COMP4161
Advanced Topics in Software Verification

fun

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Content

- Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

General Recursion

The Choice

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The Choice

- ➜ Limited expressiveness, automatic termination
  - primrec
- ➜ High expressiveness, termination proof may fail
  - fun
- ➜ High expressiveness, tweakable, termination proof manual
  - function

fun — examples

```
fun sep :: ”’a ⇒ ’a list ⇒ ’a list”
where
    ”sep a (x # y # zs) = x # a # sep a (y # zs)” |
    ”sep a xs = xs”
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```
fun ack :: "nat ⇒ nat ⇒ nat"
where
    "ack 0 n = Suc n" |
    "ack (Suc m) 0 = ack m 1" |
    "ack (Suc m) (Suc n) = ack m (ack (Suc m) n)"
```

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  - pattern matching in all parameters
  - nested, linear constructor patterns
  - reads equations sequentially like in Haskell (top to bottom)
  - proves termination automatically in many cases (tries lexicographic order)

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- ➜ Generates more theorems than **primrec**
- ➜ May fail to prove termination:
  - use **function (sequential)** instead
  - allows you to prove termination manually

Demo

fun — induction principle

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- Example **sep.induct**:

$$
\begin{array}{l}
[\![\bigwedge a.\ P\ a\ [];\\
\quad \bigwedge a\ w.\ P\ a\ [w]\\
\quad \bigwedge a\ x\ y\ zs.\ P\ a\ (y\#zs) \Longrightarrow P\ a\ (x\#y\#zs);\\
]\!] \Longrightarrow P\ a\ xs
\end{array}
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- ➜ For most functions this works with a lexicographic termination relation.
- ➜ Sometimes not ⇒ error message with unsolved subgoal
- ➜ You can prove termination separately.

function (sequential) quicksort **where**
quicksort [] = [] |
quicksort $(x\#xs)$ = quicksort $[y \leftarrow xs. y \leq x]$@$[x]$@ quicksort $[y \leftarrow xs. x < y]$
by pat_completeness auto

termination
by (relation "measure length") (auto simp: less_Suc_eq_le)

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- recursion operator for datatype D_rec, defined via THE.
- primrec: apply datatype recursion operator

How does fun/function work?

Similar strategy for **fun**:

- a new inductive definition for each **fun** *f*
- extract *recursion scheme* for equations in *f*
- define graph *f_rel* inductively, encoding recursion scheme
- prove totality (= termination)
- prove uniqueness (automatic)
- derive original equations from *f_rel*
- export induction scheme from *f_rel*

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- $f\_dom = acc\ f\_rel$
- acc = accessible part of f_rel
- the part that can be reached in finitely many steps
- termination $= \forall x.\ x \in f\_dom$
- still have conditional equations for partial functions

Demo

Proving Termination

termination fun_name sets up termination goal $\forall x.\ x \in \mathit{fun\_name\_dom}$

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- ➜ **size_change** (automated translation to simpler size-change graph[1])
- ➜ **relation R** (manual proof via well-founded relation)

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Well Founded Orders

Definition

$<_r$ is well founded if well founded induction holds

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Alternative definition (equivalent):

there are no infinite descending chains, or (equivalent):

every nonempty set has a minimal element wrt $<_r$

$$\begin{array}{lcl} \min\ (<_r)\ Q\ x & \equiv & \forall y \in Q.\ y \not<_r x \\ \mathrm{wf}\ (<_r) & = & (\forall Q \neq \{\}.\ \exists m \in Q.\ \min\ r\ Q\ m) \end{array}$$

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- $A <_r B = A \subset B \wedge$ finite B is well founded
- $\subseteq$ and $\subset$ in general are **not** well founded

More about well founded relations: *Term Rewriting and All That*

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- **fun** f **where** f x = (if x = 0 then 0 else f (x - 1) * 2)

  Recursion: $x \neq 0 \Longrightarrow x \leadsto x - 1$

Extracting the Recursion Scheme

Higher Order:

- **datatype** 'a tree = Leaf 'a | Branch 'a tree list

  fun treemap :: ('a ⇒ 'a) ⇒ 'a tree ⇒ 'a tree **where**
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How does Isabelle extract context information for the call?

Extracting the Recursion Scheme

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$\Rightarrow$

Congruence Rules!

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Recall rule **if_cong**:

$$[|\ \mathrm{b} = \mathrm{c};\ \mathrm{c} \Longrightarrow \mathrm{x} = \mathrm{u};\ \neg\ \mathrm{c} \Longrightarrow \mathrm{y} = \mathrm{v}\ |] \Longrightarrow$$
$$(\text{if b then x else y}) = (\text{if c then u else v})$$

Read: for transforming x, use b as context information, for y use $\neg b$.

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$$(\mathrm{if\ b\ then\ x\ else\ y}) = (\mathrm{if\ c\ then\ u\ else\ v})$$

Read: for transforming x, use b as context information, for y use $\neg b$.
In fun_def: for recursion in x, use b as context, for y use $\neg b$.

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(if_cong already added by default)

Another example (higher-order):
$[\![\, xs = ys;\ \bigwedge x.\ x \in \text{set } ys \Longrightarrow f\ x = g\ x \,]\!] \Longrightarrow \text{map } f\ xs = \text{map } g\ ys$

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Another example (higher-order):
$[\![\ \text{xs} = \text{ys};\ \bigwedge \text{x}.\ \text{x} \in \text{set ys} \Longrightarrow \text{f x} = \text{g x}\ ]\!] \Longrightarrow \text{map f xs} = \text{map g ys}$

Read: for recursive calls in f, f is called with elements of xs

Demo

Further Reading

Alexander Krauss,
Automating Recursive Definitions and Termination Proofs in Higher-Order Logic.
PhD thesis, TU Munich, 2009.

`https://www21.in.tum.de/~krauss/papers/krauss-thesis.pdf`

We have seen today ...

- General recursion with **fun**/**function**
- Induction over recursive functions
- How **fun** works
- Termination, partial functions, congruence rules