COMP4161
Advanced Topics in Software Verification

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Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7[b]]
  - Proof automation, Isar (part 2) [8]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

Last Time

- Sets
- Type Definitions
- Inductive Definitions

Inductive Definitions

How They Work

The Nat Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

- ➜ N is the set of natural numbers $\mathbb{N}$
- ➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$
- ➜ $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

- ➜ Objective: **no junk**. Only what must be in X shall be in X.
- ➜ Gives rise to a nice proof principle (rule induction)

Formally

$$\text{Rules } \frac{a_1 \in X \quad \dots \quad a_n \in X}{a \in X} \text{ with } a_1, \dots, a_n, a \in A$$

$$\text{define set } X \subseteq A$$

Formally: set of rules $R \subseteq A \text{ set} \times A$ (R, X possibly infinite)

Applying rules R to a set B:
$\hat{R}\ B \equiv \{x.\ \exists H.\ (H, x) \in R \wedge H \subseteq B\}$

Example:

$$\begin{array}{lcl} R & \equiv & \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\ \hat{R}\ \{3, 6, 10\} & = & \{0, 4, 7, 11\} \end{array}$$

The Set

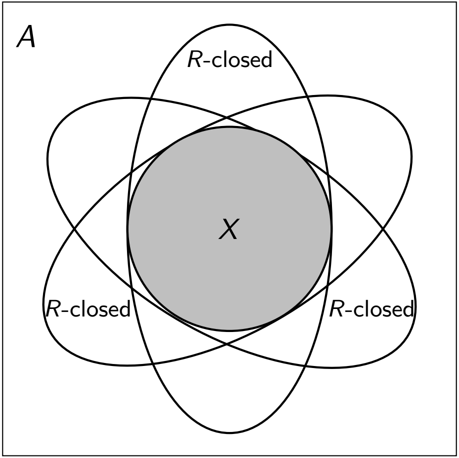
Definition: B is R-closed iff $\hat{R}\ B \subseteq B$

Definition: X is the least R-closed subset of A

This does always exist:

Fact: $X = \bigcap\{B \subseteq A.\ B\ R{-}\text{closed}\}$

Generation from Above



Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

In general:

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

Why does this work?

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

$$\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a$$

says

$\{x.\ P\ x\}$ is R-closed

but: X is the least R-closed set
hence: $X \subseteq \{x.\ P\ x\}$
which means: $\forall x \in X.\ P\ x$

qed

Rules with side conditions

$$\frac{a_1 \in X \quad \dots \quad a_n \in X \qquad C_1 \quad \dots \quad C_m}{a \in X}$$

induction scheme:

$$\begin{aligned}(\forall(\{a_1, \dots a_n\}, a) \in R.\ P\ a_1 \wedge \dots \wedge P\ a_n \wedge \\ C_1 \wedge \dots \wedge C_m \wedge \\ \{a_1, \dots, a_n\} \subseteq X \Longrightarrow P\ a)\end{aligned}$$

$$\Longrightarrow$$

$$\forall x \in X.\ P\ x$$

X as Fixpoint

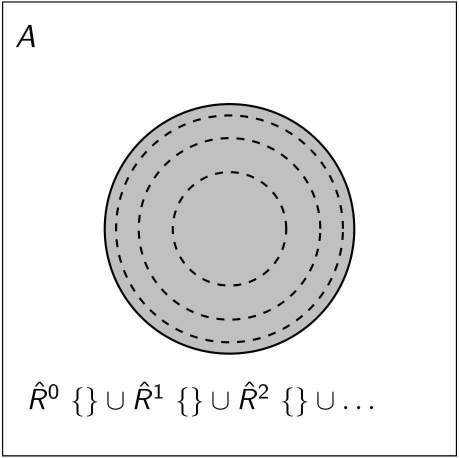
How to compute X?
$X = \bigcap\{B \subseteq A.\ B\ R-\text{closed}\}$ hard to work with.

Instead: view X as least fixpoint, X least set with $\hat{R}\ X = X$.

Fixpoints can be approximated by iteration:

$$
\begin{aligned}
&X_0 = \hat{R}^0\ \{\} = \{\} \\
&X_1 = \hat{R}^1\ \{\} = \text{rules without hypotheses} \\
&\vdots \\
&X_n = \hat{R}^n\ \{\} \\
&\\
&X_\omega = \bigcup_{n \in \mathbb{N}} (\hat{R}^n\ \{\}) = X
\end{aligned}
$$

Generation from Below



$\hat{R}^0\ \{\} \cup \hat{R}^1\ \{\} \cup \hat{R}^2\ \{\} \cup \ldots$

Does this always work?

Knaster-Tarski Fixpoint Theorem:
Let $(A, \leq)$ be a complete lattice, and $f :: A \Rightarrow A$ a monotone function.
Then the fixpoints of f again form a complete lattice.

Lattice:
Finite subsets have a greatest lower bound (meet) and least upper bound (join).

Complete Lattice:
All subsets have a greatest lower bound and least upper bound.

Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

Exercise

Formalize this lecture in Isabelle:

- Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$
- Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- Define **lfpt** f as the intersection of all f-closed sets
- Show that lfpt f is a fixpoint of f if f is monotone
- Show that lfpt f is the least fixpoint of f
- Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$
- Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R
- Show soundness of rule induction using R and lfpt $\hat{R}$

We have learned today ...

- ➜ Formal background of inductive definitions
- ➜ Definition by intersection
- ➜ Computation by iteration
- ➜ Formalisation in Isabelle