Environments (transition form)

An environment in transition form is a tuple of the form $E = \langle S_e, I_e, T, O, \pi_e \rangle$ where

1. $S_e$ is a set of states of the environment.
2. $I_e \subseteq S_e$ is the set of initial states of the environment.
3. $T \subseteq S_e \times S_e$ is a transition relation.
4. $O$ is a tuple $\langle O_1, \ldots, O_n \rangle$ such that for each $i = 1..n$, $O_i : S_e \rightarrow O$ is an observation function $O$.
5. $\pi_e : S_e \times \text{Prop} \rightarrow \{0, 1\}$ is a valuation.

Assume $T$ is serial: $\forall s \in S_e \exists t \in S_e (sTt)$

A run of an environment $E$ is an infinite sequence $\rho = s_0s_1\ldots$ of states of $E$ such that

1. $s_0 \in I_e$,
2. $s_k Ts_{k+1}$ for all $k \geq 0$,

A trace of $E$ is a finite sequence $\tau = s_0s_1\ldots s_m$ of states satisfying conditions 1 and 2.

Local state defined wrt a view

Let $\rho$ be a run of $E$. A view associates a local state with each agent at each point of time, determining a mapping $\rho^v : N \rightarrow L^n \times S_e$

In all cases $\rho_i^v(m) = \rho(m)$

Examples:

1. The observational view: $\rho_i^{obs}(m) = O_i(\rho(m))$
2. The synchronous perfect recall view: $\rho_i^{s_{pr}}(m) = O_i(\rho(0))\ldots O_i(\rho(m))$
3. The asynchronous perfect recall view: $\hat{\rho}_i^{s_{pr}}(m)$ is $\rho_i^{s_{pr}}(m)$ with consecutive repetitions removed.
System Generated by an Environment wrt a View

Let \( v \) be a view of an environment \( E \). Define \( I^v(E) = (R^v(E), \pi) \) to be the interpreted system with

1. \( R^v(E) \) the set of \( \rho^v \) such that \( \rho \) is a run of \( E \).
2. \( \pi(r, m, p) = \pi_e(r_e(m), p) \) for all \( r \in R^v(E), p \in \Phi \)

Recall, for each agent \( i \) we define the relation \( \sim_i \) on points by

\[(r, m) \sim_i (r', m') \text{ if } r_i(m) = r'_i(m).\]

Given a point \((r, m)\) of \( I^v(E)\), define

\[\text{trace}(r, m) = r_e(0) \ldots r_e(m).\]

For two traces \( \tau, \tau' \), define \( \tau \sim \tau' \) if there exist points \((r, m), (r', m')\) such that \( \text{trace}(r, m) = \tau \) and \( \text{trace}(r', m') = \tau' \) and \((r, m) \sim_i (r', m')\).

Proposition: Suppose \((r, m), (r', m)\) are points of \( I^v(E) \) and let \( \varphi \in L_{\{K_1, \ldots, K_n, C\}} \). If \( \text{trace}(r, m) = \text{trace}(r', m') \) then \( I^v(E), (r, m) \models \varphi \) iff \( I^v(E), (r', m') \models \varphi \).

If \( \tau \) is a trace of \( E \), write \( I^v(E), \tau \models \varphi \) when \( I^v(E), (r, m) \models \varphi \) for some point \((r, m)\) with \( \text{trace}(r, m) = \tau \).

Consider an environment \( E \) in which

- agent \( s \) (sender) can send the single message “hello” to agent \( r \) (receiver), but can only do this once
- agent \( s \) observes a variable that records whether or not the message has been sent
- agent \( r \) observes a variable that records whether the message has arrived
- the channel either delivers the message either immediately, or with a delay of one second
- the proposition \( p \) means “the message has arrived”
\[ I_c = \{ w \} \]  
\[ \pi_c(x, p) = \text{true} \text{ iff } x = d. \]

\[ O_s(w) = \bot, \quad O_s(t) = O_s(d) = \text{sent} \]
\[ O_r(w) = O_r(t) = \bot, \quad O_r(d) = \text{rcvd} \]

**Message transmission example (observational view)**

Suppose agent \( s \) sends the message at time 1, and the environment delivers the message immediately, then the agents wait for \( n - 1 \) ticks of the clock, i.e. consider the trace \( wd^{n-1} \)

Under the observational view,

- \( wd^{n-1} \mathrel{\sim_r} \tau \) implies \( \text{fin}(\tau) = d \)
- \( wd^{n-1} \mathrel{\sim_s} w^{n-1}t \)

Thus \( I^{\text{obs}}(E), wd^{n-1} \models K_r p \) but \( I^{\text{obs}}(E), wd^{n-1} \models \neg K_s p. \)
Message transmission example (synchronous perfect recall view)

Under the perfect recall view,
\[ \{wd^{n-1}\text{spr}_{E}\} = (\bot \cdot (\text{sent})^{n-1}) = \{wd^{n-2}\text{spr}_{E}\} \]
so \( wd^{n-1} \sim_s wd^{n-2} \).

More generally, for each length \( n \):
\[ wd^{n-1} \sim_s wd^{n-2} \sim_r w^{2d^{n-2}} \sim_s w^{2td^{n-3}} \ldots \]
\[ \ldots \sim_r w^{n-1}d \sim_s w^{n-1}t \sim_r w^n \]

S5\(_n\) Kripke Structures

An S5\(_n\) Kripke structure is a tuple \( M = \langle W, K_1, \ldots, K_n, \pi \rangle \) where

1. \( W \) is a set of worlds
2. \( K_i \) is an equivalence relation on \( W \) for each \( i = 1 \ldots n \)
3. \( \pi : W \times \Phi \rightarrow \{0, 1\} \) is an assignment

Define \( K_C = (\bigcup_i K_i)^* \)

1. \( M, w \models K_i \phi \) if \( M, w' \models \phi \) for all \( w' \mathcal{K}_iw \)
2. \( M, w \models C \phi \) if \( M, w' \models \phi \) for all \( w' \mathcal{K}_Cw \)
Given an environment $E$ and view $v$, define $M^v_E = \langle \text{traces}(E), \sim_1, \ldots, \sim_n, \pi \rangle$ where the $\sim_i$ are the equivalence relations on traces defined wrt the view and $\pi(\tau, p) = \pi_e(\text{fin}(\tau), p)$.

**Proposition**: For $\tau \in \text{traces}(E)$ and $\phi \in \mathcal{L}_{\{K_1, \ldots, K_n, C\}}$, $M^v_E, \tau \models \phi$ iff $I^v(E), \tau \models \phi$.

**Corollary**: For $\varphi \in \mathcal{L}_{\{K_1, \ldots, K_n, C\}}$, determining whether $I^\text{obs}(E), \tau \models \varphi$ can be done in time $O(|E| \cdot |\varphi|)$.

**Proposition**: For $\phi \in \mathcal{L}_{\{K_1, \ldots, K_n, C\}}$, we have $T^\text{obs}(E), \tau \models \phi$ iff $M, \text{fin}(\tau) \models \phi$.

**Progression Structures**

A *progression structure* for environment $E$ is a pair $\langle M, \sigma \rangle$ consisting of an $S5_n$ Kripke structure $M = \langle W, K_1, \ldots, K_n, \pi \rangle$ and a state mapping $\sigma : W \rightarrow S_e$ such that

$$\pi(w, p) = \pi_e(\sigma(w), p)$$

for all $w \in W$ and $p \in \text{Prop}$.

**Example**: $P_{E,n} = \langle M_n, \text{fin} \rangle$, where $M_n$ is the substructure of $M^\text{spr}_E$ consisting of the traces of length $n$. 

Model Checking at a Trace (Observational View)

Let $E = \langle S_e, I_e, T, O, \pi_e \rangle$ be a finite state environment.

A state $t \in S_e$ is reachable if $sT^*t$ for some $s \in I_e$.

Define $M = \langle W, K_1, \ldots, K_n, \pi \rangle$ by

1. $W$ is the set of reachable states of $E$.
2. $sK_it$ iff $O_i(s) = O_i(t)$
3. $\pi = \pi_e$
If $P = \langle M, \sigma \rangle$, Write $P, w \models \phi$ if $M, w \models \phi$.

The environment $E$ operates on its progression structures by

$$\langle M, \sigma \rangle * E = \langle M', \sigma' \rangle$$

where $M' = \langle W', K'_1, \ldots, K'_n, \pi' \rangle$ is the Kripke structure with

1. $W' = \{ (w, s) \mid w \in W, s \in S_c, \sigma(w)Ts \}$
2. $(w, s)K'_i(v, t)$ iff $wK_i v$ and $O_i(s) = O_i(t)$
3. $\pi'((w, s), p) = \pi_c(s)$
4. $\sigma'((w, s)) = s$

**Proposition:** Let $\tau = s_0 \ldots s_k$ be a trace of an environment $E$ and let $\phi \in L(K_1, \ldots, K_n, C)$. Then $I^{\text{spr}}(E), \tau \models \phi$ iff $P_{E,k}, w_\tau \models \phi$, where $w_\tau = ((\ldots (((s_0, s_1), s_2), s_3), \ldots, s_k)$.

**Proposition:** $P_{E,n+1}$ is isomorphic to $P_{E,n} * E$

This means we can check $I^{\text{spr}}(E), \tau \models \phi$ as follows:

1. Construct $P_{E,0}$,
2. For $i = 1 \ldots k$ construct $P_{E,k} = P_{E,k-1} * E$
3. Check $P_{E,k}, w_\tau \models \phi$ using finite state model checking.
References

Constructing finite state implementations of knowledge based programs with perfect recall, R. van der Meyden, PRICAI workshop on theoretical and practical foundations of intelligent agents, 1996