Evaluating Mismatch Probability of Activity-based Map Matching in Indoor Positioning

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Abstract—If users are known to perform specific activities at specific locations within a building, then indoor positioning could be achieved by monitoring user activities and matching them to specific locations in a preloaded floor map. This is the fundamental idea behind activity-based map matching (AMM). For example, the user’s smartphone could use the accelerometer readings to detect whether a user is using an escalator, and then match the current location of the user to the nearest escalator. AMM therefore could be used for frequently recalibrating location estimators to ground-truth values. This is especially useful for recalibrating pedestrian dead reckoning (PDR), which can estimate indoor position if started from a known location, but error grows unboundedly with time or distance traveled. However, AMM is not perfect and could potentially cause mismatches by matching the current location of the user to a wrong location. In this paper we propose a methodology and derive a closed-form expression for mismatch probability as a function of PDR sensor error and proximity between two facilities. By applying our methodology to a practical indoor complex (Sydney airport) we find several interesting results: (1) that mismatch probability is spatially non-uniform, i.e., it can be different in different parts of the floor; (2) for some specific facilities, mismatch probability can be very high (up to 80%), and (3) if escalators could be distinguished from lifts with high accuracy, we could reduce mismatch probability significantly (by up to 68%).

Index Terms—Indoor Positioning, Pedestrian Dead Reckoning, PDR Recalibration, Activity-Based Map Matching, Mismatch Probability.

I. INTRODUCTION

A recent scheme, called pedestrian dead-reckoning (PDR) [1], has demonstrated the ability to continuously estimate indoor position of a mobile phone using its accelerometer and compass if started from a known location. The key idea is simple. Based on the accelerometer readings, it is possible to count the number of steps a person has walked, and therefore derive the displacement of the person. Using the compass, the direction of each of these steps can be tracked. Merging the displacement with the direction, the user’s location can be estimated.

Conceptually, PDR is an elegant solution to indoor positioning because it is completely self-sufficient in the sense that it does not require any support from any type of infrastructure. The major problem with PDR, however, is that the dead-reckoned trajectories are accurate in the beginning, but due to noise in the mobile sensors, the accuracy diverges from the truth over time. As such, PDR cannot be used on its own for long indoor trips. Some additional mechanisms are required to occasionally obtain a position fix (also referred to as ‘recalibration’). Following a recalibration, PDR can be used once again to track location from the newly calibrated location.

Three main approaches have been proposed in the literature to recalibrate PDR. One but all rely on some form of infrastructure for the calibration.

- GPS-based: This approach uses the global positioning system (GPS) when it is available. Some Outdoor localization schemes like CompAcc [2] trigger periodic GPS measurements to recalibrate the user’s estimated location. A PDR system in [3] has used GPS as a means to calibrate and validate the PDR technology. Unfortunately, GPS is unreliable indoors, making it an inappropriate solution for indoor.

- Indoor-reference-node-based: An alternative approach uses reference nodes deployed in the infrastructure to recalibrate the user’s estimated location. The system proposed in [4] uses the radio-frequency identification (RFID) technology to calibrate the PDR system by placing RFID tags in the environment. These tags act as fiducial markers that update the PDR system by correcting positional errors and sensor inaccuracy. Another system, proposed in [5] utilizes WIFI access points (up to 30) for collecting calibration information. In the system proposed in [6], a camera was mounted on the head, pointing in the superior direction with respect to the user. Large number of markers were positioned on the ceiling and walls, arranged in a high density pattern of about 1.7 markers per square meters. Position fix was achieved by optical recognition of these markers. Unfortunately, these solutions depend on infrastructure and some of them cannot be easily achieved with smartphone (for example, RFID solution would not work without an RFID reader installed on the phone).

- Activity-based Map Matching (AMM): This recently proposed approach [7] [8] is used for recalibrating PDR systems by monitoring user activities and matching their activities to specific locations in a preloaded floor.
map. These specific locations would be one of the facilities in the building, such as an escalator, a lift, a stair, a ramp, and so on, which force the pedestrian to act differently than walking. Given that smartphone accelerometer can be used to detect user activity [9] [10], the combination of AMM and PDR provides an opportunity for the mobile device to learn its location without relying on any interactions with in-building infrastructure. This provides a completely self-sufficient indoor positioning and navigation solution, which could be readily implemented in a modern personal mobile device, such as an smartphone.

However, AMM is not perfect and potentially could introduce errors of its own. One such error could occur due to matching user location to an incorrect facility on the map. In particular, this can happen if many of such facilities exist in a building (such as many escalators within a large airport). For example, if two escalators exist in close proximity to each other, matching the current pedestrian location to the nearest escalator may actually match to the wrong one. We shall call this type of error a mismatch. Therefore, we need a method to compute mismatch probability of a given AMM solution for a given indoor complex. To the best of our knowledge, no such methods have been discussed in the literature yet. This motivates our current work.

We make the following novel contributions:

- We propose a methodology to compute mismatch probability of AMM. Our methodology can be applied to both continuous and discrete location cases.
- For continuous location, we derive a closed-form expression for mismatch probability as a function of PDR sensor error and proximity between two facilities. The analytical framework is validated by applying it to a real map with discrete locations.
- Using the proposed methodology and publicly available floor maps, we computed mismatch probability for a practical indoor complex (Sydney Airport). We obtain several interesting results:

  1) Mismatch probability is spatially non-uniform, i.e., it can be different in different parts of the floor. This is caused by non-uniform proximities between facilities located in different parts of the building.
  2) For some specific facilities, mismatch probability can be very high (up to 80%).
  3) If escalators could be distinguished from lifts with high accuracy, we could reduce mismatch probability significantly (by up to 68%).

The rest of the paper is organized as follows. An overview of AMM is provided in Section II. We present our proposed methodology to compute mismatch probability in Section III, followed by a practical application of it in Section IV. We conclude our paper in Section V.

II. OVERVIEW OF ACTIVITY-BASED MAP MATCHING

Activity-Based Map Matching (AMM) comprises of two basic modules:

1) Activity Detection (AD): The main function of this module is to detect what a person is doing at a particular instant, for example detecting whether a person is going upstairs, standing in a lift, or using an escalator.

2) Map Matching (MM): This module attempts to identify the facility on the map the user is using based on his/her detected activity, and then match the PDR estimated position to the location of the identified facility.

Both modules can introduce errors of their own. AD can introduce two types of errors. First, it can miss detecting an activity when an activity actually takes place. Second, it may confuse between two activities and incorrectly detect one when actually the other has taken place, such as mistakenly detecting a lift activity when actually the pedestrian has started using an escalator (we will show later that both escalators and lifts produce very similar accelerometer signals).

MM has another issue. AD may detect the activity correctly, but knowing the activity does not yield the exact location of the user in a large indoor complex deploying many facilities. For example, the smartphone may use accelerometer readings to correctly detect that the pedestrian is using an escalator, but it cannot work out exactly which escalator is being used if many escalators are available on the same floor. To identify the most likely escalator, one approach proposed in the literature [7] is to match the current estimated location of the user to the nearest escalator. We shall call this approach the Nearest Object Matching (NOM). However, the escalator determined by NOM (the nearest escalator) may not be the actual escalator the user is using. In this case, we say that a mismatch has occurred.

The focus of this paper is to estimate the mismatch probability for a given map and PDR sensor error. We will assume that the real location of the user is known, which helps working out whether a mismatch has occurred or not. In our work, we shall use the locations of the facility given on the map as the real positions of the user. The next section presents the details of our proposed methodology.

III. METHODOLOGY FOR COMPUTING MISMATCH PROBABILITY

We propose a methodology to compute mismatch probability for every single facility located at a known position. We have the following 3 inputs: (1) location of the facility for which mismatch probability is to be computed, (2) sensor error distribution, and (3) location of all other facilities which belong to the same class of facilities. When a certain class of pedestrian activity, such as riding an escalator, is detected, we assume that the PDR location estimate at that time can be anywhere within a certain range of the location of the escalator. This range is called PDR sensor error region, which can be derived from the sensor error distribution. We then determine mismatch probability for this escalator as the fraction of all
locations within the error region that will not match to the correct escalator (will match to another escalator instead due to shorter distance).

A. Computing Error Region

Let \( P_t = (x_t, y_t) \) be the true position of the user, i.e., the position of the correct facility to match. We need to create a region around \( P_t \) such that PDR estimates will be contained in the region with a given confidence level. This can be determined using the error distribution of the PDR sensor.

In [11], the authors assume that the errors in the \( x- \) and \( y- \) axes are jointly normally distributed with mean \((\mu_x, \mu_y)\) and variance-covariance matrix

\[
\begin{pmatrix}
\sigma_x^2 & \sigma_{xy} \\
\sigma_{yx} & \sigma_y^2
\end{pmatrix}
\]

where \( \sigma_x^2, \sigma_y^2 \) and \( \sigma_{xy} \) are the variance of the \( x- \)coordinate, the variance of the \( y- \)coordinate and their covariance, respectively. This assumption creates an elliptical region with \( P_t \) being its center and a semi-major axis having length \( a \), and the length of the semi-minor axis being \( b \). The following equations are used to estimate \( a \) and \( b \).

\[
a = \sigma_0 \sqrt{\frac{1}{2} \left( \sigma_x^2 + \sigma_y^2 + \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2} \right)} \quad (1)
\]

\[
b = \sigma_0 \sqrt{\frac{1}{2} \left( \sigma_x^2 + \sigma_y^2 - \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^2} \right)} \quad (2)
\]

where \( \sigma_0 \) is a critical constant 1.96 or 2.57 obtained from the normal tables to ensure respectively a 95% or a 0.99% probability (confidence level) that the current location estimates of the user falls in the constructed elliptical region.

We can achieve some simplification for indoor map matching by considering the errors in \( x \) and \( y \) to be independent \((\sigma_{xy} = 0)\)1 and assuming a common variance for them \((\sigma_x = \sigma_y = \sigma)\). With these simplifications, it is now possible to represent the error region by a circle of radius \( r \) defined as:

\[
r = \sigma_0 \sigma \quad (3)
\]

where \( \sigma \) is the common standard deviation for \( x \) and \( y \).

At this point, it is important to note that PDR error depends not only on the physical sensors, but also on the time it is in operation. In fact, for a given hardware, PDR error typically accumulates in time. Hence, the actual value of \( \sigma \) (and the corresponding error radius) will also depend on the time elapsed since its last recalibration. Table I shows some values of \( r \) for different combinations of sensor error standard deviation and confidence levels.

Determining the error region around the position of a facility allows us to compute mismatch probability for the facility for both discrete and continuous locations. For discrete locations, mismatch probability is obtained as:

\[
p = \frac{m}{n} \quad (4)
\]

where \( n \) is the total number of locations inside the error region and \( m \) is the number of those locations which matched to a wrong facility.

For continuous location case, i.e., if we assume infinite point locations within an indoor area, mismatch probability can be derived as a closed-form expression (see next subsection).

B. Continuous Location Case

The main reason for mismatch is due to overlap of error regions of two or more facilities. Regarding overlaps, we have three cases:

- No overlap: This occurs when \( d > 2r \), where \( d \) is the proximity between any pair of facilities (see Fig.1(a)). In this case, we have \( p = 0 \).
- Single overlap: In this case, there is only one other facility for which \( d < 2r \) (see Fig.1(b)). Let \( A \) be the overlap

### Table I

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>Standard Deviation (( \sigma ))</th>
<th>Error Radius(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>95%</td>
<td>0.5</td>
<td>0.98</td>
</tr>
<tr>
<td>99%</td>
<td>1</td>
<td>1.96</td>
</tr>
<tr>
<td>99%</td>
<td>0.5</td>
<td>1.285</td>
</tr>
<tr>
<td>99%</td>
<td>1</td>
<td>2.57</td>
</tr>
</tbody>
</table>

1Note that authors in [11] assumed a non-zero \( \sigma_{xy} \) because they had to align the ellipse along a road segment, which is difficult with \( \sigma_{xy} = 0 \).
area, i.e., the common area between the two error regions. In this case, \( p = \frac{A}{2\pi r^2} \) for each of the two facilities.

- Multiple overlaps: In this case, the facility of interest is overlapped with more than one facility. The overlap areas can be disjoint to each other as shown in Fig.1(c), or they can have mutual overlap with each other (Fig.1(d)). For disjoint overlaps, we can obtain \( p = \sum_{i=1}^{n} \frac{A_i}{2\pi r^2} \), where \( A_i \) is the area of the \( i^{th} \) overlap. The case of joint overlaps, however, will need more extensive modeling, which is left as a possible future work.

Let us proceed with the derivation of \( p \) for a single overlap case. Using geometry, it can be obtained as a closed-form expression (detail steps of the derivation are provided in the appendix):

\[
 p = \sin^{-1} \sqrt{1 - \left( \frac{d}{2r} \right)^2} - \left( \frac{d}{2r} \right) \sqrt{1 - \left( \frac{d}{2r} \right)^2} \pi \tag{5}
\]

where \( d \) is the facility proximity and \( r \) is the radius of the error region.

Fig. 2 shows that \( 0 < p < 0.5 \) when \( \left( \frac{d}{2r} \right) \) is varied between 0 and 1. This means that for single-overlap case, mismatch probability for any facility is upper bounded to 0.5. Fig. 3 further explores the impact of \( d \) and \( \sigma \) (since \( r \) is based on \( \sigma \) combined with the desired confidence level) on the mismatch probability, separately. For example, we can see (Fig. 3(a)) that for a sensor error standard deviation of half-a-location (\( \sigma = 0.5 \)), reducing proximity from 2 to 1 (50% reduction) would increase mismatch probability from 0.05 to 0.25 (a 500% increase!). This implies that mismatch probability can be very sensitive to facility proximity. Similarly, from Figure 3(b), we find that mismatch probability is also very sensitive to PDR sensor error. For example, the prospect of any mismatch is totally eliminated \( (p = 0) \) for \( \sigma < 0.5 \) when \( d = 2.5 \).

### IV. Practical Application (Sydney Airport)

In this section, we apply our mismatch probability computation methodology to a practical indoor complex, the departure level of Sydney international airport. We use the floor map of level 2 (see Fig.4), which is publicly available on the internet [12]. The map has a size of 1.5 MB, which means any smartphone can easily preload and store it in its memory. We can see from Fig.4 that the map is presented as a 2D lattice (grid map) consisting of 18 rows and 26 columns. For discrete location case, we would assume that each grid square (cell) represents a possible location of the user with the row and column numbers representing the x-coordinate and the y-coordinate, respectively.

The map shows locations of many different types of facilities. These include escalators, lifts, stairs, ATM machines, telephone booths, Internet kiosks, toilets, and so on. The more facilities an AD module can detect, the less distance is traveled
Fig. 4. Floor Map of Level 2 in Sydney Airport [12]. Square error regions with side length $2\sigma = 3$ ($\sigma = 0.5$) are shown around each of the 9 escalators.
with ‘pure’ PDR before it is recalibrated. In fact, earlier work [13] has shown that the expected distance traveled before an AMM-based recalibration occurs to the PDR is inversely proportional to the density of the detectable facilities within an indoor complex. However, not all facilities can be easily detected. In this paper, we consider escalators and lifts as two basic facilities because they can be detected using smartphone accelerometers [7]. Our methodology remains applicable to any other facilities which we learn to detect in future.

There are a total of 9 escalators and 9 lifts altogether in the map. We consider two cases. In the first case, we assume that the smartphone employs highly sophisticated signal processing and classification techniques that can distinguish between escalators and lifts with very high accuracy. In this case, when an escalator/lift is detected, there is no confusion whether the facility is an escalator or a lift. In the second case, we assume that it is difficult to rule out confusion between escalator and lift with 100% guarantee. In that case, both escalators and lifts are used as possible matching target when an escalator/lift is detected.

### A. Case of Accurate Activity Detection

Here the results are presented under the assumption that no other facility is incorrectly classified as an escalator, i.e., when calculating mismatch probability for a given escalator, we only consider the other 8 escalators as possible target for mismatch. For continuous location case, analytical results obtained using Equation (5) are shown in Tables II and III. We find that the mismatch probabilities for some escalators, 1, 4, and 8 for \( \sigma = 0.5 \) and 1 and 4 for \( \sigma = 1 \), are zero, because there are no error region overlaps. Some have single overlaps and others have multiple. Escalators with overlaps have non-zero mismatch probability as expected. We also find, as expected intuitively, that mismatch probability increases with error variance when there is an overlap. For example, for escalators 2 and 3, the mismatch probability is only 0.17 for \( \sigma = 0.5 \), but they increased to 0.33 when \( \sigma \) is increased to 1.

Next, we consider the discrete location case, where we assume that each grid square represents a possible location of the user with the row and column numbers representing the x-coordinate and the y− coordinate, respectively. For each discrete locations, we need to modify the shape of our error region from a circle to a square, because with a circle, some locations may be contained only fractionally. The square has a side of length \( 2r \), which equals to the diameter of the circle with radius \( r \). Furthermore, since our map is a 2D lattice of squared units, the side length of the square shaped error region has to be approximated to the nearest integer value. With the square error region, Table IV shows the mismatching probability of 9 escalators based on Equation (4).

Comparing the results of the discrete location case, which is more practical, to the results of the continuous location case, one can see that all the non-zero mismatch probabilities in Table IV for \( \sigma = 0.5 \) are higher than the corresponding ones in Table II. The reason for this increase is the larger error region resulting from rounding the value of \( 2r \) from 2.57 to the nearest integer, 3.

Although, some of the mismatch probabilities in Table IV for \( \sigma = 1 \) are a little bit less than the corresponding ones in Table III (escalators 2, 3, and 7), this minor reduction came due to rounding down the value of \( 2r \) from 5.14 to 5. This has created a little bit smaller error region. On the other hand, the mismatch probability in Table IV for escalator 9 is higher than its corresponding value in Table III. This can be explained by looking at Fig. 4 and realizing that part of the error region of this escalator (in case of \( \sigma = 1 \)) is chopped, since it falls outside the border of the map. This has caused a reduction in the value of \( n \), the total number of locations inside the error region, in Equation (4) and simultaneously an increase in the value of \( m \), the number of locations which matched to a wrong facility, resulting in a higher mismatch probability. Finally, this type of comparison cannot be conducted for escalators 5, 6, and 8 due to the existence of the disjoint overlaps, which we have mentioned that their calculations are left as a possible future work.

We also notice that mismatch probability varies from escalator to escalator, giving a spatially non-uniform mismatch probability for the airport. Location-dependent probability
However, AMM is not always perfect and may introduce errors of PDR or other indoor localization techniques. For example, if an escalator activity is detected, an application may decide not to recalibrate PDR with AMM due to high mismatch probability.

### B. Low Accuracy Activity Detection

By low accuracy activity detection, we refer to the case when there may be confusion between escalators and lifts. To motivate the prospect of this confusion, we carried out a small scale experiment with one person riding both an escalator and a lift in a shopping mall in Sydney while holding an Android phone in the hand. The phone was equipped with a tri-axial accelerometer. Accelerometer readings were recorded at 50 ms intervals for three different activities, walking, riding escalator, and riding lift. Fig. 5 shows the readings of accelerometer for these three different activities. It is clear that both escalator and lift can be immediately detected from walking, but escalator and lift have very similar signals. It may be difficult to separate escalator from lift with 100% accuracy, leading to confusion.

We therefore considered a total of 18 facilities, 9 escalators and 9 lifts, assuming they belong to the same class of facility. Fig. 6 compares the results under this assumption with those obtained under no-confusion case. We can see that the mismatch probability increases dramatically when lifts could be confused with escalators and vice versa. This happens due to reduced proximity, which in turn increases overlaps. For escalators, the mismatch probability could be reduced on average by 68% for \( \sigma = 0.5 \) and 55% for \( \sigma = 1 \), and for lifts by 65% for \( \sigma = 0.5 \) and 56% for \( \sigma = 1 \) if escalators and lifts could be distinguished with high accuracy. This result underscores the need for developing advanced signal processing and classification techniques for smartphones that can unambiguously detect all types of activities related to indoor facilities.

### V. Conclusion and Future Work

AMM is seen as a potential candidate to correct positioning errors of PDR or other indoor localization techniques. However, AMM is not always perfect and may introduce positioning errors of its own through mismatch. Thus, there is a degree of uncertainty for how well a map-matching algorithm will perform for a given map. We have attempted to shed light on this uncertainty by proposing a methodology to derive the mismatch probability of AMM based on the NOM algorithm. Using analytical geometry, we have demonstrated that the NOM algorithm produces good results only when the facilities are located far from each other relative to the PDR sensor error, but performance degrades rapidly as the ratio of facility proximity to sensor error reduces. The proposed methodology is applied to one of the floor maps of Sydney airport. The results show that mismatch probability for one facility can vary significantly from another in the same floor due to uneven facility density in a given floor. This knowledge can be useful for a pedestrian navigation or any other location-based applications that intend to use AMM as a viable technology for indoor positioning. For example, if the mismatch probability is above a target threshold at a given floor area, the application may choose not to use AMM in that area and use other options instead. We have also found that in Sydney airport, lifts and escalators are usually located in the same area. However, this may lead to large increase in mismatch probability unless we can find ways to distinguish between an escalator and a lift from their corresponding accelerometer signals, which appear to be very similar to each other.

The current work can be extended in several ways. First, the circular error region does not take into consideration the topology of interior. For example, with walls and other separators, the actual walking distance may significantly vary from the Euclidean distance between any two locations. One could consider more realistic error regions by incorporating interior topology information (if available) into the model. Second, the NOM algorithm makes use of only the estimated position information to achieve the matching. We are likely to achieve better matching accuracy (reduced mismatch probability) if the matching algorithm uses the heading information as well. In that case, the current methodology to compute mismatch probability would have to be extended to accommodate heading information. Finally, we have not considered the case when different facility classes are confused with each other with different probabilities (in the AD module). Extending the

![Fig. 5](image_url) Acceleration signals along x-, y- and z- axes for different activities (a) Escalator, (b) Lift, and (c) Walking

Information such as Table IV can be precomputed and stored in a smartphone, which can be used by some advanced decision making process during run-time. For example, if the PDR location estimate is within the error region of escalator 9 when an escalator activity is detected, an application may decide not to recalibrate PDR with AMM due to high mismatch probability.
current model to consider such confusion matrix would be an interesting future work.

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REFERENCES

APPENDIX

DERIVATION OF MISMATCH PROBABILITY

This appendix presents a proof of the mismatching probability formula given in (5).

In Fig. 7(a), let \( P \) and \( Q \) be the centers of the two circles with equal radius \( r \), \( d \) be the distance \( PQ \), the separation between the two centers, and \( \theta \) be the central angle \((BPC)\).

In Fig. 7(b), let \( M \) be the mid-point of the chord \( BC \), and \( \phi \) be the measure of the angle \((BPM)\). We note that:

\[
PM = \frac{d}{2}, \quad BP = r, \quad BM = \sqrt{\frac{4r^2 - d^2}{2}}, \quad \theta = 2\phi, \quad \text{and}
\]
the overlapping area, \( A \), as shaded in Fig. 7(a) is twice the area of the shaded segment in Fig. 7(b).

We also observe that, The shaded area in Fig. 7(a) is the difference between the area of the minor sector \( PBC \) and the triangle \( PBC \), which is given by:

\[
\frac{1}{2}r^2(\theta - \sin \theta) = \frac{1}{2}r^2(2\phi - \sin 2\phi) = r^2(\phi - \sin \phi \cos \phi) \quad (6)
\]

Hence,

\[
A = 2r^2(\phi - \sin \phi \cos \phi) \quad (7)
\]

Now, from the right- angle triangle \( PMB \), we can find:

\[
\sin \phi = \frac{BM}{BP} = \frac{\sqrt{4r^2 - d^2}}{2r} = \sqrt{1 - \left(\frac{d}{2r}\right)^2} \quad (8)
\]

and

\[
\cos \phi = \frac{d}{2r} \quad (9)
\]

Hence,

\[
\phi = \sin^{-1} \sqrt{1 - \left(\frac{d}{2r}\right)^2} \quad (10)
\]

Substituting for \( \sin \phi, \cos \phi, \) and \( \phi \) from (8), (9), and (10) into (7) gives:

\[
A = 2r^2 \left[ \sin^{-1} \sqrt{1 - \left(\frac{d}{2r}\right)^2} - \left(\frac{d}{2r}\right) \sqrt{1 - \left(\frac{d}{2r}\right)^2} \right] \quad (11)
\]

Finally, it follows that:

\[
p = \frac{A}{2\pi r^2} \quad (12)
\]

Substituting for \( A \) from (11) into (12), gives:

\[
p = \frac{\sin^{-1} \sqrt{1 - \left(\frac{d}{2r}\right)^2} - \left(\frac{d}{2r}\right) \sqrt{1 - \left(\frac{d}{2r}\right)^2}}{\pi} \quad (13)
\]