COMP4161
Advanced Topics in Software Verification

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Gerwin Klein, Miki Tanaka, Johannes Åman Pohjola, Rob Sison

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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

- ➜ Sets
- ➜ Type Definitions
- ➜ Inductive Definitions

Inductive Definitions

How They Work

The Nat Example

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Why the smallest set?

- Objective: **no junk**. Only what must be in X shall be in X.
- Gives rise to a nice proof principle (rule induction)

Formally

$$\text{Rules } \frac{a_1 \in X \quad \dots \quad a_n \in X}{a \in X} \text{ with } a_1, \dots, a_n, a \in A$$

define set $X \subseteq A$

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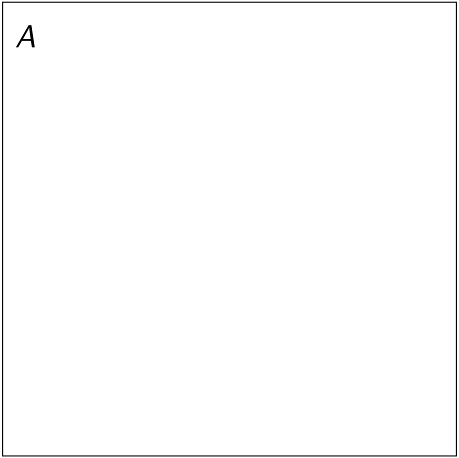
This does always exist:

Fact: $X = \bigcap\{B \subseteq A.\ B\ R\mathrm{-closed}\}$

Generation from Above



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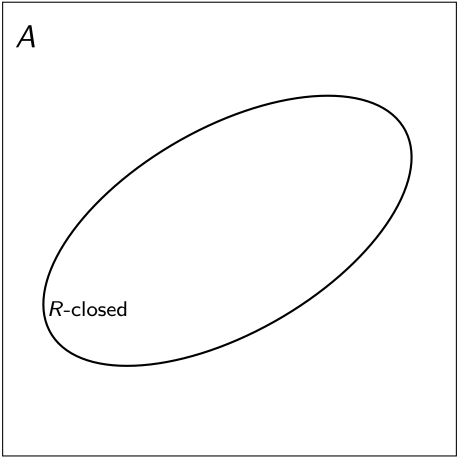

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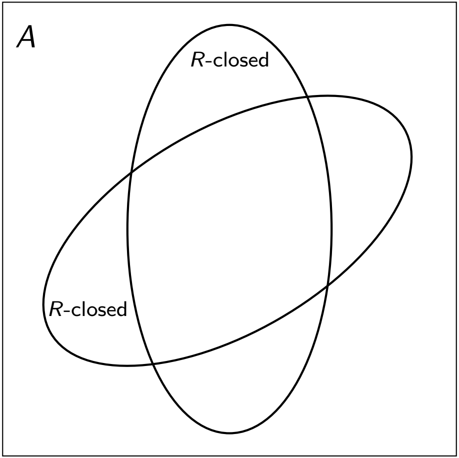

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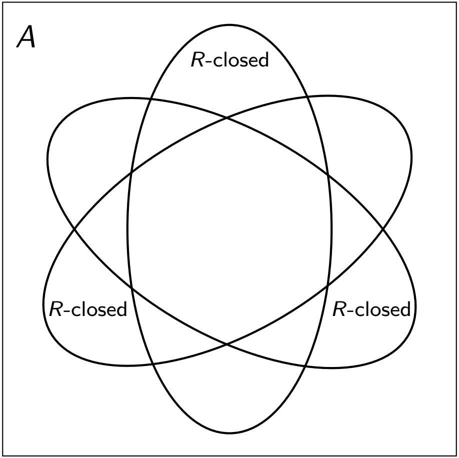

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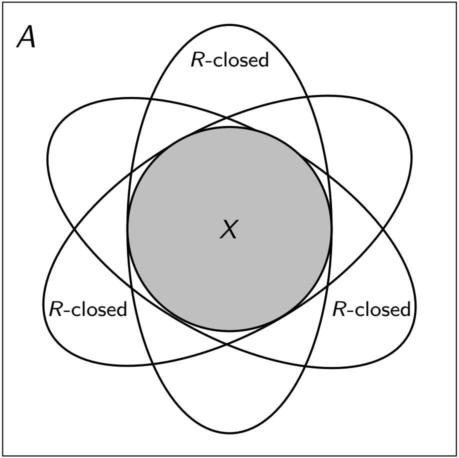

Rule Induction

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induces induction principle

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In general:

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Rules with side conditions

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induction scheme:

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$$\Longrightarrow$$

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X as Fixpoint

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&X_\omega = \bigcup_{n \in \mathbb{N}} (\hat{R}^n\ \{\}) = X
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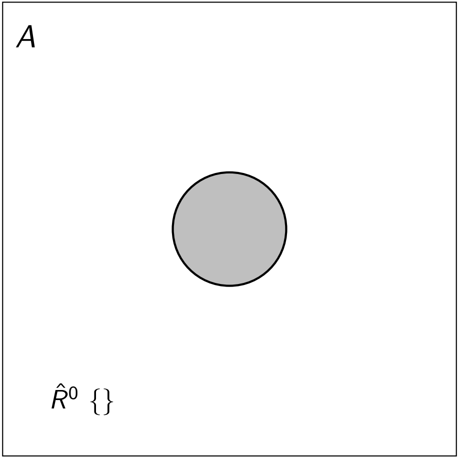

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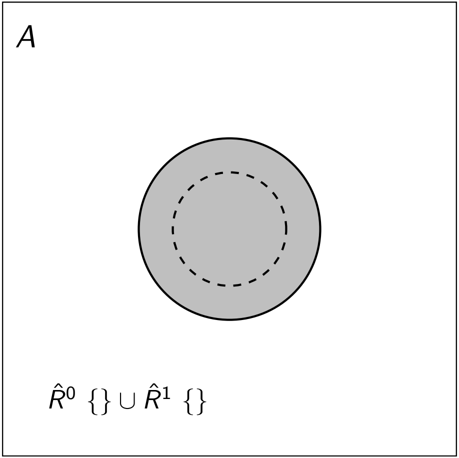

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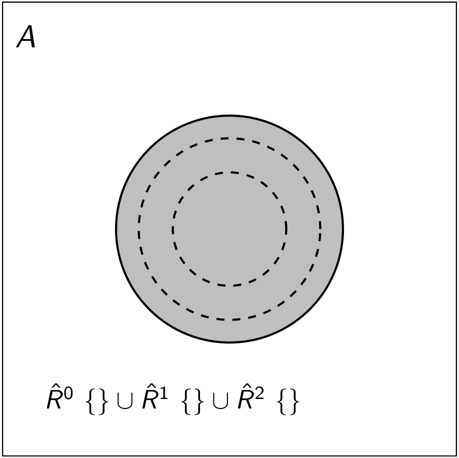

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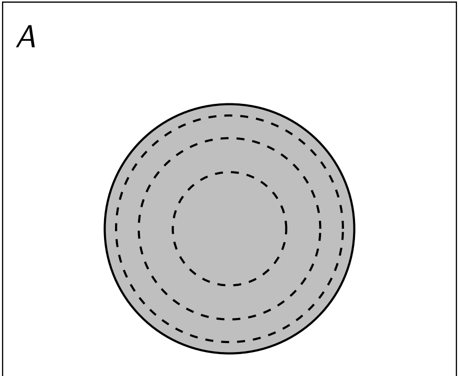

$\hat{R}^0\ \{\} \cup \hat{R}^1\ \{\} \cup \hat{R}^2\ \{\} \cup \ldots$

Does this always work?

Knaster-Tarski Fixpoint Theorem:
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Then the fixpoints of f again form a complete lattice.

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Implications:

➜ least and greatest fixpoints exist (complete lattice always non-empty).

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Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

Exercise

Formalize this lecture in Isabelle:

- ➜ Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$
- ➜ Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- ➜ Define **lfpt** f as the intersection of all f-closed sets
- ➜ Show that lfpt f is a fixpoint of f if f is monotone
- ➜ Show that lfpt f is the least fixpoint of f
- ➜ Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$
- ➜ Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R
- ➜ Show soundness of rule induction using R and lfpt $\hat{R}$

We have learned today ...

- ➜ Formal background of inductive definitions
- ➜ Definition by intersection
- ➜ Computation by iteration
- ➜ Formalisation in Isabelle