COMP4161
Advanced Topics in Software Verification

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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7]
- Proof automation, Isar (part 2) [8[b]]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Datatypes

Example:

datatype 'a list = Nil | Cons 'a "'a list"

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- ➜ Constructors:

Nil	::	'a list
Cons	::	'a ⇒ 'a list ⇒ 'a list

- ➜ Distinctness: Nil ≠ Cons x xs
- ➜ Injectivity: (Cons x xs = Cons y ys) = (x = y ∧ xs = ys)

More Examples

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Mutual Recursion:

datatype even = EvenZero | EvenSucc odd
and odd = OddSucc even

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➜ Recursive call is under a type constructor.

The General Case

$$
\begin{array}{lcl}
\textbf{datatype}\ (\alpha_1, \dots, \alpha_n)\ \tau & = & \mathsf{C}_1\ \tau_{1,1}\ \dots\ \tau_{1,n_1} \\
 & \mid & \dots \\
 & \mid & \mathsf{C}_k\ \tau_{k,1}\ \dots\ \tau_{k,n_k}
\end{array}
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Distinctness and Injectivity applied automatically

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More detail: Tutorial on (Co-)datatypes Definitions at isabelle.in.tum.de

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Because: Cantor's theorem (α set is larger than α)

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- ➜ BNF defines well-behaved type constructors, ie where recursion is allowed
- ➜ datatypes automatically are BNFs (that's how they are constructed)
- ➜ can register other type constructors as BNFs — not covered here**

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** *Defining (Co)datatypes and Primitively (Co)recursive Functions in Isabelle/HOL*

Case

Every datatype introduces a **case** construct, e.g.

$$(\text{case } xs \text{ of } [] \Rightarrow \ldots \mid y \,\#ys \Rightarrow \text{...}\ y\ \text{...}\ ys\ \text{...})$$

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- ➜ Binds weakly, needs () in context

Cases

apply (case_tac t)

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creates k subgoals

$$\llbracket t = C_i\ x_1 \ldots x_p; \ldots \rrbracket \Longrightarrow \ldots$$

one for each constructor C_i

Demo

Recursion

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

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! All functions in HOL must be total !

Primitive Recursion

primrec guarantees termination structurally

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primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

The General Case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$\begin{array}{lcl} f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\ \vdots & & \\ f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k \end{array}$$

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The recursive calls in r_i must be **structurally smaller**
(of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

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Example: rec_list :: "'a ⇒ ('b ⇒ 'b list ⇒ 'a ⇒ 'a) ⇒ 'b list ⇒ 'a"

$$\begin{array}{lll} \text{rec\_list}\ f_1\ f_2\ \text{Nil} & = & f_1 \\ \text{rec\_list}\ f_1\ f_2\ (\text{Cons}\ x\ xs) & = & f_2\ x\ xs\ (\text{rec\_list}\ f_1\ f_2\ xs) \end{array}$$

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$$\text{app} \equiv \text{rec\_list}\ (\lambda ys.\ ys)\ (\lambda x\ xs\ xs'.\ \lambda ys.\ \text{Cons}\ x\ (xs'\ ys))$$

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
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$$\mathsf{rec\_list}\ f_1\ f_2\ xs \equiv \mathsf{THE}\ y.\ (xs, y) \in \mathsf{list\_rel}\ f_1\ f_2$$

Automatic proof that set def indeed is total function

(the equations for rec_list are lemmas!)

Predefined Datatypes

nat is a datatype

datatype nat = 0 | Suc nat

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Functions on nat definable by primrec!

primrec
$f\ 0 \quad = \quad \ldots$
$f\ (\text{Suc}\ n) \quad = \quad \ldots\ f\ n\ \ldots$

Option

$$\textbf{datatype}\ \text{'a option} = \text{None} \mid \text{Some 'a}$$

Important application:

$$\text{'b} \Rightarrow \text{'a option} \quad \sim \quad \text{partial function:}$$

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Important application:

'b ⇒ 'a option	$\sim$	partial function:
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Example:
primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
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Example:
primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
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lookup k [] = None |
lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

Demo

primrec

Induction

Structural induction

$P\ xs$ holds for all lists xs if

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- ➜ and for arbitrary x and xs, $P\ xs \Longrightarrow P\ (x \# xs)$

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- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

Basic heuristics

Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

Example

A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
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itrev [] *ys* = |

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itrev [] $\quad ys = ys \mid$
itrev $(x\#xs)$ $\quad ys =$ itrev xs $(x\#ys)$

lemma itrev xs [] = rev xs

Demo

Proof Attempt

Generalisation

Replace constants by variables

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lemma ∀*ys*. itrev *xs* *ys* = rev *xs*@*ys*

Or: **apply (induct xs arbitrary: ys)**

We have seen today ...

- ➜ Datatypes
- ➜ Primitive recursion
- ➜ Case distinction
- ➜ Structural Induction

Exercises

- define a primitive recursive function **lsum** :: nat list ⇒ nat that returns the sum of the elements in a list.
- show "$2 * \text{lsum } [0.. < \mathit{Suc}\ n] = n * (n + 1)$"
- show "$\text{lsum (replicate } n\ a) = n * a$"
- define a function **lsumT** using a tail recursive version of listsum.
- show that the two functions are equivalent: $\text{lsum } xs = \text{lsumT } xs$

