COMP4161
Advanced Topics in Software Verification

$$\longrightarrow$$

Gerwin Klein, Miki Tanaka, Johannes Åman Pohjola, Rob Sison

T3/2023

Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]
- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7[b]]
  - Proof automation, Isar (part 2) [8]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

➜ Equations and Term Rewriting

Last Time

➜ Equations and Term Rewriting

Last Time

- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems

Last Time

- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

Applying a Rewrite Rule

➜ $l \longrightarrow r$ **applicable** to term $t[s]$

Applying a Rewrite Rule

- $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$

Applying a Rewrite Rule

- $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$
- **Result:** $t[\sigma\ r]$

Applying a Rewrite Rule

- $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$
- **Result:** $t[\sigma\ r]$
- **Equationally:** $t[s] = t[\sigma\ r]$

Example:

Applying a Rewrite Rule

- $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$
- **Result:** $t[\sigma\ r]$
- **Equationally:** $t[s] = t[\sigma\ r]$

Example:

Rule: $0 + n \longrightarrow n$

Term: $a + (0 + (b + c))$

Applying a Rewrite Rule

- ➜ $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$
- ➜ **Result:** $t[\sigma\ r]$
- ➜ **Equationally:** $t[s] = t[\sigma\ r]$

Example:

Rule: $0 + n \longrightarrow n$

Term: $a + (0 + (b + c))$

Substitution: $\sigma = \{n \mapsto b + c\}$

Applying a Rewrite Rule

- ➜ $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$
- ➜ **Result:** $t[\sigma\ r]$
- ➜ **Equationally:** $t[s] = t[\sigma\ r]$

Example:

Rule: $0 + n \longrightarrow n$

Term: $a + (0 + (b + c))$

Substitution: $\sigma = \{n \mapsto b + c\}$

Result: $a + (b + c)$

Conditional Term Rewriting

Rewrite rules can be conditional:

$$\llbracket P_1 \ldots P_n \rrbracket \Longrightarrow l = r$$

Conditional Term Rewriting

Rewrite rules can be conditional:

$$[\![P_1 \ldots P_n]\!] \Longrightarrow l = r$$

is **applicable** to term $t[s]$ with σ if

- ➜ $\sigma\ l = s$ and
- ➜ $\sigma\ P_1, \ldots, \sigma\ P_n$ are provable by rewriting.

Rewriting with Assumptions

Last time: Isabelle uses assumptions in rewriting.

Rewriting with Assumptions

Last time: Isabelle uses assumptions in rewriting.

Can lead to non-termination.

Example:

lemma "$f\ x = g\ x \wedge g\ x = f\ x \Longrightarrow f\ x = 2$"

Rewriting with Assumptions

Last time: Isabelle uses assumptions in rewriting.

Can lead to non-termination.

Example:

lemma "$f\ x = g\ x \wedge g\ x = f\ x \Longrightarrow f\ x = 2$"

simp	**use and simplify** assumptions
(simp (no_asm))	**ignore** assumptions
(simp (no_asm_use))	**simplify**, but do **not use** assumptions
(simp (no_asm_simp))	**use**, but do **not simplify** assumptions

Preprocessing

Preprocessing (recursive) for maximal simplification power:

$$
\begin{array}{rcl}
\neg A & \mapsto & A = \mathit{False} \\
A \longrightarrow B & \mapsto & A \Longrightarrow B \\
A \wedge B & \mapsto & A,\ B \\
\forall x.\ A\ x & \mapsto & A\ ?x \\
A & \mapsto & A = \mathit{True}
\end{array}
$$

Preprocessing

Preprocessing (recursive) for maximal simplification power:

$$
\begin{array}{rcl}
\neg A & \mapsto & A = \mathit{False} \\
A \longrightarrow B & \mapsto & A \Longrightarrow B \\
A \wedge B & \mapsto & A,\ B \\
\forall x.\ A\ x & \mapsto & A\ ?x \\
A & \mapsto & A = \mathit{True}
\end{array}
$$

Example: $(p \longrightarrow q \wedge \neg r) \wedge s$

$$\mapsto$$

Preprocessing

Preprocessing (recursive) for maximal simplification power:

$$
\begin{array}{rcl}
\neg A & \mapsto & A = \mathit{False} \\
A \longrightarrow B & \mapsto & A \Longrightarrow B \\
A \wedge B & \mapsto & A,\ B \\
\forall x.\ A\ x & \mapsto & A\ ?x \\
A & \mapsto & A = \mathit{True}
\end{array}
$$

Example:

$$(p \longrightarrow q \wedge \neg r) \wedge s$$

$$\mapsto$$

$$p \Longrightarrow q = \mathit{True} \qquad p \Longrightarrow r = \mathit{False} \qquad s = \mathit{True}$$

Demo

Case splitting with simp

$$P\ (\text{if } A \text{ then } s \text{ else } t)$$
$$=$$
$$(A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$$

Case splitting with simp

$$P\ (\text{if } A \text{ then } s \text{ else } t)$$
$$=$$
$$(A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$$

Automatic

Case splitting with simp

$$P\ (\text{if } A \text{ then } s \text{ else } t)$$
$$=$$
$$(A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$$

Automatic

$$P\ (\text{case } e \text{ of } 0\ \Rightarrow\ a \mid \text{Suc } n\ \Rightarrow\ b)$$
$$=$$
$$(e = 0 \longrightarrow P\ a) \wedge (\forall n.\ e = \text{Suc } n \longrightarrow P\ b)$$

Case splitting with simp

$$P\ (\text{if } A \text{ then } s \text{ else } t)$$
$$=$$
$$(A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$$

Automatic

$$P\ (\text{case } e \text{ of } 0\ \Rightarrow\ a \mid \text{Suc } n\ \Rightarrow\ b)$$
$$=$$
$$(e = 0 \longrightarrow P\ a) \wedge (\forall n.\ e = \text{Suc } n \longrightarrow P\ b)$$

Manually: apply (simp split: nat.split)

Case splitting with simp

$$P\ (\text{if } A \text{ then } s \text{ else } t)$$
$$=$$
$$(A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$$

Automatic

$$P\ (\text{case } e \text{ of } 0\ \Rightarrow\ a \mid \text{Suc } n\ \Rightarrow\ b)$$
$$=$$
$$(e = 0 \longrightarrow P\ a) \wedge (\forall n.\ e = \text{Suc } n \longrightarrow P\ b)$$

Manually: apply (simp split: nat.split)

Similar for any data type t: **t.split**

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

For $\Longrightarrow$ hardwired (assumptions used in rewriting)

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

For $\Longrightarrow$ hardwired (assumptions used in rewriting)

For other operators expressed with conditional rewriting.

Example: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \longrightarrow Q) = (P' \longrightarrow Q')$

Read: to simplify $P \longrightarrow Q$

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

For $\Longrightarrow$ hardwired (assumptions used in rewriting)

For other operators expressed with conditional rewriting.

Example: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \longrightarrow Q) = (P' \longrightarrow Q')$

Read: to simplify $P \longrightarrow Q$

- ➜ first simplify P to P'

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

For $\Longrightarrow$ hardwired (assumptions used in rewriting)

For other operators expressed with conditional rewriting.

Example: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \longrightarrow Q) = (P' \longrightarrow Q')$

Read: to simplify $P \longrightarrow Q$

- ➜ first simplify P to P'
- ➜ then simplify Q to Q' using P' as assumption

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

For $\Longrightarrow$ hardwired (assumptions used in rewriting)

For other operators expressed with conditional rewriting.

Example: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \longrightarrow Q) = (P' \longrightarrow Q')$

Read: to simplify $P \longrightarrow Q$

- ➜ first simplify P to P'
- ➜ then simplify Q to Q' using P' as assumption
- ➜ the result is $P' \longrightarrow Q'$

More Congruence

Sometimes useful, but not used automatically (slowdown):

conj_cong: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

More Congruence

Sometimes useful, but not used automatically (slowdown):

conj_cong: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

Context for if-then-else:

if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$

More Congruence

Sometimes useful, but not used automatically (slowdown):

conj_cong: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

Context for if-then-else:

if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$

Prevent rewriting inside then-else (default):

if_weak_cong: $b = c \Longrightarrow (\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } x \text{ else } y)$

More Congruence

Sometimes useful, but not used automatically (slowdown):

conj_cong: $\llbracket P = P'; P' \Longrightarrow Q = Q' \rrbracket \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

Context for if-then-else:

if_cong: $\llbracket b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v \rrbracket \Longrightarrow$
$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$

Prevent rewriting inside then-else (default):

if_weak_cong: $b = c \Longrightarrow (\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } x \text{ else } y)$

➜ declare own congruence rules with **[cong]** attribute

More Congruence

Sometimes useful, but not used automatically (slowdown):

conj_cong: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

Context for if-then-else:

if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$

Prevent rewriting inside then-else (default):

if_weak_cong: $b = c \Longrightarrow (\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } x \text{ else } y)$

- ➜ declare own congruence rules with **[cong]** attribute
- ➜ delete with **[cong del]**

More Congruence

Sometimes useful, but not used automatically (slowdown):

conj_cong: $\llbracket P = P'; P' \Longrightarrow Q = Q' \rrbracket \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

Context for if-then-else:

if_cong: $\llbracket b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v \rrbracket \Longrightarrow$
$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$

Prevent rewriting inside then-else (default):

if_weak_cong: $b = c \Longrightarrow (\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } x \text{ else } y)$

- ➜ declare own congruence rules with **[cong]** attribute
- ➜ delete with **[cong del]**
- ➜ use locally with e.g. **apply** (simp cong: <rule>)

Ordered rewriting

Problem: $x + y \longrightarrow y + x$ does not terminate

Ordered rewriting

Problem: $x + y \longrightarrow y + x$ does not terminate

Solution: use permutative rules only if term becomes lexicographically smaller.

Example:

Ordered rewriting

Problem: $x + y \longrightarrow y + x$ does not terminate

Solution: use permutative rules only if term becomes lexicographically smaller.

Example: $b + a \leadsto a + b$ but not $a + b \leadsto b + a$.

Ordered rewriting

Problem: $x + y \longrightarrow y + x$ does not terminate

Solution: use permutative rules only if term becomes lexicographically smaller.

Example: $b + a \rightsquigarrow a + b$ but not $a + b \rightsquigarrow b + a$.

For types nat, int etc:

- lemmas **add_ac** sort any sum $(+)$
- lemmas **mult_ac** sort any product $(*)$

Example: **apply** (simp add: add_ac) yields
$(b + c) + a \rightsquigarrow \cdots \rightsquigarrow a + (b + c)$

AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$

Commutative: $x \odot y = y \odot x$

AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$

Commutative: $x \odot y = y \odot x$

These 2 rules alone get stuck too early (not confluent).

Example: $(z \odot x) \odot (y \odot v)$

AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$

Commutative: $x \odot y = y \odot x$

These 2 rules alone get stuck too early (not confluent).

Example: $(z \odot x) \odot (y \odot v)$

We want: $(z \odot x) \odot (y \odot v) = v \odot (x \odot (y \odot z))$

AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$

Commutative: $x \odot y = y \odot x$

These 2 rules alone get stuck too early (not confluent).

Example: $(z \odot x) \odot (y \odot v)$

We want: $(z \odot x) \odot (y \odot v) = v \odot (x \odot (y \odot z))$

We get: $(z \odot x) \odot (y \odot v) = v \odot (y \odot (x \odot z))$

AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$

Commutative: $x \odot y = y \odot x$

These 2 rules alone get stuck too early (not confluent).

Example: $(z \odot x) \odot (y \odot v)$

We want: $(z \odot x) \odot (y \odot v) = v \odot (x \odot (y \odot z))$

We get: $(z \odot x) \odot (y \odot v) = v \odot (y \odot (x \odot z))$

We need: **AC rule** $x \odot (y \odot z) = y \odot (x \odot z)$

AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$

Commutative: $x \odot y = y \odot x$

These 2 rules alone get stuck too early (not confluent).

Example: $(z \odot x) \odot (y \odot v)$

We want: $(z \odot x) \odot (y \odot v) = v \odot (x \odot (y \odot z))$

We get: $(z \odot x) \odot (y \odot v) = v \odot (y \odot (x \odot z))$

We need: **AC rule** $x \odot (y \odot z) = y \odot (x \odot z)$

If these 3 rules are present for an AC operator
Isabelle will order terms correctly

Demo

Back to Confluence

Last time: confluence in general is undecidable.

Back to Confluence

Last time: confluence in general is undecidable.
But: confluence for terminating systems is decidable!

Back to Confluence

Last time: confluence in general is undecidable.
But: confluence for terminating systems is decidable!
Problem: overlapping lhs of rules.

Back to Confluence

Last time: confluence in general is undecidable.
But: confluence for terminating systems is decidable!
Problem: overlapping lhs of rules.

Definition:

Let $l_1 \longrightarrow r_1$ and $l_2 \longrightarrow r_2$ be two rules with disjoint variables.

They form a **critical pair** if a non-variable subterm of l_1 unifies with l_2.

Back to Confluence

Last time: confluence in general is undecidable.
But: confluence for terminating systems is decidable!
Problem: overlapping lhs of rules.

Definition:

Let $l_1 \longrightarrow r_1$ and $l_2 \longrightarrow r_2$ be two rules with disjoint variables.

They form a **critical pair** if a non-variable subterm of l_1 unifies with l_2.

Example:

Rules: (1) $f\ x \longrightarrow a$ (2) $g\ y \longrightarrow b$ (3) $f\ (g\ z) \longrightarrow b$
Critical pairs:

Back to Confluence

Last time: confluence in general is undecidable.
But: confluence for terminating systems is decidable!
Problem: overlapping lhs of rules.

Definition:

Let $l_1 \longrightarrow r_1$ and $l_2 \longrightarrow r_2$ be two rules with disjoint variables.
They form a **critical pair** if a non-variable subterm of l_1 unifies with l_2.

Example:

Rules: (1) $f\ x \longrightarrow a$ (2) $g\ y \longrightarrow b$ (3) $f\ (g\ z) \longrightarrow b$
Critical pairs:

$$
\begin{array}{lll}
(1)+(3) & \{x \mapsto g\ z\} & a \xleftarrow{(1)} f\ (g\ z) \xrightarrow{(3)} b \\
(3)+(2) & \{z \mapsto y\} & b \xleftarrow{(3)} f\ (g\ y) \xrightarrow{(2)} f\ b
\end{array}
$$

Completion

$$(1)\ f\ x \longrightarrow a \quad (2)\ g\ y \longrightarrow b \quad (3)\ f\ (g\ z) \longrightarrow b$$

is not confluent

,

Completion

$$(1)\ f\ x \longrightarrow a \quad (2)\ g\ y \longrightarrow b \quad (3)\ f\ (g\ z) \longrightarrow b$$

is not confluent

But it can be made confluent by adding rules!

,

Completion

$$(1)\ f\ x \longrightarrow a \quad (2)\ g\ y \longrightarrow b \quad (3)\ f\ (g\ z) \longrightarrow b$$

is not confluent

But it can be made confluent by adding rules!

How: join all critical pairs

,

Completion

$$(1)\ f\ x \longrightarrow a \quad (2)\ g\ y \longrightarrow b \quad (3)\ f\ (g\ z) \longrightarrow b$$

is not confluent

But it can be made confluent by adding rules!

How: join all critical pairs

Example:

$$(1){+}(3) \qquad \{x \mapsto g\ z\} \qquad a \xleftarrow{(1)} f\ (g\ z) \xrightarrow{(3)} b$$

shows that $a = b$ (because $a \stackrel{*}{\longleftrightarrow} b$),

Completion

$$(1)\ f\ x \longrightarrow a \quad (2)\ g\ y \longrightarrow b \quad (3)\ f\ (g\ z) \longrightarrow b$$

is not confluent

But it can be made confluent by adding rules!

How: join all critical pairs

Example:

$$(1)+(3) \qquad \{x \mapsto g\ z\} \qquad a \xleftarrow{(1)} f\ (g\ z) \xrightarrow{(3)} b$$

shows that $a = b$ (because $a \xleftrightarrow{*} b$), so we add $a \longrightarrow b$ as a rule

Completion

$$(1)\ f\ x \longrightarrow a \quad (2)\ g\ y \longrightarrow b \quad (3)\ f\ (g\ z) \longrightarrow b$$

is not confluent

But it can be made confluent by adding rules!

How: join all critical pairs

Example:

$$(1){+}(3) \qquad \{x \mapsto g\ z\} \qquad a \xleftarrow{(1)} f\ (g\ z) \xrightarrow{(3)} b$$

shows that $a = b$ (because $a \stackrel{*}{\longleftrightarrow} b$), so we add $a \longrightarrow b$ as a rule

This is the main idea of the Knuth-Bendix completion algorithm.

Demo: Waldmeister

Orthogonal Rewriting Systems

Definitions:

Orthogonal Rewriting Systems

Definitions:
A **rule** $l \longrightarrow r$ is **left-linear** if no variable occurs twice in l.

Orthogonal Rewriting Systems

Definitions:
A **rule** $l \longrightarrow r$ is **left-linear** if no variable occurs twice in l.
A **rewrite system** is **left-linear** if all rules are.

Orthogonal Rewriting Systems

Definitions:
A **rule** $l \longrightarrow r$ is **left-linear** if no variable occurs twice in l.
A **rewrite system** is **left-linear** if all rules are.

A system is **orthogonal** if it is left-linear and has no critical pairs.

Orthogonal Rewriting Systems

Definitions:
A **rule** $l \longrightarrow r$ is **left-linear** if no variable occurs twice in l.
A **rewrite system** is **left-linear** if all rules are.

A system is **orthogonal** if it is left-linear and has no critical pairs.

Orthogonal rewrite systems are confluent

Orthogonal Rewriting Systems

Definitions:
A **rule** $l \longrightarrow r$ is **left-linear** if no variable occurs twice in l.
A **rewrite system** is **left-linear** if all rules are.

A system is **orthogonal** if it is left-linear and has no critical pairs.

Orthogonal rewrite systems are confluent

Application: functional programming languages

We have learned today ...

➜ Conditional term rewriting

We have learned today ...

- ➜ Conditional term rewriting
- ➜ Congruence rules

We have learned today ...

➜ Conditional term rewriting
➜ Congruence rules
➜ AC rules

We have learned today ...

- ➜ Conditional term rewriting
- ➜ Congruence rules
- ➜ AC rules
- ➜ More on confluence