COMP4415 Assignment 3 – 2003
Due: Fri April 4

(None of these problems should require much beyond working with definitions and constructions discussed in class. See pp. 56-60 of the text.)

1. Consider a general logic, given by defining the formulas ϕ, the models M and a relation of satisfaction of a formula by a model, written $M \models \phi$. As usual, if M is a model define satisfaction of a set S of formulas in M, written $M \models S$, by $M \models \phi$ for all $\phi \in S$. Additionally, define a formula ϕ to be a logical consequence of a set S of formulas by $S \models \phi$ when for all models M, if $M \models S$ then $M \models \phi$.

   The logic is compact if $S \models \phi$ implies that there exists a finite subset S' of S such that $S' \models \phi$. Show that the converse *always* holds, i.e. if there exists a finite subset S' of S such that $S' \models \phi$, then $S \models \phi$.

2. Let S be the set of clauses $\{\ \{A, B, C\},\ \{\neg A\},\ \{\neg B, C\},\ \{\neg C\},\ \}$. Construct the tree of $S^{l_1,\ldots,l_n}$ stopping at leaves containing the empty clause, in two different ways: (1) ordering the basic propositions A, B, C, and (2) ordering the basic propositions C, B, A.

3. (See Theorem 8.22 in the text, we'll discuss this some more at the lecture on April 2).

   We started to discuss in class the following lemma: if $S^l \vdash \Box$ and $S^{\bar{l}} \vdash \Box$ then $S \vdash \Box$. The proof goes as follows. Consider the resolution proof tree showing that $S^l \vdash \Box$. If all of the leaves of this are labelled by clauses in S then we already have a proof tree showing $S \vdash \Box$. Otherwise, replace each clause C occuring in this proof tree that *is above a leaf not in* S by the clause $C \cup \{\bar{l}\}$, keeping the tree structure intact. This gives a resolution proof tree showing that $S \vdash \{\bar{l}\}$. Similarly, (but swapping l and $\bar{l}$) we can construct a resolution proof tree showing that $S \vdash \{l\}$ from the resolution proof tree for $S^{\bar{l}} \vdash \Box$. Pasting these two new resolution proof trees together at the root with a single resolution step deriving $\Box$ from $\{l\}$ and $\{\bar{l}\}$ gives a resolution proof tree showing that $S \vdash \Box$.

   Inductively, apply this construction to the two trees of $S^{l_1,\ldots,l_n}$ from (1) and (2) of the previous question, working from the leaves to the root, to obtain two resolution proofs of $\Box$ from the set S. Show each step of the construction. (Note that at the leaves we have $\Box$ in $S^{l_1,\ldots,l_n}$, so the resolution proof tree corresponding to those nodes consists just of the root labelled $\Box$. At each non-leaf node labelled $X = S^{l_1,\ldots,l_n}$, we have children $S^{l_1,\ldots,l_n,p_n} = X^{p_n}$ and $S^{l_1,\ldots,l_n,\neg p_n} = X^{\neg p_n}$, fitting the pattern of the previous paragraph.)