COMP4161
Advanced Topics in Software Verification

fun

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Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

General Recursion

The Choice

- ➜ Limited expressiveness, automatic termination
  - primrec
- ➜ High expressiveness, termination proof may fail
  - fun
- ➜ High expressiveness, tweakable, termination proof manual
  - function

fun — examples

```
fun sep :: "'a ⇒ 'a list ⇒ 'a list"
where
    "sep a (x # y # zs) = x # a # sep a (y # zs)" |
    "sep a xs = xs"
```

```
fun ack :: "nat ⇒ nat ⇒ nat"
where
    "ack 0 n = Suc n" |
    "ack (Suc m) 0 = ack m 1" |
    "ack (Suc m) (Suc n) = ack m (ack (Suc m) n)"
```


fun

- More permissive than **primrec**:
  - pattern matching in all parameters
  - nested, linear constructor patterns
  - reads equations sequentially like in Haskell (top to bottom)
  - proves termination automatically in many cases
    (tries lexicographic order)
- Generates more theorems than **primrec**
- May fail to prove termination:
  - use **function (sequential)** instead
  - allows you to prove termination manually

Demo

fun — induction principle

- Each **fun** definition induces an induction principle
- For each equation:

  show P holds for lhs, provided P holds for each recursive call on rhs
- Example **sep.induct**:

$$\llbracket\ \bigwedge a.\ P\ a\ [];\\ \quad \bigwedge a\ w.\ P\ a\ [w]\\ \quad \bigwedge a\ x\ y\ zs.\ P\ a\ (y\#zs) \Longrightarrow P\ a\ (x\#y\#zs);\\ \rrbracket \Longrightarrow P\ a\ xs$$

Termination

Isabelle tries to prove termination automatically

- ➜ For most functions this works with a lexicographic termination relation.
- ➜ Sometimes not $\Rightarrow$ error message with unsolved subgoal
- ➜ You can prove termination separately.

function (sequential) quicksort **where**
quicksort [] = [] |
quicksort ($x\#xs$) = quicksort [$y \leftarrow xs.y \leq x$]@[x]@ quicksort [$y \leftarrow xs.x < y$]
by pat_completeness auto

termination

by (relation "measure length") (auto simp: less_Suc_eq_le)

Demo

How does fun/function work?

Recall **primrec**:

- ➜ defined one recursion operator per datatype D
- ➜ inductive definition of its graph $(x, f\ x) \in D\_rel$
- ➜ prove totality: $\forall x.\ \exists y.\ (x, y) \in D\_rel$
- ➜ prove uniqueness: $(x, y) \in D\_rel \Rightarrow (x, z) \in D\_rel \Rightarrow y = z$
- ➜ recursion operator for datatype D_rec, defined via THE.
- ➜ primrec: apply datatype recursion operator

How does fun/function work?

Similar strategy for **fun**:

- ➜ a new inductive definition for each **fun** *f*
- ➜ extract *recursion scheme* for equations in *f*
- ➜ define graph *f_rel* inductively, encoding recursion scheme
- ➜ prove totality (= termination)
- ➜ prove uniqueness (automatic)
- ➜ derive original equations from *f_rel*
- ➜ export induction scheme from *f_rel*

How does fun/function work?

function can separate and defer termination proof:

- ➜ skip proof of totality
- ➜ instead derive equations of the form: $x \in f\_dom \Rightarrow f\ x = \ldots$
- ➜ similarly, conditional induction principle
- ➜ $f\_dom = acc\ f\_rel$
- ➜ acc = accessible part of f_rel
- ➜ the part that can be reached in finitely many steps
- ➜ termination $= \forall x.\ x \in f\_dom$
- ➜ still have conditional equations for partial functions

Demo

Proving Termination

termination fun_name sets up termination goal
$\forall x.\ x \in fun\_name\_dom$

Three main proof methods:

- **lexicographic_order** (default tried by **fun**)
- **size_change** (automated translation to simpler size-change graph[1])
- **relation R** (manual proof via well-founded relation)

[1] C.S. Lee, N.D. Jones, A.M. Ben-Amram, *The Size-change Principle for Program Termination*, POPL 2001.

Well Founded Orders

Definition

$<_r$ is well founded if well founded induction holds

$\text{wf}(<_r) \equiv \forall P.\ (\forall x.\ (\forall y <_r x. P\ y) \longrightarrow P\ x) \longrightarrow (\forall x.\ P\ x)$

Well founded induction rule:

$$\frac{\text{wf}(<_r) \quad \bigwedge x.\ (\forall y <_r x.\ P\ y) \Longrightarrow P\ x}{P\ a}$$

Alternative definition (equivalent):

there are no infinite descending chains, or (equivalent):

every nonempty set has a minimal element wrt $<_r$

$$\begin{array}{lcl} \text{min}\ (<_r)\ Q\ x & \equiv & \forall y \in Q.\ y \not<_r x \\ \text{wf}\ (<_r) & = & (\forall Q \neq \{\}.\ \exists m \in Q.\ \text{min}\ r\ Q\ m) \end{array}$$

Well Founded Orders: Examples

- ➜ $<$ on $\mathbb{N}$ is well founded
  well founded induction $=$ complete induction
- ➜ $>$ and $\leq$ on $\mathbb{N}$ are **not** well founded
- ➜ $x <_r y = x \text{ dvd } y \wedge x \neq 1$ on $\mathbb{N}$ is well founded
  the minimal elements are the prime numbers
- ➜ $(a, b) <_r (x, y) = a <_1 x \vee a = x \wedge b <_2 y$ is well founded
  if $<_1$ and $<_2$ are well founded
- ➜ $A <_r B = A \subset B \wedge \text{finite } B$ is well founded
- ➜ $\subseteq$ and $\subset$ in general are **not** well founded

More about well founded relations: *Term Rewriting and All That*

Extracting the Recursion Scheme

So far for termination. What about the recursion scheme?
Not fixed anymore as in **primrec**.

Examples:

- **fun** fib **where**
  fib 0 = 1 |
  fib (Suc 0) = 1 |
  fib (Suc (Suc n)) = fib n + fib (Suc n)

  Recursion: Suc (Suc n) $\leadsto$ n, Suc (Suc n) $\leadsto$ Suc n
- **fun** f **where** f x = (if x = 0 then 0 else f (x - 1) * 2)

  Recursion: $x \neq 0 \Longrightarrow x \leadsto x - 1$

Extracting the Recursion Scheme

Higher Order:

- ➜ **datatype** 'a tree = Leaf 'a | Branch 'a tree list

  fun treemap :: ('a ⇒ 'a) ⇒ 'a tree ⇒ 'a tree **where**
  treemap fn (Leaf n) = Leaf (fn n) |
  treemap fn (Branch l) = Branch (map (treemap fn) l)

  Recursion: $x \in \text{set } l \Longrightarrow (fn, \text{Branch } l) \leadsto (fn, x)$

How does Isabelle extract context information for the call?

Extracting the Recursion Scheme

Extracting context for equations

$\Rightarrow$

Congruence Rules!

Recall rule **if_cong**:

$$[| \; b = c; \; c \Longrightarrow x = u; \; \neg\, c \Longrightarrow y = v \; |] \Longrightarrow$$
$$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$$

Read: for transforming x, use b as context information, for y use $\neg b$.

In fun_def: for recursion in x, use b as context, for y use $\neg b$.

Congruence Rules for fun_defs

The same works for function definitions.

declare my_rule[fundef_cong]
(if_cong already added by default)

Another example (higher-order):
$[\![\ \mathrm{xs} = \mathrm{ys};\ \bigwedge \mathrm{x}.\ \mathrm{x} \in \mathrm{set\ ys} \Longrightarrow \mathrm{f\ x} = \mathrm{g\ x}\ ]\!] \Longrightarrow \mathrm{map\ f\ xs} = \mathrm{map\ g\ ys}$

Read: for recursive calls in f, f is called with elements of xs

Demo

Further Reading

Alexander Krauss,
Automating Recursive Definitions and Termination Proofs in Higher-Order Logic.
PhD thesis, TU Munich, 2009.

`https://www21.in.tum.de/~krauss/papers/krauss-thesis.pdf`

We have seen today ...

- ➜ General recursion with **fun**/**function**
- ➜ Induction over recursive functions
- ➜ How **fun** works
- ➜ Termination, partial functions, congruence rules

