COMP4161
Advanced Topics in Software Verification

HOL

Gerwin Klein, Miki Tanaka, Johannes Åman Pohjola, Robert Sison

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Last time...

- natural deduction rules for $\wedge$, $\vee$, $\longrightarrow$, $\neg$, iff...
- proof by assumption, by intro rule, elim rule
- safe and unsafe rules

- indent your proofs! (one space per subgoal)
- prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- *prefer* and *defer*
- *oops* and *sorry*

Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Quantifiers

Scope

- Scope of parameters: whole subgoal
- Scope of $\forall, \exists, \ldots$: ends with ; or $\Longrightarrow$

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Example:

$$\bigwedge x\ y.\ \llbracket\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\quad Q\ x\ y\ \rrbracket \implies \exists x.\ Q\ x\ y$$

means

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- Scope of parameters: whole subgoal
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Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\quad Q\ x\ y\ ]\!]\ \Longrightarrow\ \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ [\![\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\quad Q\ x\ y\ ]\!]\ \Longrightarrow\ (\exists x_1.\ Q\ x_1\ y)$$

Natural deduction for quantifiers

$$\frac{}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x}{R}\ \text{allE}$$

$$\frac{}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x}{R}\ \text{allE}$$

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Natural deduction for quantifiers

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Natural deduction for quantifiers

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$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

- **allI** and **exE** introduce new parameters ($\bigwedge x$).
- **allE** and **exI** introduce new unknowns ($?x$).

Instantiating Rules

apply (rule_tac x = "*term*" in *rule*)

Like **rule**, but ?*x* in *rule* is instantiated by *term* before application.

Similar: **erule_tac**

! *x* **is in** *rule***, not in goal** **!**

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

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apply (rule allI)

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best practice

apply (rule_tac x = "x" in exI)

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exploration

apply (rule exI)

$1.\ \bigwedge x.\ x = ?y\ x$

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$?y \mapsto \lambda u.u$

Two Successful Proofs

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best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

apply (rule refl)

simpler & clearer

exploration

apply (rule exI)

1. $\bigwedge x.\ x = ?y\ x$

apply (rule refl)

$?y \mapsto \lambda u.u$

shorter & trickier

Two Unsuccessful Proofs

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apply (rule allI)

$1.\ \bigwedge x.\ x = ?y$

Two Unsuccessful Proofs

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1. $\forall x. x = ?y$

apply (rule allI)

1. $\bigwedge x. x = ?y$

apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'. x' = x$

Two Unsuccessful Proofs

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apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

Principle:

$?f\ x_1 \ldots x_n$ can only be replaced by term t

if $params(t) \subseteq x_1, \ldots, x_n$

Safe and Unsafe Rules

Safe and Unsafe Rules

Safe	allI, exE
Unsafe	allE, exI

Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Create parameters first, unknowns later

Demo: Quantifier Proofs

Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

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apply (rule_tac x = "x" in exI)

Brittle!

Renaming parameters

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Renaming parameters

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apply (rule allI)

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apply (rename_tac N)

1. $\bigwedge N.\ \exists y.\ N = y$

Renaming parameters

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apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

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1. $\bigwedge N.\ \exists y.\ N = y$

apply (rule_tac x = "N" in exI)

In general:
(rename_tac $x_1 \ldots x_n$) renames the rightmost (inner) n parameters to $x_1 \ldots x_n$

Forward Proof: frule and drule

apply (frule < *rule* >)

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: 1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Forward Proof: frule and drule

apply (frule < *rule* >)

Rule:	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
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Substitution:	$\sigma(B_i) \equiv \sigma(A_1)$

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Rule:	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Subgoal:	1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$
Substitution:	$\sigma(B_i) \equiv \sigma(A_1)$
New subgoals:	1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_2)$
	$\vdots$
	m-1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_m)$
	m. $\sigma([\![B_1; \ldots; B_n; A]\!] \Longrightarrow C)$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

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Subgoal: 1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Substitution: $\sigma(B_i) \equiv \sigma(A_1)$

New subgoals: 1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_2)$

$\vdots$

m-1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_m)$

m. $\sigma([\![B_1; \ldots; B_n; A]\!] \Longrightarrow C)$

Like **frule** but also deletes B_i: **apply** (drule $<$ *rule* $>$)

Examples for Forward Rules

$$\frac{P \wedge Q}{P}\ \text{conjunct1} \qquad \frac{P \wedge Q}{Q}\ \text{conjunct2}$$

$$\frac{P \longrightarrow Q \quad P}{Q}\ \text{mp}$$

$$\frac{\forall x.\ P\ x}{P\ ?x}\ \text{spec}$$

Forward Proof: OF

$$r \ [\textbf{OF} \ r_1 \ldots r_n]$$

Prove assumption 1 *of theorem* r *with theorem* r_1*, and assumption* 2 *with theorem* r_2*, and* ...

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Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

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r [OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

Example:

dvd_add : $[\![\ ?a\ dvd\ ?b; ?a\ dvd\ ?c\ ]\!] \Longrightarrow ?a\ dvd\ ?b + ?c$
dvd_refl : $?a\ dvd\ ?a$

$dvd\_add[OF\ dvd\_refl]$: $[\![\ ?a\ dvd\ ?c\ ]\!] \Longrightarrow ?a\ dvd\ ?a + ?c$

Forward proofs: THEN

$$r_1 \text{ [THEN } r_2] \quad \text{means} \quad r_2 \text{ [OF } r_1]$$

Demo: Forward Proofs

Hilbert's Epsilon Operator

(David Hilbert, 1862-1943)

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$$\frac{P\ ?x}{P\ (\text{SOME}\ x.\ P\ x)}\ \text{someI}$$

More Epsilon

$\mathcal{E}$ implies Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

More Epsilon

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Existential and universal quantification can be defined with ε.

Isabelle also knows the definite description operator **THE** (aka ι):

$$\frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules

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apply fast	another automatic search tactic

Epsilon and Automation Demo

We have learned so far...

➜ Proof rules for predicate calculus

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- ➜ Proof rules for predicate calculus
- ➜ Safe and unsafe rules

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Isar (Part 1)

A Language for Structured Proofs

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

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YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```

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apply (erule notE)
apply assumption
done
```

or `by blast`

OK it's true. But WHY?

Motivation

WHY is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

Demo

Isar

apply scripts

➜ unreadable

Isar

apply scripts

➜ unreadable
➜ hard to maintain

Isar

apply scripts

- ➜ unreadable
- ➜ hard to maintain
- ➜ do not scale

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No structure.

Isar

apply scripts	**What about..**
➜ unreadable	➜ Elegance?
➜ hard to maintain	
➜ do not scale	

No structure.

Isar

apply scripts

- ➜ unreadable
- ➜ hard to maintain
- ➜ do not scale

What about..

- ➜ Elegance?
- ➜ Explaining deeper insights?

No structure.

Isar

apply scripts	**What about..**
➜ unreadable	➜ Elegance?
➜ hard to maintain	➜ Explaining deeper insights?
➜ do not scale	➜ Large developments?

No structure.

Isar

apply scripts	What about..
➜ unreadable	➜ Elegance?
➜ hard to maintain	➜ Explaining deeper insights?
➜ do not scale	➜ Large developments?

No structure. **Isar!**

A typical Isar proof

proof
  assume $\mathit{formula}_0$
  have $\mathit{formula}_1$ **by** simp
  $\vdots$
  have $\mathit{formula}_n$ **by** blast
  show $\mathit{formula}_{n+1}$ **by** $\ldots$
qed

A typical Isar proof

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  assume $formula_0$
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proves $formula_0 \Longrightarrow formula_{n+1}$

A typical Isar proof

proof
  assume *formula*$_0$
  have *formula*$_1$ **by** simp
  ⋮
  have *formula*$_n$ **by** blast
  show *formula*$_{n+1}$ **by** ...
qed

proves $\mathit{formula}_0 \Longrightarrow \mathit{formula}_{n+1}$

(analogous to **assumes**/**shows** in lemma statements)

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

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proof = **proof** [method] statement* **qed**
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method = (simp ...) | (blast ...) | (rule ...) | ...

Isar core syntax

$$
\begin{array}{lll}
\text{proof} & = & \textbf{proof}\ [\text{method}]\ \text{statement}^*\ \textbf{qed} \\
 & \mid & \textbf{by}\ \text{method}
\end{array}
$$

$$
\text{method} = (\text{simp} \ldots) \mid (\text{blast} \ldots) \mid (\text{rule} \ldots) \mid \ldots
$$

$$
\begin{array}{llll}
\text{statement} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\
 & \mid & \textbf{assume}\ \text{proposition} & (\Longrightarrow) \\
 & \mid & [\textbf{from}\ \text{name}^+]\ (\textbf{have} \mid \textbf{show})\ \text{proposition proof} & \\
 & \mid & \textbf{next} & \text{(separates subgoals)}
\end{array}
$$

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

method = (simp ...) | (blast ...) | (rule ...) | ...

statement = **fix** variables $(\bigwedge)$
  | **assume** proposition $(\Longrightarrow)$
  | [**from** name$^+$] (**have** | **show**) proposition proof
  | **next** (separates subgoals)

proposition = [name:] formula

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption

proof and qed

proof [method] statement* **qed**

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  assume A: "A"
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  assume B: "B"
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proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
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  assume A: "A"
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  assume B: "B"
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proof and qed

proof [method] statement* **qed**

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  assume A: "A"
  from A **show** "A" **by** assumption
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  assume B: "B"
  from B **show** "B" **by** assumption
qed

➜ **proof** (<method>) applies method to the stated goal

proof and qed

proof [method] statement* **qed**

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➜	**proof** (<method>)	applies method to the stated goal
➜	**proof**	applies a single rule that fits

proof and qed

proof [method] statement* **qed**

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qed

➜	**proof** (<method>)	applies method to the stated goal
➜	**proof**	applies a single rule that fits
➜	**proof** -	does nothing to the goal

How do I know what to Assume and Show?

Look at the proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)

How do I know what to Assume and Show?

Look at the proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)

- ➜ **proof** (rule conjI) changes proof state to
  1. $\llbracket A; B \rrbracket \Longrightarrow A$
  2. $\llbracket A; B \rrbracket \Longrightarrow B$

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proof (rule conjI)

- ➜ **proof** (rule conjI) changes proof state to
  1. $\llbracket A; B \rrbracket \Longrightarrow A$
  2. $\llbracket A; B \rrbracket \Longrightarrow B$
- ➜ so we need 2 shows: **show** "A" and **show** "B"

How do I know what to Assume and Show?

Look at the proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
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- so we need 2 shows: **show** "A" and **show** "B"
- We are allowed to **assume** A,
  because A is in the assumptions of the proof state.

The Three Modes of Isar

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goal has been stated, proof needs to follow.

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  assume A: "A" **[state]**

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lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
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lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** ...

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Can be used to make intermediate steps.

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lemma "$(x :: \text{nat}) + 1 = 1 + x$"

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lemma "$(x :: \text{nat}) + 1 = 1 + x$"
proof -
  have A: "$x + 1 = \text{Suc}\ x$" **by** simp
  have B: "$1 + x = \text{Suc}\ x$" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

Demo

Backward and Forward

Backward reasoning: ... **have** "$A \wedge B$" **proof**

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General case: from $A_1 \dots A_n$ **have** R **proof**

➜ first n assumptions of rule must unify with $A_1 \dots A_n$

➜ conclusion of rule must unify with R

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Introduces new variables together with property

Fancy Abbreviations

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hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this
?thesis	=	the last enclosing goal statement

Demo

Moreover and Ultimately

have X_1: P_1 ...
have X_2: P_2 ...
$\vdots$
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{ ... } is a proof block similar to **proof** ... **qed**

{ **assume** P_1 ... **have** P <proof> }
stands for $P_1 \Longrightarrow P$

Mixing proof styles

from ...
have ...
  apply -   make incoming facts assumptions
  apply (...)
  ⋮
  apply (...)
  done

We have learned so far...

➜ Isar style proofs

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- proof, qed

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