An apparent paradox:
Common Knowledge seems to be essential for coordinated action,
BUT, in many contexts (e.g. asynchronous message passing,
unreliable message passing, unbounded message delivery), common
knowledge is hard to attain.

Say that $\mathcal{I}$ is a finite state interpreted system if for all agents $i$, the set
$L_i = \{ r_i(n) \mid (r,n) \text{ a point of } \mathcal{I} \}$
is finite.
(Note that the number of points is still infinite, as may be the
number of environment states $r_e(n)$.
Example: Let $E$ be a finite state environment, as defined in Lecture
5. Then $\mathcal{I}^{\text{obs}}(E)$ is a finite state interpreted system.

Theorem: Let $\mathcal{I}$ be an interpreted finite state system, let $l$ be the
minimum of $|L_i|$, and define $k = 2l - 1$. Then
$\mathcal{I} \models C_{\mathcal{G}} \phi \iff E_{\mathcal{G}}^k \phi$.
So common knowledge collapses to a finite level of knowledge in such systems.
Recall from Lecture 5:
In a system where agents have perfect recall, and a “hello” message is either delivered immediately, or with a delay of one tick of the clock, the fact “the message has been delivered” is never common knowledge.

More generally,

Scenario 1: if a message sent at time $m$ is delivered at a random time between $m + 1$ and $m + \epsilon$, then for a “hello” message, the fact “the message has been delivered” is never common knowledge.

A variant:

Scenario 2: Suppose a message sent at time $m$ is delivered by time $m + \epsilon$. Suppose Alice and Bob use the same global clock and have synchronous perfect recall.

At time $m$, Alice sends to Bob: “this message is being sent at time $m$: hello”.

Then at time $m + \epsilon$, it is common knowledge to Alice and Bob that the message has been delivered.

What is the difference in the two scenarios?
In the first, Bob has uncertainty about when the message was sent.
In the second, Bob is certain about the sending time, and there is a simultaneous event common to Alice and Bob: time $= m + \epsilon$.

Events
An event in a system is a subset $e$ of the set of points
Event $e$ is local to agent $i$ if there is a set $X \subseteq L_i$ of agent $i$’s local states such that $(r, m) \in e$ iff $r_i(m) \in X$.
A state event is an event $e$ such that there exists a set $X$ of global states such that $(r, m) \in e$ iff $r(m) \in X$. 
Given a state event \( e \), we can define an atomic proposition \( \phi_e \) such that \( (I, r, m) \models \phi_e \) iff \( (r, m) \in e \).

Conversely, for every formula \( \phi \), there is an event

\[
e_{\exists}(\phi) = \{(r, m) \mid (I, r, m) \models \phi\}
\]

**Lemma:** An event \( e \) is local to process \( i \) in a system \( I \) iff \( I \models \psi_e \Rightarrow K_i \psi_e \).

**Lemma:** \( I \models \phi \Rightarrow K_i \psi_e \) iff \( ev_I(\phi) \) is local to process \( i \) in \( I \).

**Theorem:**
(a) For every formula \( \phi \) the ensemble for \( G \) defined by \( e(i) = ev_I(K_i C_G \phi) \) is perfectly coordinated.
(b) If \( e \) is a perfectly coordinated ensemble for \( G \), then for every \( i \in G \) we have \( I \models \psi_{e(i)} \Rightarrow K_i C_G \psi_{e(i)} \).

**Temporal Imprecision**

In real systems, clock drift is an issue.

A system \( R \) has temporal imprecision if for all points \( (r, m) \) and processes \( i \neq j \), there exists a point \( (r', m') \) such that \( r'_j(m') = r_i(m) \) but \( r'_j(m') = r_j(m + 1) \).

(\( i \) is unsure about \( j \)'s rate of progress.)
Define ensemble \( e \) to be \textit{nontrivial} if there exists a run \( r \) and times \( m, m' \) such that \((r, m) \in \bigcup_{i \in G} e(i)\) but \((r, m') \not\in \bigcup_{i \in G} e(i)\)

**Proposition:** In a system with temporal imprecision, there are no nontrivial perfectly coordinated ensembles for \( G \), if \( |G| \geq 2 \).

**Corollary:** If \( I \) is a system with temporal imprecision, \( \phi \) is any formula and \( |G| \geq 2 \) then for all runs \( r \) and times \( m \), if \((I, r, m) \models C_G \phi\) then \((I, r, 0) \models C_G \phi\).

**Fixpoints**

Consider the equivalence

\[ C_G \phi \iff E_G (\phi \land C_G \phi) \]

We can understand this as saying that \( C_G \phi \) is a solution of the “equation”

\[ x \iff E_G (\phi \land x) \]

This can be made precise using functions

\[ f : \mathcal{P}(\text{Points}(I)) \to \mathcal{P}(\text{Points}(I)) \]

Extend the language by adding a proposition \( x \)

For each formula \( \phi \) in the extended language, define

\[ f_{\phi} : \mathcal{P}(\text{Points}(I)) \to \mathcal{P}(\text{Points}(I)) \]

such that intuitively, \( f_{\phi}(A) \) is the set of points satisfying \( \phi \) if we interpret \( x \) as being true at a point \((r, m)\) iff \((r, m) \in A\).

Inductively,

\[
\begin{align*}
  f_{x}(A) &= A \\
  f_{\phi}(X) &= \{(r, m) \mid \pi(r(m), p) = 1\} \\
  f_{\neg \phi}(A) &= \text{Points}(I) \setminus f_{\phi}(A) \\
  f_{\phi \land \psi}(A) &= f_{\phi}(A) \cap f_{\psi}(A) \\
  f_{K, i \phi}(A) &= \{(r, m) \mid (r, m) \sim_i (r', m') \Rightarrow (r', m') \in f_{\phi}(A)\}
\end{align*}
\]

\[
\begin{align*}
  f_{E_G \phi}(A) &= \bigcap_{i \in G} f_{K, i \phi}(A) \\
  f_{C_G \phi}(A) &= \bigcap_{l > 0} f_{l E_G \phi}(A)
\end{align*}
\]

where \( f^0(A) = A \) and \( f^{l+1}(A) = f(f^l(A)) \).
The following justifies the intuitive interpretation of $f_\phi$ given above.

Write $\phi^I$ for $\{(r,m) \mid (I, r, m) \models \phi\}$. Then for every formula $\phi$ that does not contain $x$, we have

$$f_\phi(A) = \phi^I$$

More generally, for formulas $\phi$ that contain $x$, write $\phi[x/\psi]$ for the result of replacing all occurrences of $x$ in $\phi$ by $\psi$. Then

$$f_\phi(\psi^I) = (\phi[x/\psi])^I$$

Now note that

$$I \models C_G \phi \iff E_G (\phi \land C_G \phi)$$

is equivalent to

$$(C_G \phi)^I = (E_G (\phi \land C_G \phi))^I$$

which is equivalent to

$$f_{E_G (\phi \land x)}(A) = A$$

where $A = (C_G \phi)^I$.

Say: $(C_G \phi)^I$ is a fixpoint of $f_{E_G (\phi \land x)}$

Suppose $f : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ and $A \subseteq S$

A is a fixpoint of $f$ if $f(A) = A$

B is a greatest fixpoint of $f$ if $B$ is a fixpoint of $f$ and for all fixpoints $A$ of $f$, we have $A \subseteq B$.

B is a least fixpoint of $f$ if $B$ is a fixpoint of $f$ and for all fixpoints $A$ of $f$, we have $B \subseteq A$.

$f$ is monotone if for all sets $A \subseteq B$, we have $f(A) \subseteq f(B)$.

**Lemma:** If every occurrence of $x$ in $\phi$ occurs in the scope of an even number of negation symbols, then the function $f_\phi$ is monotone.

(Example: $x$ occurs in the scope of an even number (viz. 0) of negation symbols in $E_G (\phi \land x)$.)

**Lemma:** Every monotone function has a greatest and a least fixpoint.
Mu-calculus
Extend the language by adding operators for least and greatest fixpoints.
For every monotone formula \( \phi \) (such that \( x \) occurs in the scope of an even number of negation symbols), the following are formulas:

- \( \mu x[\phi] \) - the least fixpoint of \( \phi(x) \)
- \( \nu x[\phi] \) - the greatest fixpoint of \( \phi(x) \)

Semantically:
- \( f_{\mu x}[\phi](A) \) is the least fixpoint of \( f_{\phi} \)
- \( f_{\nu x}[\phi](A) \) is the greatest fixpoint of \( f_{\phi} \)

(\( I, r, m \)) \( \models \mu x[\phi] \) iff \( (r, m) \in f_{\mu x}[\phi](\emptyset) \)
(\( I, r, m \)) \( \models \nu x[\phi] \) iff \( (r, m) \in f_{\nu x}[\phi](\emptyset) \)

Common Knowledge expressed in mu-calculus

\[ \models C_G(\phi) \iff \nu x(E_G(\phi \land x)) \]

Properties of mu-calculus operators

**Lemma:** let \( \psi \) be a monotone formula and let \( I \) be an interpreted system. Then

(a) \( \models \psi[x/\nu x[\psi]] \iff \nu x[\psi] \), and
(b) if \( I \models \phi \Rightarrow \psi[x/\phi] \) then \( I \models \phi \Rightarrow \nu x[\psi] \).

Note that if we take \( \psi \) to be \( E_G(\phi \land x) \), where \( C_G \phi = \nu x(\psi) \), (a) says:

\[ \models E_G(\phi \land C_G \phi) \iff C_G \phi \]

If we take \( \psi \) to be \( E_G(\psi \land x) \), then (b) says: if \( I \models \phi \Rightarrow E_G(\psi \land \phi) \) then \( I \models \phi \Rightarrow C_G \psi \).

Approximations to Common Knowledge

We have seen
- simultaneous agreement requires common knowledge of agreement
- common knowledge is sometimes impossible to obtain.

Questions:
- Can we make do with weaker notions of agreement?
- What types of knowledge do such weaker forms of agreement require?
Epsilon-coordination

Say that an ensemble \( e \) for \( G \) is \( \varepsilon \)-coordinated if the local events in \( e \) never happen more than \( \varepsilon \) time units apart, i.e., if \( (r, m) \in e(i) \) for some \( i \in G \), then there exists an interval \( I = [m', m' + \varepsilon] \) such that \( m \in I \) and for all \( j \in G \) there exists \( m_j \in I \) such that \( (r, m_j) \in e(j) \).

Example: suppose that messages sent by \( A \) at time \( m \) are guaranteed to be delivered to \( B \) by time \( m + \varepsilon \).

Then the event \( e \) for \( \{A, B\} \) defined by \( e(i) = (K_i \text{sent}_A(\mu))^T \) for \( i \in \{A, B\} \) is \( \varepsilon \)-coordinated.

Epsilon-group knowledge

Define \( E_G^\varepsilon \phi \) by

\[ (I, r, m) \models E_G^\varepsilon \phi \text{ if there exists an interval } I = [m', m' + \varepsilon] \text{ such that } m \in I \text{ and for all } j \in G \text{ there exists } m_j \in I \text{ such that } (I, r, m_j) \models K_j \phi. \]

In the example: \( I \models \text{sent}_A(\mu) \Rightarrow E_{\{A,B\}}^\varepsilon(\text{sent}_A(\mu)) \)

Epsilon-Common Knowledge

\[ C_G^\varepsilon \phi = \nu x [E_G^\varepsilon(\phi \land x)] \]

This has the following properties:

\[ \models E_G^\varepsilon(\phi \land C_G^\varepsilon \phi) \iff C_G^\varepsilon \phi \]

If \( I \models \phi \Rightarrow E_G^\varepsilon(\psi \land \phi) \) then \( I \models \phi \Rightarrow C_G^\varepsilon \psi. \)

\[ \models C_G^\varepsilon(\phi) \Rightarrow C_G^\varepsilon(C_G^\varepsilon \phi) \text{ (positive introspection)} \]

(but not the other S5 properties!)
Relating Epsilon-coordination and Epsilon-Common Knowledge

**Proposition:** Let $\mathcal{I}$ be an interpreted system and $G$ a group of agents.

(a) For every formula $\phi$, the ensemble $e$ for $G$ defined by 
$e(i) = ev_I(K_iC^\epsilon_G\phi)$ is $\epsilon$-coordinated.

(b) If $e$ is an $\epsilon$-coordinated ensemble for $G$, then for every $i \in G$ we have $\mathcal{I} \models \psi_{e(i)} \Rightarrow K_iC^\epsilon_G\psi_{e(i)}$.

(c) If $e$ is an $\epsilon$-coordinated ensemble for $G$, then $\mathcal{I} \models \psi_e \Rightarrow K_iC^\epsilon_G\psi_e$.

Eventual Coordination

An ensemble $e$ for $G$ is *eventually-coordinated* if for every run $r$, if $(r, m) \in e(i)$ for some $i \in G$, then for all $j \in G$ there exists $m_j$ such that $(r, m_j) \in e(j)$.

This similarly corresponds to *eventual common knowledge*, defined by 

$$C^\diamond_G\phi = \nu x[E^\diamond_G(\phi \land x)]$$

where

$(\mathcal{I}, r, m) \models E^\diamond_G\phi$ if for each $i \in G$ there exists $m_i$ such that 
$(\mathcal{I}, r, m_i) \models K_i\phi$.

More coordination concepts

**Time-stamped coordination:** each agent has a local clock. There exists a time $t$ such that the event $e(i)$ happens for each $i \in G$ at time $t$ on agent $i$'s local clock.

**Probabilistic Coordination:** there exists an $\epsilon$ such that the events $e(i)$ happen simultaneously with probability $1 - \epsilon$.

A notion of common knowledge can be defined for each such that a similar correspondence holds.