COMP4161
Advanced Topics in Software Verification

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Content

- ➜ Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]

- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7]
  - Proof automation, Isar (part 2) [8[b]]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

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Disadvantages:

- ➜ Semantically equivalent programs are not obviously equal.
- ➜ e.g. "IF True THEN SKIP ELSE SKIP = SKIP" is not a true theorem.
- ➜ Many concepts already present in the logic must be reinvented.

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Today: a shallow embedding for (interesting parts of) C semantics

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Records are *n*-tuples with named components

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AutoCorres: verified translation from deeply embedded C to monadic representation

- ➜ Specifically designed for humans to do proofs over.

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modify – applies its argument to modify the state; returns ():

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Monads, Laws

Formally: a monad **M** is a type constructor with two operations.

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bind-assoc: $((a \gg= b) \gg= \text{c}) \quad = \quad (\text{a} \gg= (\lambda x.\ b\ x \gg= \text{c}))$

State Monad: Example

A fragment of C:

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void f(int *p) {
  int x = *p;
  if (x < 10) {
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record state =
    hp :: int ptr ⇒ int

f :: "int ptr ⇒ (state ⇒ (unit,state))"

f *p* ≡
do {
  x ← gets (λs. hp s p);
  if x < 10 **then**
    modify (hp_update (λh. (h(p := x + 1))))
  else
    return ()
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guard – fails when given condition applied to the state is False:
guard P $\equiv$ get $\gg=$ (λs. assert (P s))

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f p $\equiv$
  do {
    y ←guard (λs. valid s p);
    x ← gets (λs. hp s p);
    if x $<$ 10 **then**
      modify (hp_update (λh. (h(p := x + 1))))
    else
      return ()
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select – nondeterministic selection from a set:

$$\text{select}\ A \equiv \lambda\text{s}.\ \ ((A \times \{\text{s}\}),\text{False})$$

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Monadic while loop, defined **inductively**.

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$$\begin{aligned}\text{whileLoop} :: \; & ('a \Rightarrow 's \Rightarrow \text{bool}) \Rightarrow \\ & ('a \Rightarrow ('s \Rightarrow ('a \times 's)\ \text{set} \times \text{bool})) \Rightarrow \\ & ('a \Rightarrow ('s \Rightarrow ('a \times 's)\ \text{set} \times \text{bool}))\end{aligned}$$

whileLoop C B

- ➜ **condition** C: takes **loop parameter** and **state** as arguments, returns **bool**
- ➜ **monadic body** B: takes **loop parameter** as argument, return-value is the **updated** loop parameter
- ➜ **fails** if the loop body ever fails or if the loop never terminates

While Loops

Monadic while loop, defined **inductively**.

$$\begin{aligned}\text{whileLoop} :: \ & (\textit{'a} \Rightarrow \textit{'s} \Rightarrow \text{bool}) \Rightarrow \\ & (\textit{'a} \Rightarrow (\textit{'s} \Rightarrow (\textit{'a} \times \textit{'s})\ \text{set} \times \text{bool})) \Rightarrow \\ & (\textit{'a} \Rightarrow (\textit{'s} \Rightarrow (\textit{'a} \times \textit{'s})\ \text{set} \times \text{bool}))\end{aligned}$$

whileLoop C B

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- **fails** if the loop body ever fails or if the loop never terminates

Example: whileLoop $(\lambda p\ s.\ \text{hp}\ s\ p = 0)$ $(\lambda \text{p}.\ \text{return}\ (\text{ptrAdd}\ p\ 1))$ p

Defining While Loops Inductively

Two-part definition: results and termination

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Results: while_results :: $(\textit{'a} \Rightarrow \textit{'s} \Rightarrow \text{bool}) \Rightarrow$
$(\textit{'a} \Rightarrow (\textit{'s} \Rightarrow (\textit{'a} \times \textit{'s})\ \text{set} \times \text{bool})) \Rightarrow$
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Defining While Loops Inductively

Termination:

$$\text{while\_terminates} :: (\textit{'a} \Rightarrow \textit{'s} \Rightarrow \text{bool}) \Rightarrow (\textit{'a} \Rightarrow (\textit{'s} \Rightarrow (\textit{'a} \times \textit{'s})\ \text{set} \times \text{bool})) \Rightarrow \textit{'a} \Rightarrow \textit{'s} \Rightarrow \text{bool}$$

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$$\begin{aligned}&\text{whileLoop}\ C\ B \equiv \\ &\quad (\lambda r\ s.\ (\{(r',s').\ (\text{Some}\ (r,\ s),\ \text{Some}\ (r',\ s')) \in \text{while\_results}\ C\ B\}, \\ &\qquad\qquad (\text{Some}\ (r,\ s),\ \text{None}) \in \text{while\_results} \lor \\ &\qquad\qquad \neg\text{while\_terminates}\ C\ B\ r\ s))\end{aligned}$$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow \forall (r,s') \in \text{fst}\ (m\ s).\ Q\ r\ s'$

➜ Post-condition Q is a predicate of return-value and result state.

Weakest Precondition Rules

$\{\!| \qquad |\!\}\ \text{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\}$ $\{\!| \qquad |\!\}\ \text{get}\ \{\!|P|\!\}$ $\{\!| \qquad |\!\}\ \text{put}\ x\ \{\!|P|\!\}$

$\{\!| \qquad |\!\}\ \text{gets f}\ \{\!|P|\!\}$ $\{\!| \qquad |\!\}\ \text{modify}\ f\ \{\!|P|\!\}$

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$\{\!|\lambda s.\ P \longrightarrow Q\ ()\ s|\!\}$ assert P $\{\!|Q|\!\}$ $\quad$ $\{\!| \qquad |\!\}$ fail $\{\!|Q|\!\}$

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$\{\!|\lambda s.\ P \longrightarrow Q\ ()\ s|\!\}$ assert P $\{\!|Q|\!\}$ $\quad$ $\{\!|\lambda\_.\ \text{True}|\!\}$ fail $\{\!|Q|\!\}$

More Hoare Logic Rules

$$\frac{}{\{\!|\qquad\qquad\qquad\qquad|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

More Hoare Logic Rules

$$\frac{P \implies \{\!|Q|\!\}\ f\ \{\!|S|\!\} \quad \neg\ P \implies \{\!|R|\!\}\ g\ \{\!|S|\!\}}{\{\!|\lambda s.(P \longrightarrow Q\ s) \wedge (\neg P \longrightarrow R\ s)|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

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$$\frac{\bigwedge r.\ \{\!|\lambda s.\ I\ r\ s \wedge C\ r\ s|\!\}\ B\ \{\!|I|\!\} \quad \bigwedge r\ s.\ [\![I\ r\ s;\ \neg\ C\ r\ s]\!] \implies Q\ r\ s}{\{\!|I\ r|\!\}\ \text{whileLoop}\ C\ B\ r\ \{\!|Q|\!\}}$$

Demo

We have seen today

- ➜ Deep and shallow embeddings
- ➜ Isabelle records
- ➜ Nondeterministic State Monad with Failure
- ➜ Monadic Weakest Precondition Rules