COMP4161
Advanced Topics in Software Verification

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T3/2023

Content

- ➜ Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7[b]]
  - Proof automation, Isar (part 2) [8]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

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- ➜ Orthogonal Rewrite Systems

Specification Techniques

Sets

Sets in Isabelle

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- $f`A \equiv \{y.\ \exists x \in A.\ y = f\ x\}$
- ...

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- bexI: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x \in A.\ P\ x$
- bexE: $[\![\exists x \in A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

Demo

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  The harder, but safe choice.

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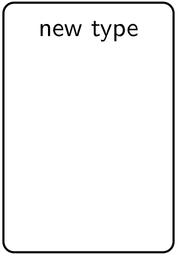
➜ **typedef**: by definiton as a set

Example: **typedef** new_type = "{some set}" <proof>
Introduces a new type as a subset of an existing type.
The proof shows that the set on the rhs in non-empty.

How typedef works



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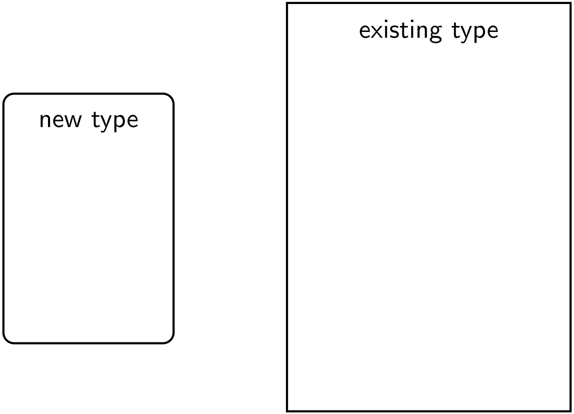

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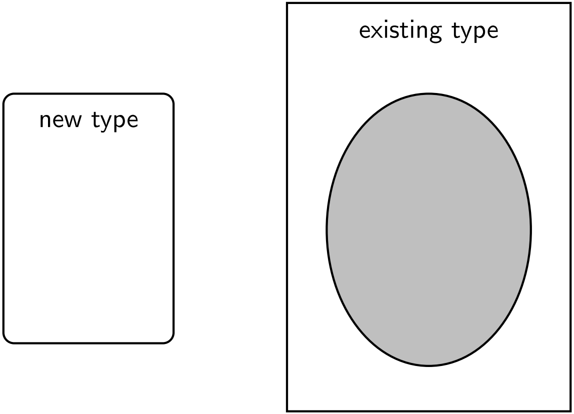

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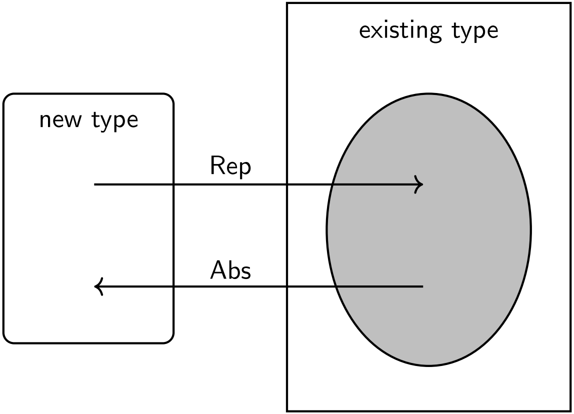

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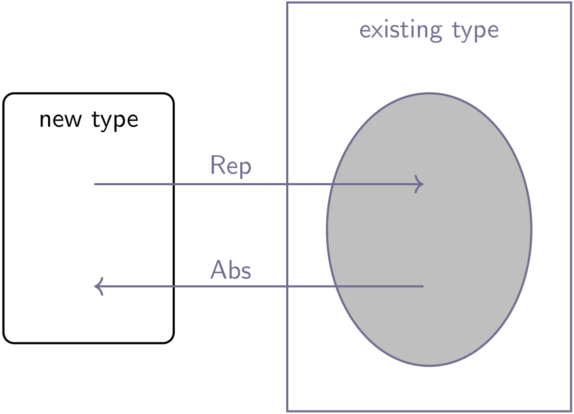

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$$(\alpha, \beta) \text{ Prod}$$

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- functions Abs_Prod, Rep_Prod
- both injective
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④ We now can:

- define constants Pair, fst, snd in terms of Abs_Prod and Rep_Prod
- derive all characteristic theorems
- forget about Rep/Abs, use characteristic theorems instead

Demo

Introducing new Types

Inductive Definitions

Example

$$\frac{}{\langle \mathsf{skip}, \sigma \rangle \longrightarrow \sigma} \qquad \frac{[\![e]\!]\sigma = v}{\langle \mathsf{x} := \mathsf{e}, \sigma \rangle \longrightarrow \sigma[x \mapsto v]}$$

$$\frac{\langle c_1, \sigma \rangle \longrightarrow \sigma' \quad \langle c_2, \sigma' \rangle \longrightarrow \sigma''}{\langle c_1; c_2, \sigma \rangle \longrightarrow \sigma''}$$

$$\frac{[\![b]\!]\sigma = \mathsf{False}}{\langle \mathsf{while}\ b\ \mathsf{do}\ c, \sigma \rangle \longrightarrow \sigma}$$

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But which set?

Simpler Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

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- Gives rise to a nice proof principle (rule induction)
- Alternative (greatest set) occasionally also useful: coinduction

Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

Demo

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We have learned today ...

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