COMP4161
Advanced Topics in Software Verification

$$\{P\} \ldots \{Q\}$$

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Last Time

- ➜ Syntax of a simple imperative language
- ➜ Operational semantics
- ➜ Program proof on operational semantics
- ➜ Hoare logic rules
- ➜ Soundness of Hoare logic

Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

Automation?

Last time: Hoare rule application is nicer than using operational semantics.

BUT:

➜ it's still kind of tedious

➜ it seems boring & mechanical

Automation?

Invariant

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Problem: While – need creativity to find right (invariant) P

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Example:

$\{M = 0 \wedge N = 0\}$
WHILE $M \neq a$ INV $\{N = M * b\}$ DO $N := N + b; M := M + 1$ OD
$\{N = a * b\}$

Weakest Preconditions

$$\textbf{pre}\ c\ Q = \textbf{weakest}\ P\ \textbf{such that}\ \{P\}\ c\ \{Q\}$$

With annotated invariants, easy to get:

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$$\text{vc } c\ Q \wedge (P \Longrightarrow \text{pre } c\ Q) \Longrightarrow \{P\}\ c\ \{Q\}$$

Syntax Tricks

- $x := \lambda\sigma.\ 1$ instead of $x := 1$ sucks
- $\{\lambda\sigma.\ \sigma\ x = n\}$ instead of $\{x = n\}$ sucks as well

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  - nice, usual syntax
  - works well if you state full program and only use vcg

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  - nice, usual syntax
  - works well if you state full program and only use vcg
- ➜ separate program variables from Hoare triple (use extensible records), indicate usage as function syntactically
  - more syntactic overhead
  - program pieces compose nicely

Demo

Arrays

Depending on language, model arrays as functions:

➜ Array access = function application:
  a[i] = a i

➜ Array update = function update:
  a[i] :== v = a :== a(i:= v)

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Use lists to express length:

- ➜ Array access = nth:

  a[i] = a ! i

- ➜ Array update = list update:

  a[i] :== v = a :== a[i:= v]

- ➜ Array length = list length:

  a.length = length a

Pointers

Choice 1

datatype	ref	= Ref int | Null
types	heap	= int ⇒ val
datatype	val	= Int int | Bool bool | Struct_x int int bool | ...

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- ➜ hp :: heap, p :: ref
- ➜ Pointer access: *p = the_Int (hp (the_addr p))
- ➜ Pointer update: *p :== v = hp :== hp ((the_addr p) := v)

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- ➜ a bit klunky
- ➜ gets even worse with structs
- ➜ lots of value extraction (the_Int) in spec and program

Pointers

Choice 2 (Burstall '72, Bornat '00)

Example: struct with next pointer and element

datatype	ref	= Ref int | Null
types	next_hp	= int ⇒ ref
types	elem_hp	= int ⇒ int

Pointers

Choice 2 (Burstall '72, Bornat '00)

Example: struct with next pointer and element

datatype	ref	$=$ Ref int $\mid$ Null
types	next_hp	$=$ int $\Rightarrow$ ref
types	elem_hp	$=$ int $\Rightarrow$ int

- ➜ next :: next_hp, elem :: elem_hp, p :: ref
- ➜ Pointer access: p$\rightarrow$next $\quad=\quad$ next (the_addr p)
- ➜ Pointer update: p$\rightarrow$next :== v $\quad=\quad$ next :== next ((the_addr p) := v)

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Example: struct with next pointer and element

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types	next_hp	= int ⇒ ref
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- ➜ next :: next_hp, elem :: elem_hp, p :: ref
- ➜ Pointer access: p→next = next (the_addr p)
- ➜ Pointer update: p→next :== v = next :== next ((the_addr p) := v)

In general:

- ➜ a separate heap for each struct field
- ➜ buys you p→next ≠ p→elem automatically (aliasing)
- ➜ still assumes type safe language

Demo

We have seen today ...

- ➜ Weakest precondition
- ➜ Verification conditions
- ➜ Example program proofs
- ➜ Arrays, pointers