Applied Logic I

Introduction to AGM Approach I

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Overview

- Belief Expansion
- Belief Contraction
  - First Construction—maximal non-implying subsets
  - Maxichoice contraction
  - Full meet contraction
  - Partial meet contraction
- Recovery and Withdrawal functions
- Summary

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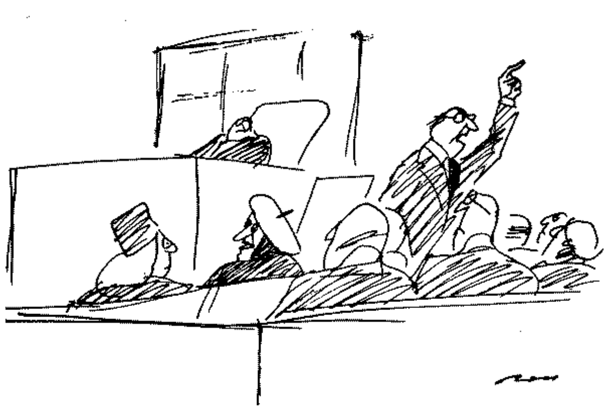

"Logic—the last refuge of a scoundrel."

Belief Expansion

- Want to add a belief(s) without giving anything up

(K+1) For any sentence α and any belief set K, $K+\alpha$ is a belief set (closure)
(K+2) $\alpha \in K+\alpha$ (success)
(K+3) $K \subseteq K+\alpha$ (inclusion)
(K+4) If $\alpha \in K$, then $K+\alpha = K$ (vacuity)
(K+5) If $K \subseteq H$, then $K+\alpha \subseteq H+\alpha$ (monotonicity)
(K+6) For all belief sets K and sentences α, $K+\alpha$ is the smallest belief set satisfying (K+1) – (K+5) (minimality)

Theorem: The expansion function + satisfies $(K+1)$ – $(K+6)$ iff $K+\alpha = Cn(K \cup \{\alpha\})$.

- There is only one AGM expansion function—classical consequence Cn
- This is not the case for AGM contraction and revision

Contraction Example

Suppose K contains the closure of the following formulae:

$rain \rightarrow wet\_grass$
$rain$
$shop\_open$
$shop\_open \rightarrow light\_on$

Consider possibilities for $K \dot{-} wet\_grass$:

$rain \rightarrow wet\_grass$	$rain$	$shop\_open$
$shop\_open$	$shop\_open$	$shop\_open \rightarrow light\_on$
$shop\_open \rightarrow light\_on$	$shop\_open \rightarrow light\_on$	

Belief Contraction

- Want to give up a belief or suspend judgement; do not want to add any beliefs

($K \dot{-} 1$)	For any sentence ϕ and any belief set K, $K \dot{-} \phi$ is a belief set	(closure)
($K \dot{-} 2$)	$K \dot{-} \phi \subseteq K$	(inclusion)
($K \dot{-} 3$)	If $\phi \notin K$, then $K \dot{-} \phi = K$	(vacuity)
($K \dot{-} 4$)	If $\not\vdash \phi$ then $\phi \notin K \dot{-} \phi$	(success)
($K \dot{-} 5$)	If $\phi \in K$, $K \subseteq (K \dot{-} \phi) + \phi$	(recovery)
($K \dot{-} 6$)	If $\vdash \phi \leftrightarrow \psi$, then $K \dot{-} \phi = K \dot{-} \psi$	(preservation)
($K \dot{-} 7$)	$K \dot{-} \phi \cap K \dot{-} \psi \subseteq K \dot{-} (\phi \wedge \psi)$	(conj. overlap)
($K \dot{-} 8$)	If $\phi \notin K \dot{-} (\phi \wedge \psi)$, then $K \dot{-} (\phi \wedge \psi) \subseteq K \dot{-} \phi$	(conj. inclusion)

- In particular, note the difference between
  - $K \dot{-} (\phi \wedge \psi)$: need only give up either ϕ or ψ
  - $K \dot{-} (\phi \vee \psi)$: must give up both ϕ and ψ

Additional Properties

The following properties follow from the AGM postulates for belief contraction.

1. If $\phi \in K$, then $(K \dot{-} \phi) + \phi \subseteq K$
2. $K \dot{-} \phi = K \cap (K \dot{-} \phi) + \neg\phi$
3. $K \dot{-} \phi \cap Cn(\{\phi\}) \subseteq K \dot{-} (\phi \wedge \psi)$
4. Either $K \dot{-} (\phi \wedge \psi) \subseteq K \dot{-} \phi$ or $K \dot{-} (\phi \wedge \psi) \subseteq K \dot{-} \psi$
5. Either $K \dot{-} (\phi \wedge \psi) = K \dot{-} \phi$ or $K \dot{-} (\phi \wedge \psi) = K \dot{-} \psi$ or $K \dot{-} (\phi \wedge \psi) = K \dot{-} \phi \cap K \dot{-} \psi$
6. If $\psi \rightarrow \phi \in K \dot{-} \phi$ and $\phi \rightarrow \psi \in K \dot{-} \psi$, then $K \dot{-} \phi = K \dot{-} \psi$

Digression—Commensurability Thesis (Levi 1991)

"Given an initial state of full belief K_1 and another state of full belief K_2, there is always a sequence of expansions and contractions, beginning with K_1, remaining within the state of potential states of full belief and terminating with K_2."

Levi Identity: $K * \phi = (K \dot{-} \neg\phi) + \phi$

- Given a contraction function we can construct a (associated) revision function: contract anything that would cause the addition of ϕ to lead to inconsistency and then expand by ϕ

Harper Identity: $K \dot{-} \phi = K \cap K * \neg\phi$

- Given a revision function we can construct a (associated) contraction function: revise by $\neg\phi$ (which would remove ϕ if it were currently believed so as to have a consistent revision) and keep those beliefs in K that were maintained by this revision

Constructing Contraction Functions

- AGM contraction function is simply a mapping from a belief set and a formula to a new belief set that satisfies certain restrictions
- How would we go about "constructing" such a function especially if we wanted to implement one in a computer program?
- Storing all the possible mappings is out of the question!
- There are a number of constructions for contraction functions that we shall investigate
- The first idea is to consider removing just enough formulae from K so that it no longer implies ϕ

Maximal Non-implying Subsets

Definition: K' is a maximal subset of K that fails to imply ϕ (a ϕ-remainder) iff
(i) $K' \subseteq K$
(ii) $\phi \notin K'$
(iii) for any $\psi \in K$ such that $\psi \notin K'$, $\psi \rightarrow \phi \in K'$

We denote by $K \perp \phi$ the set of all such maximal non-implying subsets.

Definition: A selection function $\gamma : 2^{\mathcal{K}} \rightarrow \mathcal{K}$ is a function such that for any $K \in \mathcal{K}$ and $\phi \in \mathcal{L}$, $\emptyset \neq \gamma(K \perp \phi) \subseteq K \perp \phi$ whenever $K \perp \phi \neq \emptyset$ and K otherwise. If γ always returns a singleton whenever $K \perp \phi \neq \emptyset$, then γ is referred to as an opinionated selection function.

Maxichoice Contraction

Idea: select the "best" element from $K \perp \alpha$ (minimal change).

Definition: Let γ be an opinionated selection function. A maxichoice contraction function over K may be defined as follows

(Def Max) $$K \dot{-} \phi = \begin{cases} \gamma(K \perp \phi) & \text{whenever } K \perp \phi \neq \emptyset \\ K & \text{otherwise} \end{cases}$$

Theorem: If $\dot{-}$ is a maxichoice contraction function over K, then it satisfies $(K \dot{-} 1) – (K \dot{-} 6)$.

Theorem: If a revision function $*$ is obtained from a maxichoice contraction function $\dot{-}$ via the Levi Identity, then for any ϕ such that $\neg\phi \in K$, $K * \phi$ is complete.

- Maxichoice doesn't remove enough

Full Meet Contraction

Idea: All or nothing!

Definition: A full meet contraction over K may be defined as follows

(Def Full) $$K \dot{-} \phi = \begin{cases} \bigcap(K \perp \phi) & \text{whenever } K \perp \phi \neq \emptyset \\ K & \text{otherwise} \end{cases}$$

Theorem: Any full meet contraction function satisfies $(K \dot{-} 1) – (K \dot{-} 6)$

Theorem: If a revision function $*$ is obtained from a full meet contraction function $\dot{-}$ via the Levi Identity, then for any ϕ such that $\neg\phi \in K$, $K * \phi = Cn(\phi)$.

- Full meet removes too much

Partial Meet Contraction

Idea: Compromise!

Definition: Let γ be a selection function. A partial meet contraction over K may be defined as follows

(Def Partial) $$K \dot{-} \phi = \begin{cases} \bigcap \gamma(K \perp \phi) & \text{whenever } K \perp \phi \neq \emptyset \\ K & \text{otherwise} \end{cases}$$

Theorem: For every belief set K, $\dot{-}$ is a partial meet contraction function iff $\dot{-}$ satisfies (K$\dot{-}$1) – (K$\dot{-}$6).

Selection Functions — more details

We can define a selection function as follows and then apply restrictions to see what properties result

Marking-off identitiy $\leq$
$\gamma(K \perp \phi) = \{K' \in K \perp \phi : K'' \leq K' \text{ for all } K'' \in K \perp \phi\}$

Definition: γ is a transitively relational iff it can be defined via a marking-off identity $\leq$ which is transitive.

Theorem:
For every belief set K, $\dot{-}$ is a transitively relational partial meet contraction function iff $\dot{-}$ satisfies (K$\dot{-}$1) – (K$\dot{-}$8).

Recovery

The opposite half of (K$\dot{-}$5) follows from (K$\dot{-}$1) – (K$\dot{-}$4) giving the following property

$$\text{If } \phi \in K, \text{ then } K = (K \dot{-} \phi) + \phi$$

Counterexample?: (Hansson 1991)

George is a murderer (m)
George is a law breaker (b)
George is a tax evader (t)

$K = Cn(\{m\} \cup \{b\})$
$m \notin K \dot{-} b$
$\neg t \notin K \supseteq K \dot{-} b$
$K \subseteq Cn(K \dot{-} b \cup \{b\}) \subseteq Cn(K \dot{-} b \cup \{t\})$
$m \in Cn(K \dot{-} b \cup \{t\})$

Withdrawals (Makinson 1986)

Idea: Let's consider functions that don't satisfy (Recovery)

Definition: A function $\dot{-}$ is a withdrawal function iff it satisfies postulates (K$\dot{-}$1) – (K$\dot{-}$4) and (K$\dot{-}$6) for contraction over K.

Definition: Two withdrawal functions $-$ and $\dot{-}$ are revision equivalent iff they generate the same revision function via the Levi Identity.

Theorem: Let K be any belief set. Then for each withdrawal operation $-$ on K, there is a unique contraction function $\dot{-}$ on K that is revision equivalent to $-$ and this $\dot{-}$ is the greatest element of $[-]$.

In other words, withdrawal functions can be partitioned into equivalence classes where the revision function associated with each member of a class behaves the same. The maximal element of each class (the one removing fewest beliefs) is an AGM contraction function (and there is only one per class).

Summary

- AGM framework characterised in terms of (intuitive) postulates and constructions
- Operations: expansion, contraction and revision (we shall look at this operation in the next lecture)
- Belief contraction and revision can be related in terms of the Levi and Harper identities
  - Essentialy this means that we only need expansion plus one of contraction or revision
- Belief contraction characterised in terms of postulates and our first construction (maximal non-implying subsets); we shall see other constructions in the following lecture
- Some postulates—such as Recovery—are open to question

