COMP4415 Assignment 6 – 2003
Due: Wed May 7

1. (EASY)

   (a) Show that $\forall x(P(x) \rightarrow Q(f(x))) \wedge \forall x(P(x)) \wedge \exists x(\neg Q(x))$ is satisfiable. (i.e. provide a structure in which it is true.) (1 mark)

   (b) Using the semantics of predicate logic, prove that $\mathcal{A} \models \neg\exists x(\phi(x))$ iff $\mathcal{A} \models \forall x(\neg\phi(x))$. (2 marks)

   (c) Using the semantics of predicate logic, prove that if ψ is a sentence, then $\mathcal{A} \models \psi \rightarrow \exists x(\phi(x))$ iff $\mathcal{A} \models \exists x(\psi \rightarrow \phi(x))$. (2 marks)

   (d) Using the semantics of predicate logic, prove that if ψ is a sentence, then $\mathcal{A} \models \exists x(\phi(x)) \rightarrow \psi$ iff $\mathcal{A} \models \forall x(\phi(x) \rightarrow \psi)$. (2 marks)

2. (HARDER) A formula ϕ of predicate logic that is not a sentence (i.e., has free variables) may be considered to be valid if $\forall x_1 \ldots \forall x_n(\phi)$ is valid, where $x_1, \ldots, x_n$ are the free variables of ϕ.

   The following problems yield a proof of Theorem 4.8 in the book, which states:

   "Let ϕ be a quantifier free formula of predicate logic. We may view ϕ as a formula ϕ' of propositional logic by regarding every atomic subformula of ϕ as a propositional letter. With this correspondence, ϕ is valid (in the sense above) iff ϕ' is valid as a formula of propositional logic."

   Note that this yields an algorithm for deciding the validity of predicate logic formulas of this form!

   (a) Let $\phi(\mathbf{x})$ be a formula of a language $\mathcal{L}$ with free variables $\mathbf{x} = x_1, \ldots, x_n$. Let $\mathbf{c} = c_1, \ldots, c_n$ be a sequence of new constants not in $\mathcal{L}$. Prove that $\forall x_1 \ldots \forall x_n(\phi(\mathbf{x}))$ is valid iff $\phi(\mathbf{c})$ is. (That is, for each i, substitute c_i for x_i.) (1 mark)

   (b) Prove Theorem 4.8 for formulas with no free variables. (Hint: convert between structures that satisfy ϕ or $\neg\phi$ and propositional assignments.) (1 mark)

   (c) Combine the previous two exercises to prove Theorem 4.8. (1 mark)