Rules for Equality, Applications of Reasoning in Predicate Calculus

Recall that in the semantics of predicate calculus, whereas the meaning of ordinary predicate symbols can be any relation on the universe of the model, we defined the equality symbol "=" to be a special predicate, required to have its interpretation $=^{\mathcal{M}}$ in a structure $\mathcal{M}$ be equal to

$$\{(x, x) \mid x \in A^{\mathcal{M}}\}$$

We need some proof rules to capture this definition ...

A rule for introducing equality

$$\frac{}{t = t} \quad =i$$

where t is a term

A rule for eliminating equality:

$$\frac{t_1 = t_2 \qquad \phi[X \mapsto t_1]}{\phi[X \mapsto t_2]} \quad =e$$

provided t_1 and t_2 are free for X in ϕ.

Some properties of equality provable using these rules:

$t_1 = t_2 \longrightarrow t_2 = t_1$ (Symmetry)

$(t_1 = t_2 \wedge t_2 = t_3) \longrightarrow t_1 = t_3$ (transitivity)

Example:

Suppose f and g are surjective functions from the universe to itself. Then $f \circ g$ is a surjective function.

$$\forall X \exists Y (f(Y) = X)$$
$$\forall X \exists Y (g(Y) = X)$$
$$\vdash$$
$$\forall X \exists Y (f(g(Y)) = X)$$

Suppose we have a database of family relations, with the following integrity constraints:

1. Everybody has a father.
2. Nobody has more than one father.

Then the following queries are equivalent (i.e., always yield the same result):

1. find me all people, all of whose fathers are Greek.
2. find me all people who have a Greek father.

$$\forall \mathtt{X} \exists \mathtt{Y}(\mathtt{father}(\mathtt{X}, \mathtt{Y}))$$
$$\forall \mathtt{X} \forall \mathtt{Y}_1 \forall \mathtt{Y}_2(\mathtt{father}(\mathtt{X}, \mathtt{Y}_1) \wedge \mathtt{father}(\mathtt{X}, \mathtt{Y}_2) \longrightarrow \mathtt{Y}_1 = \mathtt{Y}_2)$$
$$\vdash$$
$$\forall \mathtt{X}(\forall \mathtt{Y}(\mathtt{father}(\mathtt{X}, \mathtt{Y}) \longrightarrow \mathtt{Greek}(\mathtt{Y})) \longleftrightarrow \exists \mathtt{Y}(\mathtt{father}(\mathtt{X}, \mathtt{Y}) \wedge \mathtt{Greek}(\mathtt{Y})))$$