COMP4415 Assignment 8 – 2003
Due: Wed May 28

The compactness theorem for predicate logic follows easily from what we have done in class. That is, if $S \models \phi$, then there exists a finite subset S' of S such that $S' \models \phi$. For, by completeness, if $S \models \phi$ then there exists a proof, which is a finite tree. For at most a finite number of formulas $\alpha \in S$ can we have inserted $T\alpha$ in this finite tree. Let S' be the set of such formulas. Then the proof is also a proof of ϕ from S'. By soundness it follows that $S' \models \phi$. Note that a consequence of compactness is that if every finite subset of S is satisfiable, then S is satisfiable.

We can now use compactness to get some interesting results. Given a structure $\mathcal{A}$, write $Th_{\mathcal{L}}(\mathcal{A})$ for the set of all sentences of language $\mathcal{L}$ that are true in $\mathcal{A}$.

1. (EASY, 3 marks) Let $\mathcal{L}$ be the language of arithmetic, with constants $0, 1$, binary function symbols $+, \cdot$ and binary predicate $<$. Let $\mathcal{N}$ be the structure with domain the natural numbers that gives these symbols their usual interpretation. Show that $Th_{\mathcal{L}}(\mathcal{N})$ has a model containing an "infinite number", i.e., an element that is greater than every natural number n, which is denoted by the term $1 + \ldots + 1$ (n times).

   Hint: Let c be a new constant symbol, and consider the set of formulas $Th_{\mathcal{L}}(\mathcal{N}) \cup \{c > 0,\ c > 1,\ c > 1+1,\ c > 1+1+1, \ldots\}$. Use compactness.

2. (EASY, 4 marks) If $R^{\mathcal{A}}$ is a binary relation over a domain A, its transitive closure $T(R^{\mathcal{A}})$ is the set of all pairs $(a, b) \in A \times A$ such that there exists a sequence $a = x_0, x_1, \ldots, x_n = b$ with $(x_i, x_{i+1}) \in R^{\mathcal{A}}$ for all $i = 0 \ldots n-1$.

   Say that a formula $TC(x, y)$ with two free variables x, y of the language $\mathcal{L}$ with only the binary relation symbol R, *expresses the transitive closure of* R if for all structures $\mathcal{A}$ for $\mathcal{L}$, we have

   $$\{(x, y) \in A \times A \mid \mathcal{A} \models TC(c_x, c_y)\} \ = \ T(R^{\mathcal{A}})$$

   where c_x and c_y are constants interpreted as x and y respectively and A is the domain of $\mathcal{A}$. (That is, the set of pairs of domain elements satisfying the formula is guaranteed to be equal to the transitive closure.)

   Use compactness to show that there is no formula of $\mathcal{L}$ that expresses the transitive closure.

Hint: Show first that it is possible to write a formula $\rho_n(x, y)$ that expresses "there is a path from x to y of length n. Then consider the set of formulas $\{TC(x, y)\} \cup \{\neg\rho_1(x, y),\ \neg\rho_2(x, y), \ldots\}$.

3. (HARDER, 3 marks) Show that every language $\mathcal{L}$ based on a countable number of constants and a countable number of function symbols of each arity $n > 0$, has a countable set of ground terms.

   Hint: Associate a weight with each symbol, and define the weight of a ground term to be the sum of the weights of all the occurrences of symbols it contains. Find a way to define the basic weights such that for each weight n, the total number of ground terms of weight n is finite. Then use this fact to enumerate the set of ground terms.