\ y \end{aligned}$$

$$\texttt{if true}\ x\ y \longrightarrow_\beta^* x$$
$$\texttt{if false}\ x\ y \longrightarrow_\beta^* y$$

Now, `not`, `and`, `or`, etc is easy:

$$\begin{aligned} \texttt{not} &\equiv \lambda x.\ \texttt{if}\ x\ \texttt{false true} \\ \texttt{and} &\equiv \lambda x\ y.\ \texttt{if}\ x\ y\ \texttt{false} \\ \texttt{or} &\equiv \lambda x\ y.\ \texttt{if}\ x\ \texttt{true}\ y \end{aligned}$$

More Examples

Encoding natural numbers (Church Numerals)

$$
\begin{aligned}
0 &\equiv \lambda f\ x.\ x \\
1 &\equiv \lambda f\ x.\ f\ x \\
2 &\equiv \lambda f\ x.\ f\ (f\ x) \\
3 &\equiv \lambda f\ x.\ f\ (f\ (f\ x)) \\
\ldots
\end{aligned}
$$

Numeral n takes arguments f and x, applies f n-times to x.

$$
\begin{aligned}
\texttt{iszero} &\equiv \lambda n.\ n\ (\lambda x.\ \texttt{false})\ \texttt{true} \\
\texttt{succ} &\equiv \lambda n\ f\ x.\ f\ (n\ f\ x) \\
\texttt{add} &\equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)
\end{aligned}
$$

Fix Points

$$(\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ \ t \longrightarrow_\beta$$
$$(\lambda f.\ f\ ((\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ f))\ \ t \longrightarrow_\beta$$
$$t\ \ ((\lambda x\ f.\ f\ (x\ x\ f))\ (\lambda x\ f.\ f\ (x\ x\ f))\ t)$$

$$\mu = (\lambda x\ f.\ f\ (x\ x\ f))\ (\lambda xf.\ f\ (x\ x\ f))$$
$$\mu\ t \longrightarrow_\beta t\ (\mu\ t) \longrightarrow_\beta t\ (t\ (\mu\ t)) \longrightarrow_\beta t\ (t\ (t\ (\mu\ t))) \longrightarrow_\beta \ldots$$

$(\lambda xf.\ f\ (x\ x\ f))\ (\lambda xf.\ f\ (x\ x\ f))$ is Turing's fix point operator

Nice, but ...

As a mathematical foundation, λ does not work. **It resulted in an inconsistent logic.**

- ➜ **Frege** (Predicate Logic, $\sim$ 1879):
  allows arbitrary quantification over predicates
- ➜ **Russell** (1901): Paradox $R \equiv \{X | X \notin X\}$
- ➜ **Whitehead & Russell** (Principia Mathematica, 1910-1913):
  Fix the problem
- ➜ **Church** (1930): λ calculus as logic, `true`, `false`, $\wedge$, ... as λ terms

Problem:

with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x$	$x \in M \equiv M\ x$
you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$	
and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$	
because	$(R\ R) = (\lambda x.\ \texttt{not}\ (x\ x))\ R \longrightarrow_\beta \texttt{not}\ (R\ R)$	

We have learned so far...

- λ calculus syntax
- free variables, substitution
- β reduction
- α and η conversion
- β reduction is confluent
- λ calculus is very expressive (turing complete)
- λ calculus results in an inconsistent logic