The Role of Abductive Reasoning within the Process of Belief Revision

by

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Statement

Apart from standard techniques and results, the use of the work of other authors has been indicated by references in the text. Parts of chapters 4, 5 and 6 have been published under joint authorship with my supervisor Professor Norman Y. Foo and associate supervisor Doctor Abhaya C. Nayak as indicated. In conformity with the regulations of the University of Sydney it is claimed that the work presented as part of this dissertation is my own.

Abstract

An inquiring agent is concerned with obtaining as much new, error-free, information as possible. One way of doing this is to simply incorporate information presented to an agent as is. This strategy is adopted by many belief revision frameworks including the popular AGM framework. A more natural strategy would be for the agent to first seek an explanation or justification for the new information. After doing so, it could incorporate the explanation into its epistemic state together with the new information. Such a strategy would be particularly effective if the agent's situation does not allow it to obtain new information easily. We model this strategy through the use of abductive reasoning. This allows us to then investigate the role of abductive inference within a belief revision framework based on the AGM. We not only look at the incorporation of new information but also at the removal of information.

We begin by looking at some logical aspects of abduction and to contrast it, in a pragmatic sense, with the process of induction as performed by inverse resolution. We proceed to develop an account of an abductive expansion operator in the vein of the AGM framework. A definition, postulates and several constructions, reminiscent of the AGM, are developed together with a number of representation theorems. It is also shown how abductive expansion is related to nonmonotonic inference, in particular, default reasoning. The process of contraction is then investigated and we note how abduction can already be viewed as an active part of this operation. However, abductive expansion and AGM contraction do not exhibit the dual behaviour one might expect. This leads us into an investigation of an alternate operation known as Levi-contraction. We suggest a Grove style semantic modelling and provide additional postulates in order to obtain a complete characterisation. Our emphasis on expansion and contraction is guided to a large extent by Levi's commensurability thesis which states that any revision can be achieved through a series of expansion and contraction operations. However, using our work on expansion and contraction, we briefly investigate the repercussions for an abductive revision operator determined through the Levi identity. It turns out that this problem relies heavily on that of iterated revision.

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To my parents, family, friends and those I have met along the way

Of bodies changed to other forms I tell;
You Gods, who have yourselves wrought every change,
Inspire my enterprise and lead my lay
In one continuous song from nature's first
Remote beginnings to our modern times.

Ovid, *Metamorphoses* [89] I:1–5

Prologue

. . . suppose that on a public holiday you are standing in the street in a town that has two hamburger restaurants. . . . When you meet me, eating a hamburger, you draw the conclusion that at least one of the two restaurants is open. . . . Further, seeing from a distance that one of the two restaurants has its lights on, you believe that this particular restaurant is open. . . .

When you have reached the restaurant, however, you find a sign saying that it is closed all day. The lights are only turned on for the purpose of cleaning. . . .

In contrast, suppose you had not met me or anyone else eating a hamburger. Then your only clue would have been the lights from the restaurant.

Sven Ove Hansson [40][1]

[1] This example was used by Hansson [40] to motivate the use of belief bases rather than belief sets. We shall make use of it here in a different context; to motivate the use of abductive reasoning within the process of belief revision.

Chapter 1

Introduction

Now, Erato, thy poet's mind inspire,
And fill his soul with thy celestial fire!
Virgil, *The Aenid*, VII:52–53

An inquiring agent, reasoning about its domain or "world", must maintain a record of that information which it believes to be true. It will be confronted with new information and must decide what to add and/or delete to its current stock of beliefs to reflect this fact. It is likely that the agent will do this in such a way as to fulfil certain requirements it considers important. For instance, its current stock of beliefs should be *consistent*; they should also *cohere* in some way, etc. The study of the way in which an agent should modify its stock of beliefs to deal with new information is known as *belief revision* or *belief change*. This dissertation concerns itself with the area of belief revision and the role that a particular form of reasoning known as *abductive inference* may play in it. Abductive inference (or simply abduction) is a fundamental form of logical inference alongside deduction and induction that aims to derive plausible explanations for data (the *explanandum*) in light of certain background or domain knowledge. For example, suppose you know that everyone suffering from measles develops a red rash on their skin. If someone comes to you with a red rash on their skin you might hypothesise that they are suffering from measles, for this would account for your observation. Of course, other explanations may be possible — the person may be suffering from an allergic reaction for instance. Given that an agent seeks to gain as much information about its world as possible, an abductive strategy in the belief change process can prove very useful. This is especially true when the agent's ability to acquire new information is severely limited. It is this type of belief change that we study here, although not only regarding addition to the agent's stock of beliefs but also when beliefs are to be deleted.

As noted above, abduction is only one of a larger class of important methods of inference. In contrast to abduction, deductive inference (or deduction) seeks to derive logical consequences from given facts. Modifying the previous example, from the facts that everyone suffering from measles develops a red rash and John has the measles, one can deduce that John has a red rash. Inductive inference (induction), on the other hand, attempts to extract general rules from individual instances. From instances of people suffering measles and having a red rash, one might induce the rule that everyone suffering from measles develops a red rash.

We begin by elucidating the concepts briefly touched upon above, providing a more complete description of our aims through the use of an example. The following propositions will be useful and relate partly to the example in the Prologue.

e	*I am eating a hamburger*
ph_1/ph_2	*I purchased a hamburger from Restaurant 1/Restaurant 2*
o_1/o_2	*Restaurant 1/Restaurant 2 is open*
c_1/c_2	*Restaurant 1/Restaurant 2 is being cleaned*
l_1/l_2	*The lights are on in Restaurant 1/Restaurant 2*
co_1/co_2	*The cook of Restaurant 1/Restaurant 2 is in his restaurant*
ch	*I cooked a hamburger at home*

At any particular point in time the agent will have a certain stock of beliefs. We call this stock of beliefs (expressed in a suitable language), together with any (extralogical) relationships between them that we wish to represent, the agent's *epistemic state* (alternatively, *belief state*). Consider yourself to be an agent in the situation outlined in the Prologue. Among the beliefs that you hold at the very beginning of that scenario, might be the following:

$$o_1 \wedge ph_1 \rightarrow e$$
$$o_2 \wedge ph_2 \rightarrow e$$
$$ch \rightarrow e$$
$$o_1 \rightarrow l_1$$

$o_2 \rightarrow l_2$

$c_1 \rightarrow l_1$

$c_2 \rightarrow l_2$

$co_1 \rightarrow o_1$

$co_2 \rightarrow o_2$

Of course, these beliefs have certain (logical) consequences and we assume, for the purposes of the work here, that they are also included in the agent's epistemic state. That is, we are interested in those beliefs the agent is, in a sense, committed to regardless of whether or not it is feasible to determine them in practice. Levi [65] terms this the agent's *epistemic commitment*. Such an agent is referred to as *logically omniscient*. We shall use the symbol K to refer to your beliefs (above) and their consequences. Those expressions of your language in your epistemic state K are termed *beliefs*; they are currently *believed*.[1] Similarly, those expressions whose negations are believed, are *disbelieved* while those which do not appear in K and nor do their negations are neither believed nor disbelieved (i.e., indetermined). In this way, you have different *epistemic attitudes* towards expressions in your language at certain times. More fine-grained or discerning epistemic attitudes — some form of *degree of belief*, for instance — are also possible.

As an agent you will also receive new information; again, expressed in a suitable language. This is referred to as an *epistemic input*. This epistemic input, α say, precipitates a change in epistemic state from the current state K to a new state K'_α. Belief revision is the study of how this change is performed (see Figure 1.1); in particular, the nature of the new epistemic state.[2] In the belief revision framework developed by Alchourrón, Gärdenfors and Makinson [1, 31] (henceforth referred to as the AGM framework) three types of belief change are identified:

belief expansion the epistemic input is added to the current epistemic state without removal of any existing beliefs

[1] We avoid the controversy surrounding whether the objects of beliefs are linguistic or propositional.

[2] Gärdenfors also uses the term *belief dynamics* although Levi [65] claims that the term *comparative statics* is more accurate.

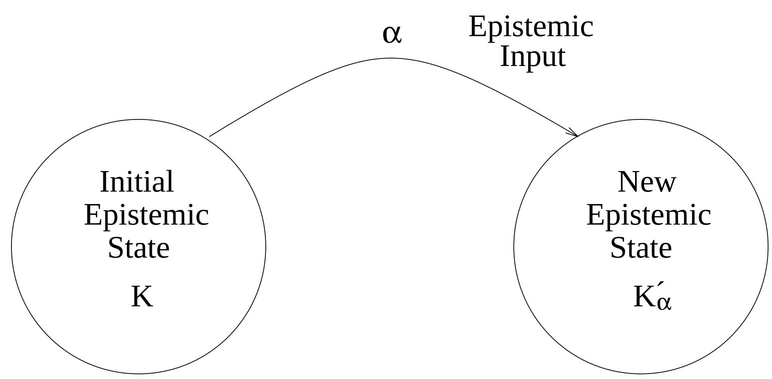
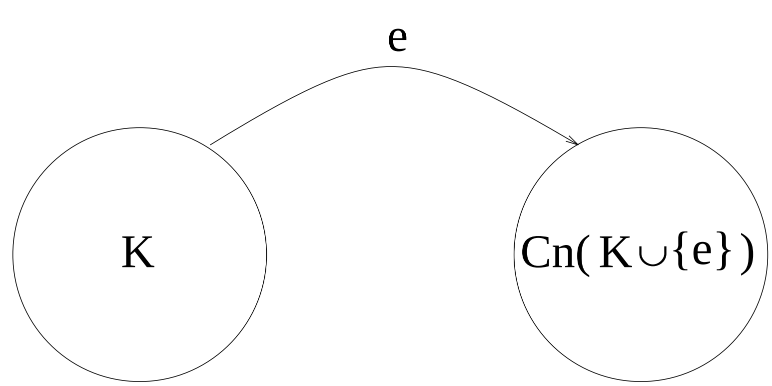


Figure 1.1: Belief revision — the basic idea (cf. [31] p.13).

belief contraction beliefs are removed from the current epistemic state in order to effect the removal of the epistemic input

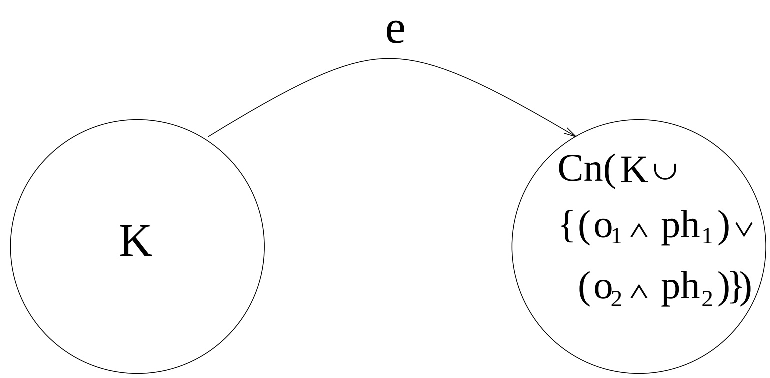
belief revision the epistemic input is incorporated into the current epistemic state but some existing beliefs may also need to be removed to maintain consistency.

Suppose that you receive the input that I am eating a hamburger, as in the example. It is likely that you would like to add this to your current stock of beliefs and, given that you do not believe anything which contradicts this input, you can expand your current belief state by adding the input (e) to your current stock of beliefs together with any consequences that result.



Many belief revision frameworks closely follow this scenario; they aim to solely incorporate the epistemic input and any resulting consequences.

However, it is our contention that a more natural and advantageous approach is for the agent to first seek some explanation or justification for the epistemic input in light of their currently held beliefs and to incorporate this explanation together with the epistemic input into their new epistemic state. That is where abductive reasoning comes in. Abduction provides us with a way of determining explanations for the epistemic input given our currently held beliefs. For example, when you see me "eating a hamburger" (e) and given your current beliefs (K) you might formulate as an explanation: "at least one of the two restaurants is open" and I bought my hamburger from it ($(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2)$). Your thirst for information could lead you to include this explanation and the original input into your new epistemic state.



Of course, several explanations may be possible (e.g., I had cooked the hamburger at home) — some possibly inconsistent — and it is up to the agent to determine which to include.

The original belief change strategy outlined clearly makes use of deductive inference. We claim that it makes a certain amount of sense to augment this strategy with abductive and inductive inference also. Levi [65] claims that an agent is interested in acquiring new information while avoiding error. If one were to use only deduction, then the amount of new information acquired is limited. More information can be acquired through the use of abduction and induction, as also noted by Levi.[3] There is also psychological evidence [108] suggesting that, for human agents, this is in fact the case although we emphasise at the outset that we are interested in developing a normative account of such belief change rather than a psychological one. An agent also seeks to avoid error which precludes the acquisition of too much information. A cautious or skeptical agent would include little

[3]In fact, Peirce [96] identifies abduction, deduction and induction as fundamental in the process of inquiry.

extra information while an incautious or credulous agent is likely to include much more. In this dissertation we take the AGM framework for belief revision as a guide and attempt to develop an account of belief change operators that use abduction. We pay a lot of attention to abductive belief expansion — which is no longer a trivial operation as in the AGM — since it, together with contraction, will be considered a primitive belief change operation and can be used to construct a revision operator. Our main aim is to investigate what role abductive inference can play in the process of belief change.

It is not only the incorporation of new information (expansion and revision) in an abductive manner that interests us here but also the removal of information (contraction). Suppose that your epistemic state has now evolved to also include the following beliefs:

l_1

o_1

co_1

That is, seeing the lights of restaurant 1 on, you concluded it was open and the cook is present. However, upon reaching restaurant 1, you "find a sign saying that it is closed all day". You wish to retract the statement "restaurant 1 is open and I bought my hamburger there" ($o_1 \wedge ph_1$). You may also retract certain beliefs that are "responsible" for this statement because the statement is a consequence of these beliefs or because your confidence in them has diminished — in this instance, that the cook is present in restaurant 1 for example. Again, abduction can be used to single out such culprits. In fact, the AGM account can be interpreted as already working in this fashion. We investigate another form of contraction, however, that can be considered a dual of our account of abductive expansion. We also discuss other work in this regard.

The main aim of the present dissertation is to investigate, in a formal manner, the uses that may be made of abduction for the purposes of belief revision. Our enterprise is guided, to a large extent, by the account offered by the AGM and to the principles and insights identified therein. Abduction was chosen because it can be rendered logically in a way that fits nicely with the AGM and because it has been demonstrated to be an effective technique in many problem areas, including: database updates [57, 56], diagnostic reasoning [105], text understanding [123] and vision [14]. We begin with a brief logical investigation of the notion of abduction. Using these insights we investigate the use of abduction in belief expansion, contraction and revision. Our emphasis here is on the normative aspects of such belief change. That is, we are more interested in the way belief revision "should" be performed rather than the way in which it is carried out by specific agents (human agents, for example).

1.1 Related Work

We briefly review some of the more pertinent work related to the aims of this dissertation. At other times we shall also have occasion to review related work, usually of a more technical nature, more specifically important to particular sections. The emphasis here will be on abduction as it relates to the incorporation of new information in belief change since that is what concerns most of the previous literature in the area.

1.1.1 Levi's Routine and Deliberate Expansion[4]

Levi [65] discusses two important forms of expansion: *routine expansion* and *deliberate expansion*.[5] However, it is first important to understand that Levi's agents are concerned with two tasks:

(*i*) acquiring new information which is free from error; and

(*ii*) incorporating new information into their current stock of beliefs.

For Levi, (*i*) is not restricted to epistemic inputs *per se* but these may be elaborated upon through the use of other forms of inference or deliberation; in particular, abduction and induction. The results of this first process are then taken to be the input to the second process — they are incorporated into the agent's epistemic state in the manner prescribed by (*ii*). The agent's beliefs are held in such high regard exactly because they are the result

[4] I am indebted to Abhaya Nayak for discussions regarding the significance of Levi's work in this respect.

[5] He also mentions *expansion by choice* but since this is only discussed briefly and is not essential to our concerns, we shall not consider it.

of such a costly process of inquiry. The AGM on the other hand is mainly concerned with (*ii*). It is assumed that epistemic inputs have been filtered in some way — which is not made entirely clear in their framework — and that the agent is concerned with incorporating only that new information. In this dissertation we extend the formal AGM framework so that new information can be obtained through the process of abduction.

In routine expansion an epistemic state is expanded "in conformity with some habit, program or routine" [65] (p. 43) as a response to some input. This routine is supposed, by the agent, to be adequately reliable. That is, there is only a low chance that it will lead to error. This does not mean that error will not occur however. If it does, contraction can be used to restore consistency. The routine can of course be modified over time through a reasoned process.

Deliberate expansion, on the other hand, is expansion through a deliberative or inferential process. Essentially, the agent uses abduction to determine potential expansions of the initial epistemic state (the exact details of the abductive procedure used are not important for our purposes here). The potential expansions thus determined constitute what is known as an *ultimate partition*. The agent then evaluates the elements of the ultimate partition to ascertain their informational value and the risk of error if that element were to be adopted. These values are combined in a ratio determined by the agent's *degree of caution* or *boldness* to determine an element's *expected epistemic utility*. The new epistemic state is then given by the join of those elements of the ultimate partition with maximal expected epistemic utility. Since deliberate expansion is the result of a deliberative process, the agent can only expand into inconsistency in this manner inadvertently unlike routine expansion where error is possible though considered improbable by the agent. Levi offers a probabilistic account of some of the notions referred to above but this shall not be of great concern to us here.

1.1.2 Abduction via Belief Revision

A different idea within the belief revision literature is the use of revision to determine abductions or, more precisely, explanations.[6] This idea would appear to be in some sense

[6]We do not equate the two notions here though one can think of abduction as producing explanations or potential explanations.

related to the Ramsey test [107] for conditional sentences. Basically, the Ramsey test states that a conditional sentence is acceptable precisely when incorporation of the antecedent into the current epistemic state, modified in a minimal way so as to maintain consistency, leads to belief in the consequent. In more contemporary work, Spohn [121] gives the following definition, for one proposition to be *a reason for* another, which may be seen as an extension of the Ramsey test:

> α is a reason for β for the person X at time s iff X's believing α at s would raise the epistemic rank of β for X at s.[7]

Here, *epistemic rank* refers to the agent's epistemic attitudes. In Spohn's framework, a greater variety of epistemic attitudes are possible than the three mentioned in the introduction above (i.e, believed, disbelieved and neither believed nor disbelieved). Each proposition has an associated *degree of firmness* indicating how firmly it is believed in relation to other propositions. In this light, the condition above says that one proposition is a reason for another if and only if believing the former more firmly leads to believing the latter more firmly. Of course, the way in which epistemic rank is altered is related to the belief change process in use.

Gärdenfors [32] adopts Spohn's definition and renders it in terms of the AGM framework. Essentially, Gärdenfors' definition says that α is a reason for β if and only if either β is included in the (minimal) revision by α but not in the revision by the negation of α, or the negation of β is included in the revision by the negation of α but not in that by α. However, the AGM allows only three epistemic attitudes (*viz.* believed, disbelieved, neither believed nor disbelieved) and would thus appear to be less discerning than Spohn's original framework. In fact, Gärdenfors notes that multiple reasons cannot be captured adequately by such a definition and that it leads to undesirable circularity in reasons in the sense that there would be beliefs $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_i$ such that α_1 is a reason for α_2, α_2 is a reason for α_3, $\ldots$, α_i is a reason for α_1.

Boutilier and Becher [9] pursue a similar argument. They maintain three reasons for adopting such an approach over a more traditional abductive view of explanation: explanations

[7] We modify Spohn's notation to maintain consistency with that adopted here.

should not be required to deductively entail the given data but may do so nonmonotonically or defeasibly; some notion of preference should exist to discriminate between potential explanations; and, it is not possible to explain data inconsistent with the background theory using an abductive view. Although we do not address the relationship between abduction and an intuitive notion of explanation here[8] we shall see through the course of this dissertation that the latter two points do not turn out to be very problematic in a traditional abductive view (in particular, for the second point refer to Chapter 5 and for the third, to Chapter 7). Although perhaps attractive initially, the case for the first point is not convincing. Boutilier and Becher [9] (p. 44) provide the following example:

> The sprinkler being on can explain the wet grass; but the sprinkler being on with a water main broken is not a reasonable explanation.

A lot of information is left implicit here but it is possible to adequately specify this information so that the sprinkler being on and the water mains being broken (at the same time) are incompatible and thus not a possible abduction.[9] Moreover, abduction will lead to nonmonotonicity. They then consider two types of explanation: *predictive explanations* where, if the agent were to accept the explanation, it would be compelled to accept the explanandum and *nonpredictive explanations*. In the former category they classify three types of explanation:

factual if the explanandum is currently believed, then the explanation should be a belief

hypothetical if the explanandum is not a current belief then a non-belief should be adopted as the explanation (two further cases can be distinguished; one requiring disbeliefs to explain disbeliefs and the other requiring propositions that are neither believed nor

[8] The interested reader is referred to Salmon [116] where, although abduction is not specifically mentioned, parallels can be drawn from the discussion of work concerning Hempel and Oppenheim's deductive-nomological model of explanation [50].

[9] Presumably, in this example "the sprinkler being on" refers to the tap (to which one presumes the sprinkler is connected via a hose etc.) being open. In which case, the sprinkler could be on (i.e., the tap open) and the water mains broken yet the grass not wet. However, the fact remains that this formulation of what is (implicitly) presumed to be the agent's background knowledge is inadequately specified. It is possible to give a proper specification of the background knowledge and, adopting an abductive view, obtain the desired result. One must be careful when formalising the problem.

disbelieved to be adopted as explanations for an explanandum that is neither believed nor disbelieved)

counterfactual the explanation is something which, if brought about, would lead to the occurrence of the explanandum (this generalises factual and hypothetical explanations only in the case where the explanandum is neither believed nor disbelieved).

Williams *et al.* [132] provide a rendering of Spohn's condition in a setting that is much closer in spirit to Spohn's original framework (in fact, slightly more general — see Williams [130] for details). Using a process representing an absolute minimal change[10] they show a clear relationship with a definition of abduction like the one we adopt here. Moreover, it does not suffer from the problems noted by Gärdenfors and is also able to capture notions identified by Boutilier and Becher.

Note, however, that part of the endeavour in this dissertation has different aims to the idea outlined in this section. The approaches here use the revision process to determine explanations or reasons and in particular cases identify this with abduction. We, among other things, investigate the use of abduction as part of the revision process itself. That is, in the framework developed here, abduction is "internal" to the belief change process and helps determine the resulting beliefs whereas the approaches inspired by Spohn use the belief change process to produce explanations.

1.1.3 Abduction in Database Updates

Kakas and Mancarella [57] use abduction for calculating updates to logical databases in work that is similar to part of our proposal here. They suggest a number of reasons why such a method is useful. Firstly, it makes explicit certain information that is in some sense implicit in the representation and could otherwise be lost. Also, if a knowledge base is organised in a particular way — information is represented or stored in terms of certain predicates or propositions, for instance — then it may be possible to use abduction to ensure that subsequent information is also organised in this way.

[10] Spohn is inclined towards one using a relative minimal change.

The setting they use is that of representing logical or deductive databases as logic programs (i.e., essentially Horn clauses) [68]. The databases itself is divided into two parts:

extensional database (EDB) set of ground facts (over base relations) describing state of domain

intensional database (IDB) set of rules (over view relations) from which new facts can be obtained.

The base predicates (or abducibles) are a demarcated set of predicates that can be used in forming abductions. An update request (*insert*(α) or *delete*(α)) is treated as an explanandum (i.e., explain α or explain $\neg\alpha$ respectively) and an attempt is made to determine an abduction which, when added to the EDB, would satisfy the request. Note that the IDB remains fixed; changes are only made to the EDB (in terms of ground base relations) to effect the desired result. That is because the IDB can be thought of as the domain information.

An update request *insert*(α) can be achieved by determining an abduction for new information α with respect to the IDB. That is, some formula consistent with the IDB such that, together with it, α follows as a logical consequence. This abduction is formulated in terms of base predicates so the desired result can be achieved by inserting the abduction into the EDB. For example, suppose the IDB contains *bird*$(x) \rightarrow$ *flies*(x) where *bird* is a base predicate and the EDB is empty (the EDB plays little part in the abductive process) and we receive an update request *insert*$($*flies*$($*tweety*$))$. This can be achieved by inserting *bird*$($*tweety*$)$ into the EDB. The existence of negation can cause some problems but Kakas and Mancarella show how to solve this by translating the database into an alternative form. They also show that a *delete*(α) request can be achieved by an *insert*$(\neg\alpha)$ request.

The method described is promising. It can handle *insert* (and *delete*) update requests of information inconsistent with the database (possible through the translation process alluded to above). In terms of belief revision, we note that this procedure is specific to the logic programming domain and no general logical treatment is attempted (at least not in the terms we shall attempt here: providing rationality postulates, constructions and representation theorems). Moreover, although one could identify the notion of epistemic state with the database (i.e., IDB $\cup$ EDB), one would need both to give a clear epistemic motivation for dividing the database up into two separate parts, and to explain their epistemic status. In its defence, the approach is motivated by more pragmatic concerns rather than epistemic concerns. We adopt a different setting and do not assume epistemic states to be divided in such a way. We also find the use of base predicates (abducibles) objectionable and shall return to this point later. Interesting extensions of this work can be found in Teniente [126] and Fung [51].

1.2 Overview

The following two chapters provide an overview to the two main areas with which this dissertation is concerned: belief revision and abductive inference. They survey important concepts which will be useful in the subsequent chapters. Chapter 4 presents a logical treatment of abduction including some of the notions discussed in Chapter 3. It also briefly contrasts abduction and induction with particular regard to the manner in which they are popularly dealt with in the field of artificial intelligence. In Chapter 5 we extensively investigate an operation for performing abductive belief expansion. This is performed in a fashion reminiscent of the AGM framework: through definitions, rationality postulates and a number of constructions motivated by those for AGM contraction and revision. Its relationship with the area of default reasoning is also discussed. This chapter forms the cornerstone of the dissertation and is important for the work that succeeds it. Chapter 6 is devoted to the role of abduction in the process of belief contraction. We begin with a semantic investigation of a form of contraction first suggested by Levi that can be considered in some respects a dual of our account of abductive expansion. Other ways in which abduction is related to belief contraction are also discussed. Chapter 7 presents a brief look at abductive revision. Using the results of the previous sections on expansion and contraction, an abductive revision operator is constructed. In the final chapter we present a summary of the results obtained in this dissertation, discussing their significance. We end with suggestions for future research possibilities arising from this work. The appendices at the end of this dissertation contain the formal proofs of claims made throughout.

1.3 Technical Preliminaries

We shall primarily consider a propositional language $\mathcal{L}$ with finitely many propositional symbols. We shall restrict ourselves to a finite language to simplify this exposition. Strictly speaking, however, this restriction is only necessary in certain circumstances which we shall identify. $\mathcal{L}$ will also be assumed to contain the standard logical connectives, namely $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ (with the understanding that they can be interdefined in the usual way) and the propositional constants $\top$ (truth) and $\bot$ (falsum). We identify $\mathcal{L}$ with the set of all its well formed formulae. We use the notation $\mathcal{L}(\Gamma)$ to refer to the (smallest) language over which a set of formulae Γ is formulated.

We normally adopt the following linguistic conventions:

- upper case Greek letters $\Delta,\ \Gamma,\ \ldots$ denote sets of formulae (in any particular form — the relevant form shall be evident from the context)
- lower case Greek letters $\alpha,\ \beta,\ \ldots$ denote formulae where the syntactic form is unimportant
- upper case Roman letters $A,\ B,\ C,\ \ldots$ denote clauses
- upper case Roman letters $H,\ K,\ \ldots$ denote belief sets (i.e., deductively closed sets of formulae)
- lower case Roman letters $j,\ k,\ l,\ \ldots$ denote (positive or negative) literals (i.e., propositional symbols)

The underlying logic will be identified with its *consequence operator* Cn. A consequence operator is a function $Cn : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ with the following properties:

(*i*)	$\Gamma \subseteq Cn(\Gamma)$	(Inclusion)
(*ii*)	If $\Gamma \subseteq \Delta$, then $Cn(\Gamma) \subseteq Cn(\Delta)$	(Monotonicity)
(*iii*)	$Cn(\Gamma) = Cn(Cn(\Gamma))$	(Iteration)

Moreover, we assume that Cn satisfies the following conditions:

(*iv*) If α can be derived from Γ by classical truth-functional logic, then $\alpha \in Cn(\Gamma)$ (Supraclassicality)

(*v*) $\beta \in Cn(\Gamma \cup \{\alpha\})$ if and only if $(\alpha \rightarrow \beta) \in Cn(\Gamma)$ (Deduction)

(*vi*) If $\alpha \in Cn(\Gamma)$, then $\alpha \in Cn(\Gamma')$ for some finite subset $\Gamma' \subseteq \Gamma$ (Compactness)

The following properties of a consequence operator Cn follows from those above and will be useful in some of the proofs.

(*vii*) If $\alpha \rightarrow \beta \in Cn(\Gamma)$ and $\alpha \in Cn(\Gamma)$, then $\beta \in Cn(\Gamma)$ (Modus Ponens)

(*viii*) If $\alpha \rightarrow \beta \in Cn(\Gamma)$, then $\neg\beta \rightarrow \neg\alpha \in Cn(\Gamma)$ (Contraposition)

We often write $\Gamma \vdash \alpha$ to mean $\alpha \in Cn(\Gamma)$ and $\vdash \alpha$ for $\emptyset \vdash \alpha$.

A set of formulae $K \subseteq \mathcal{L}$ is a *theory* in $\mathcal{L}$ if and only if K is closed under the consequence operator Cn (i.e., $K = Cn(K)$). We shall also refer to such a set as a *belief set* (see §2). A theory in $\mathcal{L}$ is *consistent* if and only if it does not contain formulae α and $\neg\alpha$ for any $\alpha \in \mathcal{L}$. A theory in $\mathcal{L}$ is *inconsistent* if it is not consistent. In fact, there is a single inconsistent theory in $\mathcal{L}$ and we denote it $K_\perp$ (and note that $K_\perp = \mathcal{L}$). A theory K is *complete* if and only if for every $\alpha \in \mathcal{L}$, $K \vdash \alpha$ or $K \vdash \neg\alpha$. A theory K is *finitely axiomatisable* if and only if there is a finite set of formulae Γ such that for any formula $\alpha \in \mathcal{L}$, $K \vdash \alpha$ iff $\Gamma \vdash \alpha$. We denote the set of all belief sets (or theories) in $\mathcal{L}$ by $\mathcal{K}$. An important property of belief sets or theories is that their intersection is a also a belief set (i.e., $Cn(K \cap H) = K \cap H$ for belief sets K and H).

At times (especially in Chapters 3 and 4 when considering material from the logic programming literature) we will have occasion to deal with clausal form logic [68]. *Atoms*, in the propositional case, are simply propositional letters. In the first-order case, terms have their usual meaning and atoms are predicate symbols applied to terms. A *positive literal* is an atom while a *negative literal* is a negated atom. A *clause* $C = \{c_1, \ldots, c_n\}$ is a finite set of literals representing their disjunction $c_1 \vee \ldots \vee c_n$. A *Horn clause* is a clause with at most one positive literal while a *definite clause* is a clause with exactly one positive literal. In first-order logic, a *substitution* θ is a finite set of pairs $\theta = \{v_1/t_1, \ldots, v_n/t_n\}$ where the

v_i are variables and t_i terms. When applied to a term, substitution replaces each occurrence of the variable v_i with the term t_i. A *unifier* of two terms (alternatively atoms) t_1 and t_2 is a substitution θ such that $t_1\theta = t_2\theta$. The unifier θ is a *most general unifier* (mgu) of t_1 and t_2 if, for each unifier δ of t_1 and t_2, there is a substitution γ such that $\delta = \theta\gamma$. An *inverse substitution* θ^{-1} maps terms to variables in $t\theta$.

The following is a list of common properties that a binary relation $R \subseteq \mathcal{D} \times \mathcal{D}$ may possess [125]. We shall adopt the familiar notation xRy for $\langle x,\ y\rangle \in R$.

Reflexive	For all $x \in \mathcal{D}$, xRx
Symmetric	For all $x,\ y \in \mathcal{D}$, if xRy, then yRx
Transitive	For all $x,\ y,\ z \in \mathcal{D}$, if xRy and yRz, then xRz
Irreflexive	For all $x \in \mathcal{D}$, $\neg(xRx)$
Asymmetric	For all $x,\ y \in \mathcal{D}$, If xRy, then yRx does not hold
Antisymmetric	For all $x,\ y \in \mathcal{D}$, if xRy and yRx, then $x = y$
(Strongly) Connected	For all $x,\ y \in \mathcal{D}$, either xRy or yRx
Equivalence	R is Reflexive, Symmetric and Transitive
Preorder	R is Reflexive and Transitive
Partial Order	R is Reflexive, Transitive and Antisymmetric
Total Order	R is a Partial Order and Connected
Simple Order	R is Transitive and Antisymmetric
Strict Partial Order	R is Asymmetric and Transitive
Strict Simple Order	R is Asymmetric, Transitive and Connected

An element $x \in \mathcal{D}$ is an R-minimal (sometimes referred to as R-first) element of a set $\mathcal{D}$ if and only if for any $y \in \mathcal{D}$ such that $x \neq y$ does xRy hold. An element $x \in \mathcal{D}$ is an R-lower bound of a set $\mathcal{D}$ if and only if xRy for all $y \in \mathcal{D}$. An element $x \in \mathcal{D}$ is an R-infimum (or R-greatest lower bound) of a set $\mathcal{D}$ if and only if x is an R-lower bound and for any R-lower bound $y \in \mathcal{D}$, yRx. An element $x \in \mathcal{D}$ is an R-upper bound of a set $\mathcal{D}$ if and only if yRx for all $y \in \mathcal{D}$. An element $x \in \mathcal{D}$ is an R-supremum (or R-least upper bound) of a set $\mathcal{D}$ if and only if x is an R-upper bound and for any R-lower bound $y \in \mathcal{D}$, xRy. A set $\mathcal{D}$ is a lattice relative to R if and only if R is a partial ordering of $\mathcal{D}$ and for any $x,\ y \in \mathcal{D}$,

$\{x,\ y\}$ has an R-supremum and R-infimum in $\mathcal{D}$.

An order $\leq$ is a transitive relation. We use $<$ to refer to the strict part of $\leq$ (i.e., $x < y$ iff $x \leq y$ and $y \not\leq x$).

Chapter 2

Belief Revision

You could not step twice into the same rivers; for other waters are ever flowing on to you.
Heracleitus of Ephesus (as quoted by Plutarch, *On the E at Delphi*)

As noted in the introduction, the concept of belief revision is, in essence, a simple one. We are interested in characterising the dynamics of epistemic states; how an agent in a particular epistemic state modifies this state upon receipt of some new information (or epistemic input). Moreover, we are interested in investigating changes of belief that are performed in a *rational* manner.

2.1 Foundationalism Versus Coherentism

Before addressing the problem of how to alter an epistemic state given an epistemic input, we shall briefly investigate the nature of the states themselves. The two foremost approaches to modelling epistemic states are the *foundational* and *coherence* theories. Pollock [101] refers to these as *doxastic* theories; they assume that the justificatory pedigree of beliefs depends solely on those beliefs held by an agent.

The major distinguishing feature of the foundational approach is that it demarcates a special class of beliefs. These are often referred to as “epistemologically basic beliefs” (or simply “basic beliefs”). Every belief in a foundational system is supposed to be justified in terms of other beliefs which are, in turn, justified by further beliefs until ultimately we reach basic beliefs which have no need of justification. In a certain sense, they can be thought of as self justifying.

Examples of foundational systems within artificial intelligence are Doyle's [21] Truth Maintenance System (TMS)[1] and its successor the Assumption Based Truth Maintenance System (ATMS) [112]. Their basic task is to record inferences passed to them by a domain dependent problem solver. Basically, the TMS consists of two structures: *nodes* representing propositions; and, *justifications* representing reasons. Each node may be in one of two states:

in	the node has a valid justification and consequently is considered a current belief
out	the node does not have a valid justification; it is currently not believed

A justification is a pair of sets of nodes: an *inlist* and an *outlist*. A justification is *valid* if all nodes on its *inlist* are *in* and all those on its *outlist* are *out*. Clearly, a proposition becomes a belief when one of its justifications is valid; it becomes a non-belief when none of its justifications are valid. The TMS takes care of creating new nodes and adding or retracting justifications. This may become a complex process as other nodes and justifications may be affected. Also, circular justifications must not be admitted. It can also mark a node as a *contradiction*. This has the effect of stating that the elements of a justification for this node are inconsistent. If such a node acquires a valid justification a process known as *dependency-directed backtracking* ensues, making sure that any justification is no longer valid.[2]

Elkan [24] (see also Reinfrank *et al.* [109, 110]) provides a logical rendering of the TMS in order to show how it relates to Gelfond and Lifschitz' [36] stable model semantics for logic programming and Moore's [74] (propositional) autoepistemic logic. A justification for a proposition c is simply represented as a (directed) propositional clause

$$a_1 \wedge \ldots \wedge a_n \wedge \neg b_1 \wedge \ldots \wedge \neg b_m \rightarrow c$$

(where $\neg$ represents negation as failure).[3] Propositions $a_1, \ldots, a_n$ represent those in the

[1] The term Truth Maintenance System has often been cited as a misnomer and the alternative Reason Maintenance System (RMS) suggested as a more appropriate alternative. However, the term Truth Maintenance System appears to have stuck and we shall use it here.

[2] This is essentially achieved by retracting beliefs known as *assumptions* — having a justification with a non-empty *outlist* — which, although not necessarily part of the justification, lead to it becoming invalid.

[3] In autoepistemic logic this justification may be rendered $a_1 \wedge \ldots \wedge a_n \wedge \neg \mathrm{L} b_1 \wedge \ldots \wedge \neg \mathrm{L} b_m \rightarrow c$.

justification's *inlist* while $b_1, \ldots, b_m$ are those in the *outlist*. A contradiction c can be represented by the clause $a_1 \wedge \ldots \wedge a_n \rightarrow c$ meaning that $a_1, \ldots, a_n$ cannot all be believed (think of a contradiction c as representing $\perp$).

One of the more popular successors of the TMS is de Kleer's [18] ATMS. It is based on the idea of keeping track of the assumptions upon which a proposition is based as well as its justifications.[4] In this case a node is composed of three parts: the *datum* representing a proposition but treated as atomic; the *label* representing sets of assumptions — called *environments* — which would allow the datum to be inferred; and, the *justifications*, each containing antecedents supporting the datum. An *assumption* is a node with an environment consisting of its datum only.[5] The nodes derivable from an environment (including those corresponding to the elements of the environment itself) are referred to as *contexts*. Sets of assumptions that cannot hold simultaneously are referred to as *nogoods*. They are similar to contradictions in the TMS and act like integrity constraints, reducing the size of the search space for any eventual query passed to the ATMS (by causing the deletion of derived justifications that violate these constraints). When an inference is passed to the ATMS it takes care of updating nodes. If the consequent is unknown, a new node is created with the consequent as the datum. The antecedent of the inference becomes a new justification for the node and labels of the antecedents are used in determining the label for the node. All combinations of an environment from every label are used in determining new environments. However, a label must be

consistent	no environment is a superset of a nogood
complete	the environment from which the datum follows is a superset of some environment in the datum's label
sound	the datum follows from each environment
minimal	no environment is a subset of another environment in the label

Soundness and completeness are guaranteed by the procedure used to compute environments. Inconsistent and non-minimal environments must be removed.

[4]Moreover, whereas the TMS concentrates on finding one support, the ATMS is geared towards finding all supports.

[5]Cf. TMS — the notions differ.

Logically, ATMS justifications are simply Horn clauses

$$a_1 \wedge \ldots \wedge a_n \rightarrow c$$

where c represents the datum and $a_1, \ldots, a_n$ a justification. Assumptions can be represented in the same manner however, in this case, $a_1, \ldots, a_n$ is taken from one of the datum's environments. Nogoods $\{a_1, \ldots, a_n\}$ are just supports for falsity, $a_1 \wedge \ldots \wedge a_n \rightarrow \perp$.

Example 2.1.1 *Using part of our previous hamburger example*

$$o_1 \wedge ph_1 \rightarrow e$$
$$o_2 \wedge ph_2 \rightarrow e$$

Suppose also that the two restaurants serve milkshakes so that, if a restaurant is open and I purchase a milkshake from it (pm_1/pm_2), then I will have a milkshake (m)

$$o_1 \wedge pm_1 \rightarrow m$$
$$o_2 \wedge pm_2 \rightarrow m$$

Moreover, suppose there is a municipal restriction stating that the two restaurants can never be open at the same time. We express this fact through the addition of the following nogood

$$o_1 \wedge o_2 \rightarrow \perp$$

Now, suppose a new inference is passed to the ATMS: I am satiated (s) after having a hamburger and a milkshake

$$e \wedge m \rightarrow s$$

All possible justifications for e and m produce the following justifications for s

$$o_1 \wedge ph_1 \wedge pm_1 \rightarrow s$$
$$o_1 \wedge ph_1 \wedge o_2 \wedge pm_2 \rightarrow s$$
$$o_1 \wedge pm_1 \wedge o_2 \wedge ph_2 \rightarrow s$$
$$o_2 \wedge ph_2 \wedge pm_2 \rightarrow s$$

However, the second and third violate the nogood and must be removed. All justifications are minimal. □

The coherence approach, unlike the foundational approach underlying the TMS and ATMS, denies the existence of any select set of basic beliefs. On this account, beliefs are justified by the way they interact or "cohere" with other beliefs. In other words, it is the relationship with other beliefs that is important when determining whether a belief is justified.

Pollock [101] distinguishes four types of coherence theories into two groups:

1a) Positive Coherence

> The agent must possess reasons for maintaining a belief. That is, each belief must have "positive support".

1b) Negative Coherence

> The agent is justified in holding a belief provided there is no reason to think otherwise. ("All beliefs are 'innocent until proven guilty"', Pollock [101] p. 72.)

2a) Linear Coherence

> The agent adopts a more traditional (i.e., foundational) view of reasons except that if we look at a reason, the reasons for holding reasons, etc., we would never stop; either we have an infinite sequence of reasons or there is some circularity in the reason structure.

2b) Holistic Coherence

> The agent is justified in holding a belief due to some relationship between the belief and all other beliefs held.

It is possible to have coherence theories which possess more than one aspect from this list.

The distinction between the foundational and coherence approaches is often illustrated through two metaphors: the foundationalist "pyramid" and the coherentist "raft". These are succinctly expressed by Sosa [119]:

> For the foundationalist every piece of knowledge stands at the apex of a pyramid that rests on stable and secure foundations whose stability and security does not derive from the upper stories or sections. For the coherentist a body of knowledge is a free-floating raft every plank of which helps directly or indirectly to keep all the others in place, and no plank of which would retain its status with no help from the others.

The latter derives from a metaphor by Neurath [86] used to express the fact that it is not possible (nor desirable) to start from scratch in developing a language for scientific discourse:

> We are like sailors who have to rebuild their ship on the open sea, without ever being able to dismantle it in dry-dock and reconstruct it from the best components.

Pollock [101] notes that this metaphor is more in keeping with the negative coherence view.

2.2 The AGM Framework for Belief Revision[6]

We shall base our study on the AGM framework for belief revision [1, 31, 32]. This approach is claimed, by Gärdenfors [32] (also [31] p. 35), to be coherentist in nature. The main reason for adopting this approach is that it is a well developed formal framework that should allow us to take advantage of a logical theory of abduction (to be discussed in Chapters 3 and 4). Such a study has not been undertaken previously whereas investigations of the relationship between TMSs and abduction have (e.g., [112]). Moreover, links have been investigated between belief revision and other areas of artificial intelligence (e.g., nonmonotonic reasoning [34, 72]).

It was mentioned earlier that we are interested in accounts of rational belief change. Gärdenfors and Rott [35] adopt the following rationality criteria or "integrity constraints":

[6] The results in this section may be found in the AGM literature; in particular, they are collected together in Gärdenfors [31] unless otherwise stated.

- Where possible, epistemic states should remain consistent
- Any sentence logically entailed by beliefs in an epistemic state should be included in the epistemic state
- When changing epistemic states, loss of information should be kept to a minimum[7]
- Beliefs held in higher regard should be retained in favour of those held in lower regard

The third criterion can be thought of as a manifestation of Occam's razor as applied to the removal of information (rather than the making of hypotheses), and is held in high regard in the AGM framework. In fact, it is often mentioned in connection with this framework. We shall see that a variant of it also applies to the acquisition of new information.

The second criterion leads to the following conception of an epistemic state within the AGM framework. Epistemic states are closed under logical consequence (Cn) and are referred to as *belief sets*.[8] The set of all belief sets is denoted $\mathcal{K}$. One special type of belief set is the absurd belief set $K_\perp$ which contains all formulae in $\mathcal{L}$. This rather idealistic modelling of epistemic states may best be viewed as the agent's *doxastic commitment to full recognition* of the truth of the deductive consequences of what it believes (see Levi [65] p. 8). A lot of attention has also been paid to the study of *belief bases* [30, 42, 80, 85]; sets of formulae that are not necessarily closed under the logical consequence operation.

Given any consistent belief set K, there are three types of *epistemic attitude* toward a sentence α:

(*i*) α is *accepted* (or believed) if $\alpha \in K$

(*ii*) α is *rejected* (or not believed) if $\neg\alpha \in K$

(*iii*) α is *indetermined* if $\alpha \notin K$ and $\neg\alpha \notin K$

Epistemic inputs are represented by a single sentence from the object language. More complex representations may be found in the literature (e.g., [45, 122]).

[7]This is also referred to as the Principle of Informational Economy [31] and, when informational loss is measured by set inclusion, the Principle of Conservation [49]. They are special cases of the Principle of Minimal Change [49] which states that minimal change should occur when beliefs are added as well as removed.

[8]That is, belief sets are simply theories albeit with a special interpretation in mind.

Belief change operators can be seen as prescribing how a given epistemic state is to be altered given an epistemic input. The AGM considers three types of belief change operators given a belief state K, representing the agent's current epistemic state, and epistemic input α:

Belief Expansion (K^+_α) Incorporation of new belief α into K without retraction of any existing beliefs

Belief Contraction (K^-_α) Removal of belief α from K without introduction of any new beliefs

Belief Revision (K^*_α) Incorporation of new belief α into K with possible removal of existing beliefs in order to maintain consistency

A belief change operator is essentially a function taking a belief set K and epistemic input α to a new belief set K'_α $(+, -, * : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K})$.[9] These belief change operators are investigated in a number of ways: through rationality postulates and through a variety of constructions. The postulates are then related to the constructions via *representation theorems*. The idea is to study all possible belief change functions — that is, all possible ways of expanding, contracting and revising K by α — in accord with the rationality constraints imposed by the postulates.

2.2.1 Postulates

Rationality postulates specify constraints that the respective operators should satisfy. They are guided by the rationality criteria outlined above which we adopt in this dissertation as the standards for characterising a rational agent.

[9]The restriction that the nature of an epistemic state be the same before and after undergoing change is referred to as the *Principle of Categorical Matching* [35].

Expansion

The expansion of a belief set K by an epistemic input α is denoted K^+_α. Expansion is generally recommended when α is consistent with K. An AGM expansion operator $+ : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$ satisfies the following rationality postulates:

(K^+1)	For any sentence α and any belief set K,	
	K^+_α is a belief set	(closure)
(K^+2)	$\alpha \in K^+_\alpha$	(success)
(K^+3)	$K \subseteq K^+_\alpha$	(inclusion)
(K^+4)	If $\alpha \in K$, then $K^+_\alpha = K$	(vacuity)
(K^+5)	If $K \subseteq H$, then $K^+_\alpha \subseteq H^+_\alpha$	(monotonicity)
(K^+6)	For all belief sets K and all sentences α, K^+_α is the	
	smallest belief set that satisfies (K^+1) — (K^+5)	(minimality)

The postulate of closure expresses the fact that $+$ is a function taking a belief set and a sentence as input and produces a belief set. Success states that the epistemic input is accepted in the expanded epistemic state. Inclusion says that no beliefs are retracted and is a form of the Principle of Minimal Change as phrased above. Vacuity represents a boundary case and states that nothing need be done if the epistemic input is already accepted.[10] Monotonicity says that, if one belief state contains at least the same information as another, then its expansion will contain at least the information of the expansion of the other with respect to the same epistemic input. The postulate of minimality can be considered an expression of the Principle of Minimal Change applied to the addition of new beliefs to an epistemic state K; the smallest possible change to accommodate the new information is made. The term "smallest" is understood with respect to set inclusion (of the original epistemic state relative to the expanded epistemic state). This leads to the following representation theorem.

Theorem 2.2.1 *The expansion function $+$ satisfies (K^+1) — (K^+6) if and only if $K^+_\alpha = Cn(K \cup \{\alpha\})$.*

[10] Postulate (K^+4) is superfluous as it follows from postulates (K^+1) — (K^+3), (K^+5) and (K^+6).

Therefore, to calculate an AGM expansion, one need only take the deductive closure of the initial epistemic state and the new information.

Contraction

The contraction of a belief set K by epistemic input α is denoted K^-_α. Contraction is recommended when doubt is raised about a current belief or the agent wishes to temporarily suspend belief in a proposition. It can be used together with expansion to perform revision, as we shall see, and satisfies the following rationality postulates:

(K^-1)	For any sentence α and any belief set K,	
	K^-_α is a belief set	(closure)
(K^-2)	$K^-_\alpha \subseteq K$	(inclusion)
(K^-3)	If $\alpha \notin K$, then $K^-_\alpha = K$	(vacuity)
(K^-4)	If $\nvdash \alpha$ then $\alpha \notin K^-_\alpha$	(success)
(K^-5)	If $\alpha \in K$, $K \subseteq (K^-_\alpha)^+_\alpha$	(recovery)
(K^-6)	If $\vdash \alpha \leftrightarrow \beta$, then $K^-_\alpha = K^-_\beta$	(extensionality)
(K^-7)	$K^-_\alpha \cap K^-_\beta \subseteq K^-_{\alpha\wedge\beta}$	(intersection)[11]
(K^-8)	If $\alpha \notin K^-_{\alpha\wedge\beta}$, then $K^-_{\alpha\wedge\beta} \subseteq K^-_\alpha$	(conjunction)[12]

Closure states that a contraction operation takes pairs of belief sets and formulae to belief sets ($- : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$). Inclusion says that no new beliefs should be introduced into the contracted epistemic state. Vacuity expresses the fact that nothing need be done if the epistemic input is not currently accepted. It is a manifestation of the Principle of Minimal Change. Success states that if it is possible to remove the epistemic input, it will be retracted from the current epistemic state. The only situation in which it is not possible to do so occurs when the epistemic input is a logical truth for, by the second of our rationality criteria above, it will be included in all possible epistemic states. Recovery says that if we were to retract a belief from K and then expand the result by the same formula, all original beliefs would be included in the final epistemic state. This behaviour is also due to the Principle of Minimal

[11] Also referred to as *conjunctive overlap* [44].
[12] Also referred to as *conjunctive inclusion* [44].

Change to a certain extent since this principle dictates that beliefs not be unnecessarily discarded when determining K^-_α. Recovery is arguably the most controversial of the AGM rationality postulates and there are a number of contributions discussing its removal [41, 71]. Makinson [71] refers to a contraction operation satisfying postulates (K^-1) — (K^-4) and (K^-6) as a *withdrawal*. Extensionality expresses the Principle of Irrelevance of Syntax; it is the content rather than the syntactic formulation of the epistemic input that is important in belief change. These first six postulates are often referred to as the *basic postulates for contraction over K*. The remaining two postulates are supplementary postulates. They are best motivated in the style of Nayak [80] (p. 506). Intersection states that, if one does not give up belief in γ when giving up belief in α nor in giving up belief in β, then one should not give up belief in γ when giving up belief in the conjunction $\alpha \wedge \beta$. Conjunction states that, if one were to give up α when giving up the conjunction $\alpha \wedge \beta$, then whatever one gives up in giving up α, should also be given up in giving up $\alpha \wedge \beta$.

Revision

The revision of a belief set K by epistemic input α is denoted K^*_α. Revision is particularly important when α is inconsistent with K and the agent wishes to incorporate it in such a way as to end up in a consistent epistemic state. It satisfies the following rationality postulates:

(K^*1)	For any sentence α and any belief set K, K^*_α is a belief set	(closure)
(K^*2)	$\alpha \in K^*_\alpha$	(success)
(K^*3)	$K^*_\alpha \subseteq K^+_\alpha$	(inclusion)
(K^*4)	If $\neg\alpha \notin K$, then $K^+_\alpha \subseteq K^*_\alpha$	(preservation)
(K^*5)	$K^*_\alpha = K_\perp$ if and only if $\vdash \neg\alpha$	(vacuity)
(K^*6)	If $\vdash \alpha \leftrightarrow \beta$, then $K^*_\alpha = K^*_\beta$	(extensionality)
(K^*7)	$K^*_{\alpha\wedge\beta} \subseteq (K^*_\alpha)^+_\beta$	(superexpansion)
(K^*8)	If $\neg\beta \notin K^*_\alpha$, then $(K^*_\alpha)^+_\beta \subseteq K^*_{\alpha\wedge\beta}$	(subexpansion)

(K^*1) is the familiar postulate of closure ($* : \mathcal{K} \times \mathcal{L} \to \mathcal{K}$). Success states that the new information should be included in the revised epistemic state. Inclusion says that expansion represents an “upper bound” when incorporating new beliefs (this will trivially hold in case the negation of the epistemic input is already accepted). Preservation expresses that, when the negation of the epistemic input is not accepted, revision reduces to expansion. It is the conditional converse of inclusion. Vacuity tells us that the only situation in which revision would end up in the inconsistent epistemic state occurs when the agent is asked to accept logically contradictory information. Extensionality, like its contraction counterpart, is an expression of the Principle of Irrelevance of Syntax. Again, (K*1) — (K*6) are referred to as the *basic postulates for belief revision over* K. Two supplementary postulates for revision exist and can be thought of as generalisations of Inclusion and Preservation. Superexpansion states that any belief included in the revision of K by $\alpha \wedge \beta$ should also be included if we first revise K by α and then expand the result by β. Subexpansion says that, if β is not rejected in revising K by α, then any belief included by first revising K by α and expanding the result by β should also be included in the revision of K by $\alpha \wedge \beta$. That is, $K^*_{\alpha\wedge\beta}$ and $(K^*_\alpha)^+_\beta$ consist of the same beliefs in this case. This postulate is the conditional converse of Superexpansion.

Interestingly enough, not all of the above operators are essential; some may be defined in terms of the other operators.[13] A revision operator, for instance, may be determined from a contraction operator and an expansion operator via the *Levi Identity*:

$$(\text{Def } *) \qquad\qquad K^*_\alpha = (K^-_{\neg\alpha})^+_\alpha$$

It states that a revision of K by α can be performed by first removing $\neg\alpha$ (to avoid inconsistency) and incorporating α into the result. The following theorem gives credence to this definition (and Levi’s claims).

Theorem 2.2.2 *Let* $-$ *be a contraction function satisfying postulates* (K$^-$1) — (K$^-$4) *and* (K$^-$6) *and* $+$ *an expansion function satisfying postulates* (K$^+$1) — (K$^+$6). *Then the revision function* $*$ *obtained from* $(\text{Def } *)$ *satisfies* (K*1) — (K*6). *Moreover, if* $-$ *satisfies* (K$^-$7), *then* $*$ *satisfies* (K*7) *and if* $-$ *satisfies* (K$^-$8), *then* $*$ *satisfies* (K*8).

[13] In fact, Levi [64] claims that the only “legitimate” forms of changing an epistemic state are expansion and contraction, a view to which we subscribe. He refers to this as the *commensurability thesis* [65] (p. 65). Such a view places less emphasis on the revision operator which is deemed achievable through a sequence of expansions and contractions. As a result, we place a greater emphasis on these latter two operators in this dissertation.

Notice that recovery is not required in Theorem 2.2.2. This tells us that revision operators defined, via the Levi Identity, from AGM contractions and those from withdrawal operators are *revision equivalent*. That is, they determine the same class of revision operators.

Alternatively, it is possible to define a contraction operator using a revision operator and set intersection.[14] This may be achieved by the *Harper Identity*:

$$(\text{Def } -) \qquad\qquad K^-_\alpha = K \,\cap\, K^*_{\neg\alpha}$$

which states that contracting K by α consists of those beliefs in K that are retained in revising K by $\neg\alpha$. The motivation for this definition stems from the fact that $K^*_{\neg\alpha}$ represents a minimal change of K required to incorporate $\neg\alpha$ (in a consistent manner) and should therefore include a large part of K that does not entail α.

Theorem 2.2.3 *Let $*$ be a revision function satisfying postulates* (K^*1) — (K^*6)*. Then the contraction function obtained from* $(\text{Def } -)$ *satisfies* (K^-1) — (K^-6)*. Moreover, if $*$ satisfies* (K^*7)*, then $-$ satisfies* (K^-7) *and if $*$ satisfies* (K^*8)*, then $-$ satisfies* (K^-8)*.*

2.2.2 Constructions

Having outlined conditions that the various belief change operators should satisfy, it is interesting to study how operators satisfying these postulates could be constructed. The AGM framework possesses four main constructions: selection functions over maximal subsets of K failing to imply α, Grove's system of spheres, epistemic entrenchment and safe contraction.[15]

Selection Functions

Given the Levi Identity and Theorem 2.2.1 regarding belief expansion, it is sufficient to concentrate on contraction. One approach to constructing a contraction of belief set K by epistemic input α is to seriously consider the Principle of Minimal Change and look at

[14] Recall that the intersection of two belief sets is also a belief set.

[15] A construction in terms of nice preorders over models (see [61, 34]) is also presented by Peppas and Williams [100] but we shall not consider it here.

subsets of K which are as big as possible without entailing α. Such a set can be defined as follows:

Definition 2.2.1 *A belief set K' is a maximal subset of K that fails to imply α if and only if*

(i) $K' \subseteq K$

(ii) $\alpha \notin K'$

(iii) for any $\beta \in \mathcal{L}$, if $\beta \in K$ and $\beta \notin K'$, then $\beta \to \alpha \in K'$

The set of all belief sets that are maximal subsets of K failing to imply α are denoted $K \perp \alpha$.

Generally, $K \perp \alpha$ contains more than one maximal subset. The first idea in constructing a contraction function is to apply a *selection function* γ to select one element from $K \perp \alpha$.[16] Intuitively, $\gamma(K \perp \alpha)$ returns the "best" element from $K \perp \alpha$ and is known as a *maxichoice selection function*. The contraction of K by α can be defined as follows

$$\text{(Def Max)} \qquad K^-_\alpha = \begin{cases} \gamma(K \perp \alpha) & \text{whenever } K \perp \alpha \text{ is nonempty}^{17} \\ K & \text{otherwise} \end{cases}$$

and is referred to as a *maxichoice contraction function over K*. Sure enough, such a function satisfies the basic postulates for belief contraction over K.

Lemma 2.2.4 *Let K be a belief set. If $-$ is a maxichoice contraction function over K, then it satisfies postulates* (K^-1) — (K^-6) *for belief contraction over K.*

Unfortunately, we obtain the following undesirable results.

Lemma 2.2.5 *Let K be a belief set and $\alpha \in \mathcal{L}$. If $\alpha \in K$ and K^-_α is defined by means of a maxichoice contraction function, then for any proposition β either $\alpha \vee \beta \in K^-_\alpha$ or $\alpha \vee \neg\beta \in K^-_\alpha$.*

[16] A selection function applied to a set X returns an element of the co-domain whenever X is nonempty.
[17] Note that $K \perp \alpha = \emptyset$ only when $\vdash \alpha$.

Corollary 2.2.6 *Let $-$ be a maxichoice contraction function over K. If a revision function $*$ is defined from $-$ by the Levi Identity, then, for any α such that $\neg\alpha \in K$, K^*_α is a complete theory.*

They suggest that maxichoice contractions retain too much information. In the resulting revision, the agent has opinions as to the truth or falsity of every proposition.

It seems natural then to consider a selection function at the other extreme; one returning all elements of $K\perp\alpha$. This is known as a *full meet selection function* and leads to a *full meet contraction function over K* which may be defined as follows.

$$\text{(Def Meet)} \qquad K^-_\alpha = \begin{cases} \bigcap(K\perp\alpha) & \text{whenever } K\perp\alpha \text{ is nonempty} \\ K & \text{otherwise} \end{cases}$$

A full meet contraction function also satisfies the basic postulates for contraction.

Lemma 2.2.7 *Let K be a belief set. If $-$ is a full meet contraction function over K, then it satisfies postulates* (K^-1) — (K^-6) *for belief contraction over K.*

However, we again have undesirable results.

Lemma 2.2.8 *Let K be a belief set and $\alpha \in \mathcal{L}$. If $\alpha \in K$ and K^-_α is defined by means of a full meet contraction function, then for any proposition β, $\beta \in K^-_\alpha$ if and only if $\beta \in K$ and $\neg\alpha \vdash \beta$.*

Corollary 2.2.9 *Let $-$ be a full meet contraction function over K. If a revision function $*$ is defined from $-$ by* (Def $*$)*, then for any α such that $\neg\alpha \in K$, $K^*_\alpha = Cn(\alpha)$.*

In a sense, too much information is removed. This is somewhat at odds with the Principle of Minimal Change.

A remedy lies in making a compromise between these two extremes. We adopt a selection function γ that returns a subset of $K\perp\alpha$. We can think of γ as returning the set of “best” elements of $K \perp \alpha$.[18] This is referred to as a *partial meet selection function*. The resulting contraction — a *partial meet contraction function over K* — may be defined as follows

$$\text{(Def Part)} \qquad K^-_\alpha = \begin{cases} \bigcap \gamma(K \perp \alpha) & \text{whenever } K \perp \alpha \text{ is nonempty} \\ K & \text{otherwise} \end{cases}$$

The following representation theorem says that such functions exactly coincide with the basic postulates for contraction.

Theorem 2.2.10 *Let K be a belief set and $-$ be a contraction function. Then $-$ is a partial meet contraction function over K if and only if it satisfies postulates* (K^-1) — (K^-6) *for contraction over K.*

It is interesting to further investigate the nature of the selection function γ and how it decides which elements of $K \perp \alpha$ are preferred. One idea is to impose a relation $\preceq$ over the elements of $K \perp \alpha$ and define γ by the following *marking-off identity* (when $K \perp \alpha \neq \emptyset$):

$$(\text{Def } \gamma) \qquad \gamma(K \perp \alpha) = \{K' \in K \perp \alpha : K'' \preceq K' \text{ for all } K'' \in K \perp \alpha\}$$

The relation $\preceq$ "marks off" the most preferred elements of $K \perp \alpha$. When γ is defined in this way, the resulting contraction function is referred to as a *relational partial meet contraction function over K*.

Lemma 2.2.11 *Let K be a belief set. Any relational partial meet contraction function over K satisfies postulate* (K^-7) *for contraction over K.*

A straightforward extension is to require $\preceq$ be transitive. In this case γ is known as *transitively relational* and the resulting contraction as a *transitively relational partial meet contraction function over K*.

Lemma 2.2.12 *Let K be a belief set. Any transitively relational partial meet contraction function over K satisfies postulate* (K^-8) *for contraction over K.*

The following theorem supports the utility of such a construction.

[18] Cf. the fourth rationality criterion. Each element of $\gamma(K \perp \alpha)$ contains beliefs held in higher regard. The beliefs held in highest regard are those common to the best elements returned of $\gamma(K \perp \alpha)$.

Theorem 2.2.13 *Let K be a belief set and $-$ be a contraction function defined over K. Then $-$ is a transitively relational partial meet contraction function over K* if and only if *it satisfies postulates* (K$^-$1) — (K$^-$8) *over K.*

The respective revision operation defined via the Levi identity satisfies postulates (K*1) — (K*8). It can also be shown that requiring $\preceq$ to be connected does not lead to further contraction postulates.

Grove's Sphere Semantics

Grove [39] developed a "sphere semantics" for the AGM framework inspired by Lewis' [67] semantics for counterfactual reasoning.[19] He concentrated on revision functions although the idea is easily extended to deal with contraction (via the Harper Identity) and expansion.

Grove views maximally consistent sets of formulae (consistent complete theories) as "possible worlds". He places an ordering over the set $\mathcal{M}_{\mathcal{L}}$ of all possible worlds. The possible worlds consistent with any set K are denoted $[K]$ and may be defined as follows.

$$[K] = \begin{cases} \{m \in \mathcal{M}_{\mathcal{L}} : K \subseteq m\} & \text{if } K \neq K_{\perp} \\ \emptyset & \text{otherwise} \end{cases}$$

In a similar fashion, the possible worlds consistent with a formula α are denoted $[\alpha]$ and defined as $[\alpha] = [\{\alpha\}]$ (i.e., $[\alpha] = \{m \in \mathcal{M}_{\mathcal{L}} : \alpha \in m\}$). We also define a function $th : 2^{\mathcal{M}_{\mathcal{L}}} \rightarrow \mathcal{K}$ mapping sets of possible worlds to belief sets. For any $X \subseteq \mathcal{M}_{\mathcal{L}}$ we have

$$\text{(Def } th) \qquad th(X) = \begin{cases} \bigcap\{m \in X\} & \text{for } X \subseteq \mathcal{M}_{\mathcal{L}} \text{ and } X \neq \emptyset \\ K_{\perp} & \text{if } X = \emptyset \end{cases}$$

We reproduce the following properties, listed by Grove [39], for reference.

Lemma 2.2.14 *Properties of th* [39].

(i) $th([K]) = K$ *for all belief sets (i.e., theories) K if the underlying logic is compact*

(ii) $th(X) \neq K_{\perp}$ *if and only if X is nonempty*

[19] Grove's idea can be viewed as a semantics insofar as it gives a "picture" for AGM belief change. Strictly speaking however, it deals with syntactic objects.

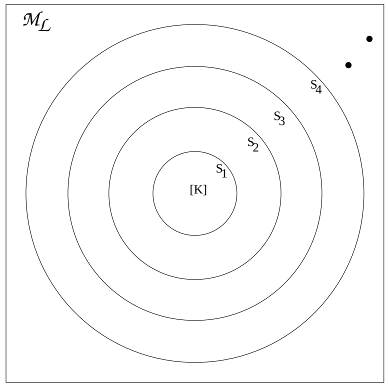


Figure 2.1: A system of spheres centred on $[K]$.

(iii) For any sentence $\alpha \in \mathcal{L}$ and $X \subseteq \mathcal{M}_{\mathcal{L}}$, $th(X \cap [\alpha]) = Cn(th(X) \cup \{\alpha\})$

(iv) For $X,\ X' \subseteq \mathcal{M}_{\mathcal{L}}$, if $X \subseteq X'$, then $th(X') \subseteq th(X)$

(v) For $K,\ K' \in \mathcal{K}$, if $K \subseteq K'$, then $[K'] \subseteq [K]$

A *sphere* is defined to be a set of possible worlds. A *system of spheres centred on K* is an ordering over sets of possible worlds where $[K]$ is the innermost sphere and $\mathcal{M}_{\mathcal{L}}$ the outermost sphere. It can be formally defined as follows.

Definition 2.2.2 [39]
Let $\mathcal{S}$ be any collection of subsets of $\mathcal{M}_{\mathcal{L}}$. We call $\mathcal{S}$ a system of spheres, centred on X for some subset $X \subseteq \mathcal{M}_{\mathcal{L}}$, if it satisfies the following conditions:

(S1) *$\mathcal{S}$ is totally ordered by $\subseteq$; that is, if $U,\ V \in \mathcal{S}$, then $U \subseteq V$ or $V \subseteq U$*

(S2) *X is the $\subseteq$-minimum of $\mathcal{S}$ (i.e., $X \in \mathcal{S}$ and if $U \in \mathcal{S}$, then $X \subseteq U$)*

(S3) *$\mathcal{M}_{\mathcal{L}}$ is in $\mathcal{S}$ (the largest element of $\mathcal{S}$)*

(S4) *If $\alpha \in \mathcal{L}$, and there is any sphere in $\mathcal{S}$ intersecting $[\alpha]$, then there is a smallest sphere in $\mathcal{S}$ intersecting $[\alpha]$ (there is a sphere $U \in \mathcal{S}$ such that $U \cap [\alpha] \neq \emptyset$, and $V \cap [\alpha] \neq \emptyset$ implies $U \subseteq V$ for all $V \in \mathcal{S}$)*

A pictorial representation of a system of spheres centred on $[K]$ is given in Figure 2.1.

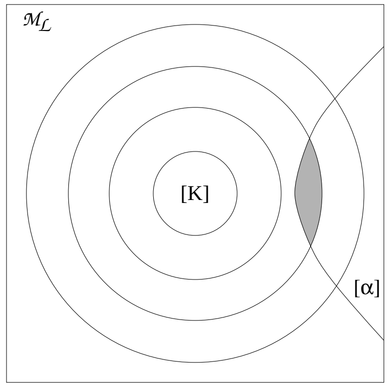


Figure 2.2: Sphere semantics for belief revision showing $[K^*_\alpha]$ shaded.

Condition (S4) guarantees that if any formula α has worlds intersecting $\mathcal{M}_\mathcal{L}$ then there is a smallest sphere (in the sense of set inclusion) or innermost sphere in $\mathcal{S}$ intersecting $[\alpha]$. We shall denote such a sphere by $c_\mathcal{S}(\alpha)$. If $[\alpha]$ does not intersect any sphere in $\mathcal{S}$ (i.e., $[\alpha] \cap \mathcal{M}_\mathcal{L} = \emptyset$), then $c_\mathcal{S}(\alpha) = \mathcal{M}_\mathcal{L}$ (note that this will only occur whenever $[\alpha] = \emptyset$ by condition (S3)).

With any system of spheres $\mathcal{S}$ centred on $[K]$, we can associate a function $f_\mathcal{S} : \mathcal{L} \to 2^{\mathcal{M}_\mathcal{L}}$ defined in the following manner for any $\alpha \in \mathcal{L}$

$$(\text{Def } f_\mathcal{S}) \qquad\qquad f_\mathcal{S}(\alpha) = [\alpha] \cap c_\mathcal{S}(\alpha)$$

Intuitively, the function $f_\mathcal{S}$ can be viewed as selecting those α-worlds[20] in $\mathcal{M}_\mathcal{L}$ that are "closest" to $[K]$. In other words, it selects the innermost α-worlds.

The sphere semantics for a revision operation is now simply specified as follows.

$$[K^*_\alpha] = f_\mathcal{S}(\alpha)$$

That is, the worlds corresponding to a revision of K by α are exactly those α-worlds closest to $[K]$. Such a choice is motivated by the Principle of Minimal Change interpreted with respect to the sphere model outlined above and taking minimality to be "proximity" to $[K]$. It is illustrated in Figure 2.2 (with $[K^*_\alpha] = f_\mathcal{S}(\alpha) = c_\mathcal{S}(\alpha) \cap [\alpha]$ shaded).

The following two representation theorems show that the given semantics is appropriate.

[20]An α-world is any world $m \in \mathcal{M}_\mathcal{L}$ in which α holds (i.e., $\alpha \in m$). $[\alpha]$ is, of course, the set of all α-worlds.

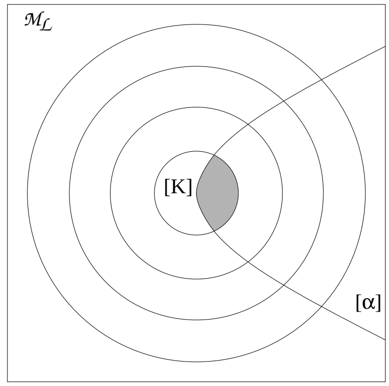


Figure 2.3: Sphere semantics for belief expansion showing $[K^+_\alpha]$ shaded.

Theorem 2.2.15 *[39]*
Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred on $[K]$ for some belief set K in $\mathcal{K}$. If one defines, for any $\alpha \in \mathcal{L}$, K^*_α to be $th(f_{\mathcal{S}}(\alpha))$, then the postulates (K*1) — (K*8) *are satisfied.*

Theorem 2.2.16 *[39]*
Let $* : \mathcal{K} \times \mathcal{L} \to \mathcal{K}$ be any function satisfying postulates (K*1) — (K*8). *Then for any (fixed) belief set K there is a system of spheres on $\mathcal{M}_{\mathcal{L}}$, $\mathcal{S}$ say, centred on $[K]$ and satisfying $K^*_\alpha = th(f_{\mathcal{S}}(\alpha))$, for all $\alpha \in \mathcal{L}$.*

The semantics for belief expansion of an epistemic state K by epistemic input α is now straightforward to determine. In the principal case where $\neg\alpha \notin K$ we have that α is consistent with K and therefore $[K] \cap [\alpha] \neq \emptyset$. That is, the closest α-worlds reside within the innermost sphere $[K]$ and the worlds consistent with the expanded epistemic state are thus given by

$$[K^+_\alpha] = [K] \cap [\alpha]$$

This situation is illustrated in Figure 2.3. In the case that $\neg\alpha \in K$, we have $K^+_\alpha = K_\perp$. However, in this case $[K] \cap [\alpha] = \emptyset$ and so again, $[K^+_\alpha] = [K] \cap [\alpha]$.

The sphere semantics for belief contraction is slightly more involved though not all that complicated. In fact, it can be easily obtained from that of revision using the Harper Identity. In this situation we are losing information and hence increasing the number of possible worlds. In contracting an epistemic state K by epistemic input α we need to

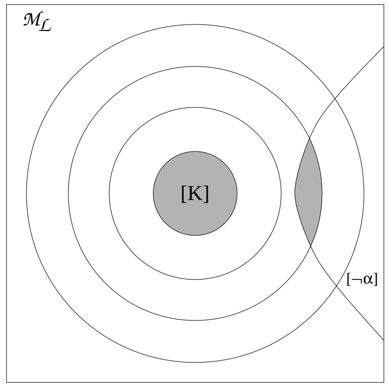


Figure 2.4: Sphere semantics for belief contraction showing $[K_\alpha^-]$ shaded.

supplement the worlds in $[K]$. Specifically, we must at least incorporate some $\neg\alpha$-worlds otherwise α would still be accepted in the contracted epistemic state and therefore violate the postulate of success for belief contraction.[21] In accordance with the Principle of Minimal Change we should add the closest $\neg\alpha$-worlds. Therefore, the worlds consistent with the new epistemic state may be obtained by

$$[K_\alpha^-] = [K] \cup f_\mathcal{S}(\neg\alpha)$$

This situation is illustrated in Figure 2.4.

Epistemic Entrenchment

It was shown by Grove [39] that an ordering over possible worlds is equivalent to an ordering over the formulae of $\mathcal{L}$. A more popular treatment along these lines was developed by Gärdenfors and Makinson [33] and is known as *epistemic entrenchment*. Intuitively, such an ordering represents a preference ordering over formulae. Epistemic entrenchment is motivated, to a large extent, by the fourth rationality criterion above.

In contraction, less entrenched formulae would be removed in preference to more deeply entrenched formulae. An epistemic entrenchment ordering may be formally defined as follows.

[21]We are discussing the principal case here in which $\alpha \in K$. If $\alpha \notin K$ then no change in worlds occurs.

Definition 2.2.3 *([33]) An ordering $\leq$ over $\mathcal{L}$ is an epistemic entrenchment ordering if it satisfies the following conditions:*

(SEE1)	*For any $\alpha,\ \beta,\ \gamma \in \mathcal{L}$, if $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$*	*(transitivity)*
(SEE2)	*For any $\alpha,\ \beta \in \mathcal{L}$, if $\{\alpha\} \vdash \beta$ then $\alpha \leq \beta$*	*(dominance)*
(SEE3)	*For any $\alpha,\ \beta \in \mathcal{L}$, either $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$*	*(conjunctiveness)*
(SEE4)	*When $K \neq K_{\perp}$, $\alpha \notin K$ iff $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$*	*(minimality)*
(SEE5)	*If $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$, then $\vdash \alpha$*	*(maximality)*

The first postulate simply states that an epistemic entrenchment ordering is transitive. The Dominance postulate is based on the rationale that, whenever a formula α entails a formula β and one or the other must be given up, a smaller change would result from abandoning α. Giving up β alone is not possible since, being a consequence of α, it would be retained in the resulting belief set. Giving up α alone, on the other hand, may be possible. Therefore, in general, giving up α would imply a smaller change than giving up β. Hence, β cannot be strictly less entrenched than α. This postulate is clearly motivated by the Principle of Minimal Change. The Conjunctiveness postulate says that removing $\alpha \wedge \beta$ can be accomplished by removing either α or β. The minimality postulate states that non-beliefs are minimally entrenched. The maximality postulate, on the other hand, says that logical truths are maximally entrenched; logical truths are the hardest to give up.[22] Essentially then, an epistemic entrenchment represents a total preorder over the formulae of the language in which tautologies are maximally entrenched and non-beliefs minimally entrenched.

The first three postulates (SEE1) — (SEE3) turn out to be quite significant and any ordering satisfying them is referred to as an *expectations ordering* [34]. Such orderings provide a strong link between the AGM account of belief revision and nonmonotonic inference [34, 72]. Gärdenfors and Makinson supply the following properties satisfied by expectations orderings, some of which will be useful in proving results later in this dissertation.

Lemma 2.2.17 *(*[33]*)*

(i) $\alpha \leq \beta$ *or* $\beta \leq \alpha$ *(Connectivity)*

[22] In fact, they cannot be given up at all given our second rationality criterion.

(ii) *If* $\beta \wedge \gamma \leq \alpha$*, then* $\beta \leq \alpha$ *or* $\gamma \leq \alpha$

(iii) $\alpha < \beta$ *iff* $\alpha \wedge \beta < \beta$

(iv) *If* $\gamma \leq \alpha$ *and* $\gamma \leq \beta$*, then* $\gamma \leq \alpha \wedge \beta$

(v) *If* $\alpha \leq \beta$*, then* $\alpha \leq \alpha \wedge \beta$

Foo [29] also investigated epistemic entrenchment and provides the following further properties related to expectations orderings and epistemic entrenchment orderings. Note that we may write $\alpha = \beta$ for $\alpha \leq \beta$ *and* $\beta \leq \alpha$. Also, $\alpha < \beta$ is a shorthand for $\alpha \leq \beta$ and $\beta \not\leq \alpha$. We denote the greater of a set of formulae Γ by $max\{\Gamma\}$ and the lesser by $min\{\Gamma\}$. Those properties in the next lemma relate to expectations orderings and some will be helpful later on.

Lemma 2.2.18 *(*[29]*)*

(i) *If* $\alpha < \beta$ *and* $\beta < \gamma$*, then* $\alpha < \gamma$

(ii) *If* $\alpha \leq \beta$ *and* $\beta < \gamma$*, then* $\alpha < \gamma$

(iii) *If* $\beta < \gamma$ *and* $\beta < \alpha$*, then* $\beta < \alpha \wedge \gamma$

(iv) $\alpha \not< \alpha$ *for any* $\alpha \in \mathcal{L}$

(v) *If* $\alpha < \beta$*, then* $\alpha < \gamma \vee \beta$ *for any* $\gamma \in \mathcal{L}$

(vi) *If* $\alpha < \beta$*, then* $\alpha < \alpha \vee \beta$

(vii) *If* $\alpha < \beta$*, then* $\alpha \wedge \gamma < \beta$ *for any* $\gamma \in \mathcal{L}$

(viii) *If* $\alpha \leq \beta$*, then* $\alpha \wedge \gamma \leq \beta$

(ix) *If* $\beta \wedge \gamma < \alpha$*, then* $\beta < \alpha$ *or* $\gamma < \alpha$

(x) *If* $\alpha \leq \beta$*, then* $\alpha \wedge \gamma \leq \beta \wedge \gamma$

(xi) *If* $\alpha = \beta$*, then* $\alpha \wedge \gamma = \beta \wedge \gamma$

(xii) $\alpha \wedge \beta = min\{\alpha,\ \beta\}$

(xiii) $\alpha \vee \beta \geq max\{\alpha,\ \beta\}$

(xiv) *If* $\alpha = \alpha \vee \beta$*, then* $max\{\alpha,\ \beta\} = \alpha$

(xv) $\alpha = \alpha \vee \beta$ *or* $\beta = \alpha \vee \beta$ *iff* $max\{\alpha,\ \beta\} = \alpha \vee \beta$

(xvi) α *and* β *are not independent*[23]*iff* $\alpha = \alpha \vee \beta$ *or* $\beta = \alpha \vee \beta$

[23]Two formulae are independent if one can be removed without affecting the other.

The following results relate to epistemic entrenchment orderings.

Lemma 2.2.19 *([29])*

(i) *If* $\nvdash \alpha$ *and* $\vdash \beta$, *then* $\alpha < \beta$

(ii)[24] *Suppose* $\beta \in K$. *If* $\alpha < \beta$, *then* $\beta \in K_\alpha^-$

(iii) *For all* α *and* $\beta \notin K$, $\alpha \leq \beta$ *and* $\beta \leq \alpha$

(iv) $\alpha \notin K$ *iff* $\alpha < \beta$ *for all* $\beta \in K$

An epistemic entrenchment ordering $\leq$ for a particular belief set K may be constructed from a contraction function $-$ using the following condition.

$$(\text{C } \leq) \qquad \alpha \leq_- \beta \text{ iff } \alpha \notin K^-_{\alpha\wedge\beta} \text{ or } \vdash \alpha \wedge \beta$$

The principal part of the condition states that β is at least as epistemically entrenched as α whenever α is removed from K in contracting K by $\alpha \wedge \beta$ since, to contract by $\alpha \wedge \beta$, only one of α or β need be removed and the fact that α has been removed means that it cannot be strictly more entrenched than β (otherwise, only β need be given up). In the case where α and β are both tautological, they are equally entrenched.

More importantly, it is possible to construct a contraction function $-_\leq$ (restricted to a particular K)[25] from an epistemic entrenchment ordering as follows.

$$(\text{C } -) \qquad \beta \in K_\alpha^{-_\leq} \text{ iff both } \beta \in K \text{ and either } \alpha < \alpha \vee \beta \text{ or } \vdash \alpha$$

Clearly, any formula not in the original epistemic state is not going to occur in the contracted epistemic state. In the situation where the epistemic input is a logical truth, it cannot be retracted and therefore no change is made. Otherwise, we note that by the recovery postulate, $\neg\alpha \vee \beta \in K_\alpha^-$ for any belief $\beta \in K$. Now, if the disjunction $\alpha \vee \beta$ of the epistemic input and some belief β is more entrenched than the epistemic input itself, then this disjunction is going to be retained. These two facts imply that β will remain in the contracted state. The

[24] The proof of this property requires condition (C$-$) which we shall introduce shortly.

[25] Note that given an epistemic entrenchment relation $\leq$, the belief set over which it is restricted is easily determined as $K = \{\alpha : \beta < \alpha \text{ for some } \beta \in \mathcal{L}\}$.

following representation theorems show the appropriateness of the epistemic entrenchment ordering and the conditions given above.

Theorem 2.2.20 *[33]*
Let $K \in \mathcal{K}$ be a belief set and $\leq$ be an epistemic entrenchment over K. If for any $\alpha \in \mathcal{L}$, we define K^-_α using (C $-$)*, then* (K$^-$1) — (K$^-$8) *are satisfied as well as the condition* (C $\leq$).

Theorem 2.2.21 *[33]*
Let $- : \mathcal{K} \times \mathcal{L} \to \mathcal{K}$ be any function satisfying (K$^-$1) — (K$^-$8)*. Then, for any belief set $K \in \mathcal{K}$, if we define $\leq$ using condition* (C $\leq$)*, then $\leq$ is an epistemic entrenchment ordering (i.e., it satisfies* (SEE1) — (SEE5)*) and also satisfies condition* (C $-$).

These three constructions are arguably the most important for the AGM framework. They are clearly related as evidenced by the representation theorems. The interested reader is referred to Gärdenfors [31] and Peppas and Williams [100] for a further discussion of these relationships.

Safe Contraction

The construction termed *safe contraction* [3, 4] combines, in a certain sense, elements common to both epistemic entrenchment orderings and partial meet contraction functions. On the one hand, it is assumed that an acyclic relation $<$ of the elements of K is given.[26] Moreover, we consider the minimal subsets of an epistemic state K that imply epistemic input α (in partial meet contraction functions, however, note that we deal with maximal subsets of K failing to imply α). Such a subset may be defined as follows.

Definition 2.2.4 *A set K' is a minimal subset of K implying α if and only if*

(i) $K' \subseteq K$

(ii) $K' \vdash \alpha$

[26] Alchourrón and Makinson refer to an acyclic relation as a “hierarchy”. It will be irreflexive and asymmetric.

(iii) $K'' \not\vdash \alpha$ *for any* $K'' \subset K'$

The set of all minimal subsets of K implying α is denoted $K \Downarrow \alpha$ (cf. $K \perp \alpha$).

Definition 2.2.5 *Any belief $\beta \in K$ is said to be* safe *with respect to α if and only if β is not minimal under $<$ with respect to the elements of any $K' \in K \Downarrow \alpha$. The set of all safe elements of K is denoted $K \setminus \alpha$.*

A belief is safe if it is not "culpable" for the presence of α. Intuitively, one element must be removed from each subset of K in $K \Downarrow \alpha$. The ordering $<$ helps us choose which element to remove from each subset. The remaining beliefs are safe and can be used to determine the *safe contraction* of a belief set K by α (modulo $<$). Specifically, we define $K^-_\alpha = Cn(K \setminus \alpha)$.

Safe contraction functions satisfy the six basic postulates for contraction. It is interesting to investigate particular types of hierarchies $<$ over beliefs.

Definition 2.2.6 *If K is a belief set and $<$ is a hierarchy then, for all $\alpha,\ \beta,\ \gamma \in K$*

(i) $<$ continues up $\vdash$ over K *if and only if* $\alpha < \beta$ *and* $\beta \vdash \gamma$ *imply* $\alpha < \gamma$

(ii) $<$ continues down $\vdash$ over K *if and only if* $\alpha \vdash \beta$ *and* $\beta < \gamma$ *imply* $\alpha < \gamma$

(iii) $<$ is virtually connected over K *if and only if* $\alpha < \beta$ *implies either* $\alpha < \gamma$ *or* $\gamma < \beta$

It can be shown [3] that, if $<$ continues up or down $\vdash$, then the resulting safe contraction function satisfies the postulate of intersection (K$^-$7) and, if $<$ is virtually connected over K, it satisfies the postulate of conjunction (K$^-$8) over K. The following representation theorem holds at least when K consists of a finite number of logically equivalent sentences (i.e., when K is partitioned into a finite number of equivalence classes by the consequence relation $\vdash$).

Theorem 2.2.22 *Let K be a belief set. A safe contraction function $-$ is generated by a hierarchy $<$ that continues up and down $\vdash$ over K and is virtually connected if and only if $-$ is a transitively relational partial meet contraction function over K.*

2.3 Spohn — Ordinal Conditional Functions

An alternative approach to the problem of belief revision has been proposed by Spohn [122]. We shall only give a brief outline here as we do not make much use of this approach in this dissertation.

Spohn bases his account on possible worlds although one need not identify these with the possible worlds considered by Grove [39][27]. The set of all possible worlds is denoted W. Possible worlds are considered to be ordered by a *grading of disbelief*. An *ordinal conditional function* k is used for this purpose, assigning an ordinal to each world $w \in W$. The smaller the ordinal assigned to a world, the more plausible (less disbelieved) it is (0 being the smallest ordinal). In this manner, k specifies an epistemic state.

Since propositions can be identified with sets of worlds, it is also possible to talk about the grading of disbelief of a proposition α. It is simply that of the most plausible of its α-worlds i.e., $k(\alpha) = min\{k(w) : \alpha \in w\}$. A grading of disbelief possesses two important properties

- Either $k(\alpha) = 0$ or $k(\neg\alpha) = 0$ for any proposition α
- $k(\alpha \cup \beta) = min\{k(\alpha),\ k(\beta)\}$ for consistent α and β.

We can therefore say that a proposition α is believed (or accepted) in an epistemic state induced by k if and only if $k(\neg\alpha) > 0$. This proposition α is said to be believed with *degree of firmness* $k(\neg\alpha)$. An ordinal conditional function then, allows us to say whether one proposition is more firmly believed (more plausible) than another proposition β in an epistemic state. In this way, the number of possible epistemic attitudes is greater than those possible with the AGM and therefore ordinal conditional functions are more discriminating.

Another important difference between Spohn's framework and the AGM is the manner in which belief change is effected. Spohn takes epistemic inputs to consist not only of a proposition α but also of an ordinal. Intuitively, the ordinal represents the degree of firmness α should acquire after the change takes place. This means that belief expansion,

[27] Spohn's possible worlds can be thought of as uninterpreted points.

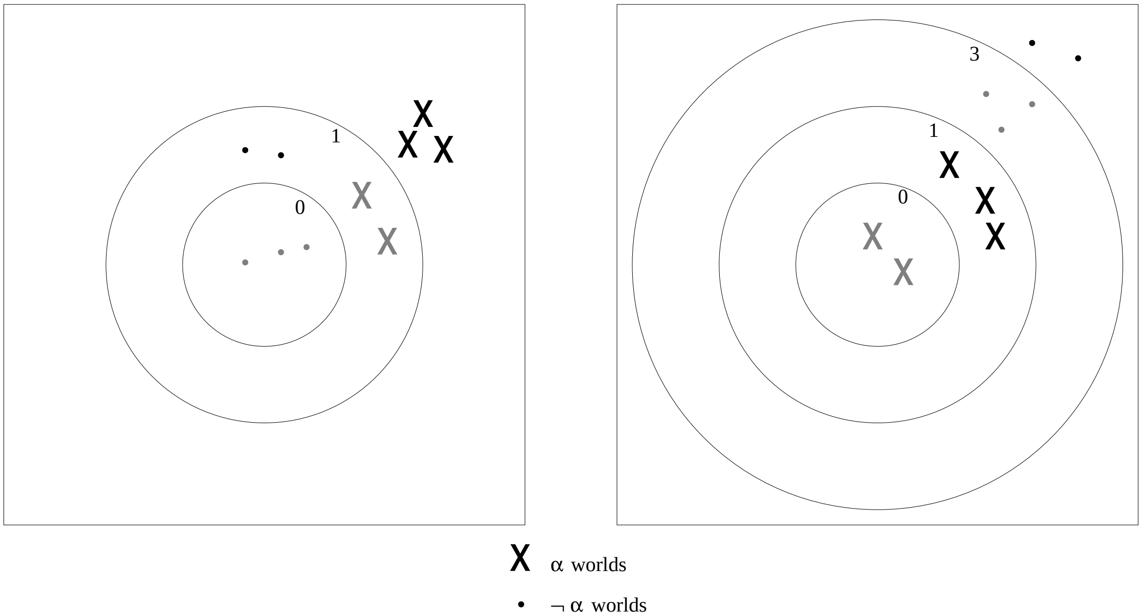


Figure 2.5: An example of Spohn's approach to belief revision.

contraction and revision, as we have come to know them in the AGM, can all be captured by a single mechanism. The actual belief change process is known as *conditionalisation*. The basic idea is that, for epistemic input α (proposition) and i (ordinal) the α and $\neg\alpha$ worlds are "shifted" relative to each other in order to assign α degree of firmness i. An example is illustrated in Figure 2.5. In the ordinal conditional function on the left $k(\alpha) = 1$ and $k(\neg\alpha) = 0$ (i.e., $\neg\alpha$ is believed with degree of firmness 1) while that on the right shows the result of conditionalisation on input $\alpha, 3$. Another advantage of Spohn's approach, evidenced by this example is that it permits *iterated revision*. That is, it is possible to perform a sequence of belief changes due to the fact that the ordering on worlds (grading of disbelief) is still defined after every change. It is not, at first, clear how this is achievable in the AGM since there is no selective mechanism (i.e., system of spheres, epistemic entrenchment, etc.) defined after a change. However, a number of authors have attacked the problem [38, 62, 81, 84, 114, 132].

2.4 Summary and Discussion

Belief revision is the study of the dynamics of epistemic states. The two main approaches to modelling epistemic states are known as foundationalism and coherentism. Essentially, foundationalism posits the existence of a select set of epistemologically basic beliefs whereas coherentism denies the existence of any such beliefs at all. Formally however, any such difference has been called into question. Dixon and Foo [20] show, in the case of contraction, how ATMS behaviour can be achieved through a particular ordering of beliefs in an epistemic entrenchment ordering. Only the relative ordering of certain formulae need be specified, giving rise to a *partial epistemic entrenchment ordering*. This ordering characterises a class of epistemic entrenchment orderings, any of which exhibit the same contraction behaviour as a particular ATMS context. Del Val [19] goes even further, showing that, for a finite propositional language, a mathematical definition of a coherence revision operator based on Katsuno and Mendelzon's [58] version of the AGM and a definition of a foundational revision operator motivated by syntax-based approaches to belief revision (see [70, 85], for example) lead to identical classes of revision operators. This result, however, only shows the equivalence of operators satisfying the definitions given and leaves open the connection between coherence and foundational theories in general. Moreover, one must keep in mind that these theories concern the nature of epistemic states not the method employed to move from one epistemic state to another. We shall stick with the more intuitive descriptions of the theories given here. Arguments for and against both theories can be found in Gärdenfors [32] and Doyle [22].

Our main concern here is with the (purportedly) coherent AGM framework, due principally to its well developed logical theory. Katsuno and Mendelzon [59] claim that the AGM is well suited to situations in which an agent is reasoning about a static world but does not have full information about it. They offer an account of an alternative belief change operator, known as *belief update*, claimed to be suited to reasoning about dynamic worlds. Peppas [99] investigates the relationship between the two approaches but since it is not central to our concerns here, we shall not consider it further.

Chapter 3

Abductive Reasoning

The surprising fact, C, is observed;
But if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true.
Charles Sanders Peirce, [96] 5.189

The term "abduction" was introduced by the philosopher Charles Sanders Peirce [96, 97]. He was the first person to distinguish it as a fundamental form of logical inference alongside deduction and induction. However, this mode of reasoning appears to have its origins in a syllogistic form of reasoning discussed by Aristotle in the Prior Analytics [7] (Book II, Chapter 25), known as *apagoge*.[1] This syllogism looks for premises that would make a given conclusion more desirable. In English, the word "abduction"[2] would seem an appropriate rendering. Peirce also refers to this form of reasoning as retroduction and presumption.

In this chapter we shall provide a brief overview of abduction, particularly as it pertains to artificial intelligence. We begin with an overview of Peirce's views on abduction. We then investigate the two main categories of approaches in artificial intelligence: set-cover based approaches and logic based approaches.

---

[1] $\alpha\pi\alpha\gamma\omega\gamma\grave{\eta} = \grave{\alpha}\pi\acute{o} + \ddot{\alpha}\gamma\omega + \gamma\grave{\eta}$

$\grave{\alpha}\pi\acute{o}$ = away, off, in return, back
$\alpha\gamma\omega$ = lead

In many texts, this word is translated as *reduction*.

[2] The word abduction derives from the Latin word *abdūcere* = *ab* + *dūcere*

ab = from, off
dūcere = to lead, to take

3.1 Peirce's Conception of Abduction[3]

Peirce's main interest in logic stemmed from his desire to develop formal methods of research especially with regard to the advancement of science. His theory of abduction is an attempt to furnish logic with a method for proposing hypotheses. Many philosophers do not agree that logic is involved in proposing a hypothesis but, rather, that it is only concerned with techniques for testing them.

Peirce classified the three fundamental forms of inference into two categories: *explicative inference* and *ampliative inference*. Explicative inference refers to inference where the conclusion follows necessarily from the premises — the conclusion explicates the premises — while ampliative inference refers to inference where the conclusion does not follow necessarily from the premises — the conclusion amplifies the premises. Therefore, deduction is a form of explicative inference whereas abduction and induction are forms of ampliative inference. Peirce notes that from explicative inference to ampliative inference the "security" (or certainty) of the inference decreases while the "uberty" (or productivity) increases.

Peirce not only identified the three fundamental forms of logical inference but also maintained that they represent three stages of scientific inquiry:

(i) abduction proposes hypotheses

(ii) deduction derives the consequences of the hypotheses; and,

(iii) induction tests or verifies hypotheses.

There are two major factors which must be considered when investigating abductive reasoning:

(i) constructing or determining hypotheses; and,

(ii) selecting the "best" or most plausible hypothesis from among these.

Concentrating on the latter, Peirce suggests three major factors that are important in selecting the best hypothesis:

[3]Our main reference for this section is Fann [28].

- a hypothesis must be capable of explaining the facts
- we must be able to verify a hypothesis — in particular, through experiment; and,
- considerations of "economy" should guide the choice of best hypothesis.

With regard to this last factor, Peirce mentions three factors that should play a part. Firstly, it is preferable to select a hypothesis which can be verified with lesser cost with respect to "money, time, energy and thought" ([28] p. 43). Secondly, we should take into account a hypothesis' effect on other "projects"; one must take into account what will happen if a hypothesis turns out to be incorrect and, in fact, attempt to avoid this eventuality. Finally, we need to consider the intrinsic value of a proposed hypothesis.

Peirce proposed a number of ways of ascertaining the intrinsic value of a hypothesis. The notion of "simplicity" is one important consideration. Initially, Peirce used the concept of logical simplicity to express this notion but later he maintained that the hypothesis which appears more "natural" or which is suggested by "instinct" should be considered simpler. Another consideration is the likelihood of a hypothesis, though Peirce suggests that care should be exercised when adopting measures of likelihood as most are subjective. When a hypothesis can be decomposed into parts an important consideration is to use "caution" because "twenty skillful hypotheses will ascertain what a million stupid ones will fail to do" ([28] p. 50). A final consideration is the "breadth of a hypothesis". Basically, a hypothesis which can explain more facts is likely to be more useful.

It is interesting to note that, with respect to the considerations presented above, Peirce is not suggesting that the best hypothesis is "truer" but rather that it should be tested first. This hypothesis is preferred because it appears to be the most easily verifiable. If, ultimately, it is found to be false then we can proceed to examine another hypothesis.

3.2 Set-Cover Based Approaches to Abduction

Set-cover based approaches consist of explicit sets of effects and causes with some representation of the interconnections between them. The basic idea is to construct some set of causes whose associated effects account for all of the effects witnessed.

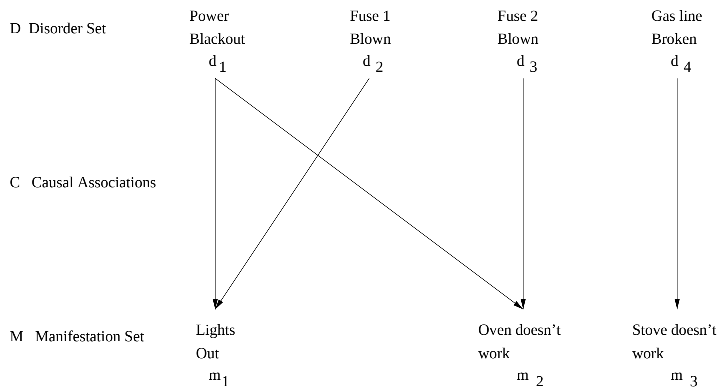


Figure 3.1: An example causal network.

3.2.1 Parsimonious Covering Theory

Peng and Reggia [98] have developed a comprehensive set-cover based approach to solving diagnostic problems through abductive inference. A diagnostic problem involves finding an explanation for the existence of a set of manifestations (observations, symptoms, effects, etc.) using existing knowledge. Peng and Reggia view the task of proposing hypotheses to form an explanation as the "resolution of two conflicting goals" ([98] p. 7):

(i) Covering goal — to explain all present manifestations; and,
(ii) Parsimony goal — to minimise the complexity of the explanation.

It should be evident that these two goals correspond to the two aspects of abduction mentioned in the previous section (i.e., constructing an explanation and selecting the best explanation).

A diagnostic problem consists of two important entities: *manifestations* and *disorders*. Manifestations refer to symptoms, effects, etc. that are observable (e.g., "the engine does not start", "John has a severe cough"). Disorders refer to diseases, malfunctions, etc. and are considered to be causes of manifestations (e.g., "the battery is dead", "John has a cold"). Manifestations are related to disorders by causal associations which can be exhibited through the use of a causal network (see Figure 3.1).

Formally, Parsimonious Covering Theory expresses these notions through the following definition.

Definition 3.2.1 *[98]*
A diagnostic problem P is a 4-tuple $\langle D,\ M,\ C,\ M^+\rangle$ where:

- $D = \{d_1, \ldots, d_n\}$ *is a finite, non-empty set of objects, called disorders*
- $M = \{m_1, \ldots, m_k\}$ *is a finite, non-empty set of objects, called manifestations*
- $C \subseteq D \times M$ *is a relation with domain$(C) = D$ and range$(C) = M$, called causation; and,*
- $M^+ \subseteq M$ *is a distinguished set of M which is said to be present.*

In an implementation based upon this definition, the sets D and M and the relation C would correspond to the knowledge base while the set M^+ would correspond to the input to the system. M^+ need not be fully specified to begin with but may be constructed incrementally.

In determining an explanation it is important to consider the effects of disorders and the causes of manifestations. Therefore, we define the following two sets:

Definition 3.2.2 *[98]*
For any $d_i \in D$ and $m_j \in M$ in a diagnostic problem $P = \langle D,\ M,\ C,\ M^+\rangle$,

effects$(d_i) = \{m_j : \langle d_i,\ m_j\rangle \in C\}$*, the set of objects directly caused by d_i; and,*

causes$(m_j) = \{d_i : \langle d_i,\ m_j\rangle \in C\}$*, the set of objects which can directly cause m_j.*

It is also possible that more than one disorder is present. Therefore, we extend the previous definition to take account of the causes and effects of groups of items.

Definition 3.2.3 *[98]*
For any $D_I \subseteq D$ and $M_J \subseteq M$ in a diagnostic problem $P = \langle D,\ M,\ C,\ M^+\rangle$,

effects$(D_I) = \bigcup_{d_i \in D_I}$ *effects*(d_i)*, and*

causes$(M_J) = \bigcup_{m_j \in M_J}$ *causes*(m_j).

Example 3.2.1 *Using the causal network in Figure 3.1 we have the following examples.*

$$\textit{effects}(d_2) = \{m_1\}$$

$$\textit{causes}(m_2) = \{d_1,\ d_3\}$$

$$\textit{effects}(\{d_2,\ d_3\}) = \{m_1,\ m_2\}$$

$$\textit{causes}(\{m_1,\ m_2\}) = \{d_1,\ d_2,\ d_3\}$$

It should be quite clear that the disorders proposed as an explanation should account for the observed manifestations. To this end, we define the notion of a *cover*.

Definition 3.2.4 *[98]*
The set $D_I \subseteq D$ is said to be a cover of $M_J \subseteq M$ if $M_j \subseteq$ effects(D_I).

In other words, a set of disorders is considered a cover for a set of manifestations if their direct effects include all these manifestations. This allows us to define what constitutes an explanation.

Definition 3.2.5 *[98]*
A set $E \subseteq D$ is said to be an explanation of M^+ for a problem $P = \langle D,\ M,\ C,\ M^+\rangle$ iff E covers M^+ and E satisfies a given parsimony criterion.

Therefore, an explanation consists of a set of disorders whose direct effects are capable of accounting for all the observed manifestations and also satisfy some selective criterion.

Examples of parsimony criteria are given in the following definition of various types of covers.

Definition 3.2.6 *[98]*

A cover D_I *of* M_J *is said to be* minimum *if its cardinality is smallest among all covers of* M_J.

A cover D_I *of* M_J *is said to be* irredundant *if none of its proper subsets is also a cover of* M_J*; it is* redundant *otherwise.*

A cover D_I *of* M^+ *is said to be* relevant *if it is a subset of* $causes(M^+)$*; it is* irrelevant *otherwise.*

Of these, irredundant covers are usually considered the most important. Even though minimum covers explain the observed manifestations by hypothesising the presence of the least number of disorders, there are many cases where this explanation may not be the most plausible. For example, it may seem more reasonable to explain a set of symptoms by hypothesising the presence of certain common diseases rather than proposing an explanation consisting of a lesser number of relatively uncommon diseases. Relevant but redundant explanations are usually not favoured because they contain more hypotheses than are necessary to explain the manifestations present.

Example 3.2.2 *Using the causal network in Figure 3.1 and supposing that neither the lights nor the oven work (i.e.,* $M^+ = \{m_1,\ m_2\}$*), we have the following example of covers:*

$\{d_1\}$	*relevant, minimal and irredundant*
$\{d_2,\ d_3\}$	*relevant and irredundant*
$\{d_1,\ d_2,\ d_3\}$	*relevant and redundant*
$\{d_1,\ d_4\}$	*irrelevant*

The solution to a diagnostic problem, then, is any cover which satisfies our selected parsimony criterion.

Definition 3.2.7 *The solution of a diagnostic problem* $P = \langle D,\ M,\ C,\ M^+\rangle$*, designated Sol*$(P)$*, is the set of all explanations of* M^+.

It is evident that a solution may consist of many explanations. Peng and Reggia also describe a probabilistic method that can be used to prune this collection. They also go on to describe an algebra for constructing covers. The interested reader is referred to [98] for further details.

An alternative set-cover based approach is that by Allemang *et al.* [5] known as *hypothesis assembly*. The proposed process consists of four phases or parts:

screening phase implausible hypotheses are ruled out of consideration

collection phase hypotheses accounting for each observation are collected

parsimony phase redundant hypotheses are removed

critique phase essential hypotheses are determined.

For each phase, algorithms are outlined which, when combined, compute abductions.

3.3 Logic Based Abduction

As the name suggests, logic based abduction attempts to capture the notion of abductive reasoning through the use of a formal language and a suitable logic over this language. It is assumed that we have a domain theory or background theory expressing some conceptualisation of the situation or world in question. This theory takes the form of a set of formulae Γ. Similarly, we are presented with a set of new data (often considered to be observations) — a set of formulae Φ — for which we are attempting to account. The purpose of abduction is to determine a set of hypotheses which, together with the domain theory, would allow us to derive the observations using the underlying logic in question. Ideally, the hypotheses should be consistent with the domain theory since any formula inconsistent with the domain theory would suffice to account for the observations (in a logic such as the one we suppose here).

Definition 3.3.1 *An abduction for a set of formulae Φ with respect to a domain theory Γ is a set of formulae Ψ such that the following two conditions are satisfied:*

(i) $\Gamma \cup \Psi \vdash \Phi$*;*

(ii) $\Gamma \cup \Psi \nvdash \perp$

Note that by $\Gamma \vdash \Phi$ we mean $\Gamma \vdash \phi$ for every $\phi \in \Phi$ which is not always the conventional interpretation of such a sequent (cf. Segerberg [117]). That is, we essentially identify Φ with the conjunction of its elements. Often, a "unit" version of the definition is adopted in which Ψ and Φ are replaced by single formulae ψ and ϕ. We shall adopt such a definition in this dissertation and will return to this issue briefly in the following chapter. Note also our use of the term "abduction". On the one hand we have used it to refer to a particular method of inference and, on the other, to the result of such an inference. An analogous practice is common in the logical treatment of deduction (cf. Enderton [25] for instance). A more popular term for the result of such an inference is "explanation". However, it is not entirely clear to what extent the definition above captures an intuitive notion of explanation.[4] To avoid inaccuracy, we shall continue to refer to the result of an abductive inference as an abduction though occasionally deferring to the term "explanation" in order to provide an intuitive understanding of the framework.

The definition above is quite general and may give rise to many abductions. More often than not however, we are interested in a single "best" abduction. Therefore, it is common to impose further restrictions on abductions, beyond the two specified above, in order to allow the selection of a single abduction or simply to prune the number of abductions that need be considered at a later stage. Such restrictions or *selection criteria* come in a number of categories and we shall briefly survey some of the more popular here (as well as investigating some common abductive frameworks).

[4] The interested reader is referred to Salmon [116] for a look at an overview of such issues. He discusses, among other things, the Hempel-Oppenheim [50] *deductive-nomological* model which bears close resemblance to the notion of abduction.

Syntactic Criteria

One of the more common types of restriction, particularly within artificial intelligence, are syntactic restrictions on the background theory Γ, new data Φ and abduction Ψ. The most frequent is to specify the background theory Γ as a set of clauses or Horn clauses.[5] As we shall see (§ 3.3.5), this can help in the calculation of abductions. In fact, the main motivation behind this restriction (and many others) is to make computation of abductions tractable. Conversion of the background theory to clausal form, even when motivated by computational considerations, indicate that one is less interested in the syntactic nature of the knowledge itself. This can be seen as (at least partial) support for the Principle of Irrelevance of Syntax (see § 2.2.1).

Another common syntactic restriction, especially among implementations, is for abductions to be conjunctions of literals. For example, in Reiter and de Kleer's Clause Management System [112] "[e]xplanations are conjunctions of ground literals" (p. 184). Harman [49] makes the following comment: "[f]urthermore, the relevant explanations are always of the form *R because P,. . . , and Q*, explaining *why* or how it is that something is so. Achinstein (1983) points out that there are other sorts of abduction".[6]

Abducibles

Another very popular restriction (especially among Logic Programming approaches [56]) is to ensure abductions consist of propositional or predicate symbols from a predetermined class. Such propositions and predicates are referred to as *abducibles*. One problem with this approach is that it may be difficult to determine which propositions and predicates should be deemed abducible. Stickel [123] also suggests that what may be considered a suitable abducible in one situation may not be suitable in others and vice versa.

[5] This is especially true of Logic Programming approaches [56].

[6] We retain Harman's emphasis.

Minimality (or Simplicity)

Restricting abductions to be minimal in some way is another very popular restriction. It is often motivated by the concern to avoid suggesting superfluous hypotheses in an abduction. This can be viewed as an adoption of the popular Occam's Razor maxim. Often, this criterion is used together with other restrictions. For instance, if we adopt the syntactic restriction mentioned above, that abductions be conjunctions of literals, then minimal abductions might be those for which no proper subset of their literals are also abductions. Alternatively, abductions consisting of a minimal number of literals may be preferred. In some cases minimality forms an integral part of a restrictive measure. In the cost-based restrictions outlined below, abductions that minimise some cost measure are preferred. In a similar fashion, Charniak [13] adopts the hypothesis which maximises the expression $\frac{E}{A}$ where E is the number of explained observations and A the number of assumptions made. Ram and Leake [106] mention a few other such criteria. Firstly, the explanation whose causal chain is shortest in overall length may be preferred. Alternatively, if one hypothesis subsumes another, then the more general one is preferred as it is likely to explain more (cf. Specificity below).

Triviality

The simplest way to explain new data is simply to hypothesise that it is true. Intuitively however, such explanations or abductions are not very compelling. One can overcome this by specifying that abductions be *non-trivial*. None of the new data should appear in the resulting abduction. Alternatively, one can specify that the abduction make some use of the background theory Γ. In other words, the abduction alone should not be able to prove the new data ($\Psi \nvdash \Phi$).

Specificity

If an abduction, with the help of the background theory Γ, is able to prove another abduction, then we say that the former is a *more specific abduction* than the latter. Considering all such interconnections of specificity among abduction allows us to talk of the *levels of specificity* of an abduction or explanation. Stickel [123] discusses three different types of abduction based on this idea. He restricts his background theory (which he calls a knowledge base) to first-order Horn clauses and considers explanations to be conjunctions of positive literals (together with a substitution) but his ideas are applicable in other settings also.

By allowing only pure literals — literals that cannot be resolved with any clauses in the background theory — to be assumed, we obtain abductions that are maximally specific in Stickel's setting. These are referred to as *most specific abductions*. This type of abduction is not uncommon. It is the type of abduction computed by Pople's [105] procedure and also by the procedure described by Cox and Pietrzykowski [17]. In fact, this type of abduction is often that identified by Abductive Logic Programming approaches [104].

Stickel claims that this form of abduction is suited to diagnostic tasks as it tends to neglect trivial explanations. For instance, suppose we are trying to explain ”the car won't start”. Suggesting the trivial explanation — the car won't start — is unlikely to be of much use whereas suggesting that the ignition system is faulty would seem to be more helpful. Of course, explanations can be too specific. Suggesting that a specific component of the ignition system is at fault may be of less use than simply hypothesising that the ignition system is faulty and should be replaced.

At the other end of the spectrum, we could simply assume what we are attempting to explain. That is, adopt the trivial abduction. Stickel refers to this as *least specific abduction*. He argues that it is best suited to natural language interpretation where we may be uninterested in the complexities underlying the situation at hand. In attempting to explain "the car won't start", one could simply hypothesise that the car won't start. The suggestion is that, when we interpret a sentence, the meaning is near the surface. One advantage of this approach is that it does not attempt to determine more specific abductions which often involve riskier assumptions.

The other type of abduction suggested by Stickel is called *predicate specific abduction* and is exactly the technique, outlined above, of adopting abducibles. That is, abductions are constructed from a subset of predicates known as abducibles or *assumables* (Stickel's terminology). He claims that this type of abduction is suited to planning and design synthesis problems.

Coherence

Notions of coherence — in an abductive setting, more commonly referred to as *explanatory coherence* — attempt to determine the best abduction or explanation by measuring how well component hypotheses "stick together" or support each other.

Ng and Mooney [88] propose an *explanatory coherence metric* based on the proof tree for the new data (which incorporates the abduction). The metric favours abductions possessing certain properties. Those leading to more connections between any two observations (new data) and fewer disjoint partitions are particularly favoured. The metric can be formally defined as follows:

$$C = \frac{\sum_{1 \leq i \leq j \leq l} N_{i,j}}{N \begin{pmatrix} l \\ 2 \end{pmatrix}}$$

where

$l =$ total number of observations;

$N =$ total number of nodes in proof graph; $\begin{pmatrix} l \\ 2 \end{pmatrix} = \frac{l(l-1)}{2}$; and,

$N_{i,j} =$ number of distinct nodes n_k in proof graph such that there is a sequence of directed edges from n_k to n_i and a sequence of directed edges from n_k to n_j, where n_i and n_j are observations.

The denominator scales the result so that it will lie within the interval $[0, 1]$. The numerator measures the total number of nodes in an explanation connecting pairs of observations, increasing with the number of nodes in an explanation that simultaneously lend support to a given connection. Ng and Mooney claim that, using a depth-first search algorithm, this value can be calculated in $O(lN + e)$ time, where l and N are as above and e is the total number of (directed) edges in the proof graph.

This approach has the advantage of being less affected by the way in which the background theory is represented. There is also a good likelihood that abductions will consist of a minimal number of assumptions. Ng and Mooney also point out that explanatory coherence can be used to determine the specificity of an abduction. The more specific abduction is adopted if it leads to an increase in the value of the coherence metric.

Thagard [127, 128] develops a comprehensive theory of explanatory coherence. His theory is based on a primitive notion of explanation[7] and develops a notion of *acceptability* of a proposition. A proposition that is coherent with our beliefs should be accepted; a proposition that is incoherent with our beliefs should be rejected; and, a proposition which is neither coherent nor incoherent with our beliefs should be treated indifferently.[8] Thagard outlines eight principles to determine coherence between propositions and the global coherence of a system of propositions. We shall not delve into Thagard's principles here but note that a certain "bias" is directed towards propositions that require fewer assumptions to be explanations, explain more observations, possess more specific explanations, or represent observations.

Thagard also presents a network based implementation. Nodes of the network represent propositions and possess a degree of acceptability. Links in the network represent coherence (or incoherence) of propositions and possess a value representing the degree of coherence (a negative value if the propositions incohere). Propositions joined by coherence links support each other while those joined by incoherence links hinder each other. Acceptability values are allowed to "propagate" through the network until they exhibit asymptotic behaviour. The network is examined to determine which propositions have positive acceptability (i.e., are accepted) and which have negative acceptability values (i.e., are rejected). Thagard demonstrates his implementation, modelling competing scientific theories (e.g., phlogiston theory versus Lavoisier's theory of combustion) and legal arguments, among others. An interesting advantage of this implementation is that new information can be added without the need to restart computation from scratch. Another advantage is that explanations need

[7] Thagard does not tell us exactly what explanation is, beyond being a relation among propositions: $P_1, \ldots, P_n$ *explain* Q. He does however tell us what it is not and clearly does not intend a logic based formulation — at least not as general as that introduced at the beginning of this section.

[8] If two propositions do not cohere then they do not necessarily incohere.

not account for all observations which sometimes appears to occur in reality. The main problem with this approach is the lack of definitions for such notions as explanation and analogy.

Probability Measures

Another way of discriminating among abductions is to attach probabilities to propositions. Such methods usually rely on Bayes' rule to combine probabilities. Often however, due to the large number of probabilities required for a problem, simplifying assumptions and heuristics are adopted to reduce the complexity of calculating the probability of an abduction. We shall not elaborate on any specific probabilistic approaches here but note that many of the simplifying assumptions and heuristics used have been criticised for being unrealistic.

Cost-based Measures

Cost-based measures [15, 123] work by associating a cost with making an assumption or using an axiom from the background theory. The best abduction(s) is usually considered to be that which, when assumed, minimises the overall cost.

Utility-based Measures

Many of the measures for determining the best hypothesis that we have discussed so far have been directly related to the structural (syntactic) properties of an explanation. Ram and Leake [106] introduce the notion of utility-based criteria which select hypotheses based on a system's intended use for an explanation. Their main use of these criteria are in explanations of anomalies (i.e., situations where our expectations were not realised).

Ram and Leake claim that an explanation of an anomaly must answer two questions:

- Why did things occur as they did in the world? This question focuses on understanding and learning about the causal structure of the domain.

- Why did I fail to predict this correctly? This question focuses on understanding and improving the organisation of the reasoner's own model of the domain.

Utility-based criteria are methods of evaluation related to these two criteria. Utility-based criteria refer to any method of evaluating explanations based on goals arising from these two questions. In order to clarify the form these criteria may take, we shall briefly discuss these questions.

The first of these questions relates to the causality of the domain and is thus called a *domain explanation*. It leads to knowledge acquisition goals which aim to collect more information about the domain. These goals are usually guided by the explainer's tasks. Some of the tasks that can give rise to knowledge acquisition goals are [106]:

- Choosing a response to an unexpected event — learn causes that allow discrimination between possible plans by predicting events or identifying current circumstances
- Repairing an undesirable state — learn repairable causes of that state
- Causing recurrence — learn achievable causes
- Preventing recurrence — learn blockable causes
- Assigning credit or blame — learn particular actors' influence on an outcome
- Replicating another actor's success — learn motivations of the observed actor's unusual planning decisions.

The second question involves the reasoning processes of the system and is called *introspective* or *meta-explanation*. It results in knowledge organisation goals which aim to improve the organisation of knowledge in memory. Some of the factors that can characterise this type of goal are [106]:

- Missing knowledge — learn new knowledge to fill gap in domain model
- Unconnected knowledge — learn new connection or new index

- Implicit assumption — learn heuristics for when to check assumption explicitly
- Calculated simplification — learn heuristics for when to check assumption in detail
- Explicit assumption — learn new knowledge to correct the assumption
- Conjunctive assumptions — learn new interactions.

3.3.1 Abduction and Default Logic

An important form of nonmonotonic inference that has become popular within artificial intelligence is *default reasoning* [111]. Poole [102] presents a logical framework for performing default reasoning, based on the THEORIST system [104], which is abductive in nature. It is claimed that this method subsumes the "intuition behind Reiter's default logic" ([102] p. 27) and argued that there is no need to go to the trouble of defining a new logic; one can alter the way logic is used. In fact, we shall find it close in spirit to part of our aims here.

The main idea behind this framework is to require a set of hypotheses, together with a set of facts, to entail some goal. Three types of formulae are distinguished:

$\mathcal{F}$ set of facts;

Δ set of possible hypotheses; and,

g observation to be explained.

The set of facts are formulae representing those things we believe to be true in the world and are not prepared to give up. In our belief change framework, $\mathcal{F}$ would essentially correspond to the agent's epistemic state and the latter proviso could be weakened in certain situations. The set of possible hypotheses are things we are prepared to accept in constructing an explanation. Any ground instances of these formulae may be assumed provided they are consistent. By thinking of these formulae as possible hypotheses or abductions, the abductive nature of this formalism should be apparent. On the other hand, thinking of them as defaults gives us default reasoning.

The first definition we require is that of a *scenario* where the connection with abduction is further borne out.

Definition 3.3.2 *(*Scenario *[102])*
A scenario *of* $\mathcal{F}, \Delta$ *is a set* $D \cup \mathcal{F}$ *where* D *is a set of ground instances of elements of* Δ *such that* $D \cup \mathcal{F}$ *is consistent.*

This allows us to introduce a notion of explanation.

Definition 3.3.3 *[102]*
If g *is a closed formula, then an explanation of* g *from* $\mathcal{F}, \Delta$ *is a scenario of* $\mathcal{F}, \Delta$ *which implies* g*.*

So, an explanation is a scenario implying the observation or new information. Note that the only way that an abduction differs from an explanation is that we do not require an abduction to include $\mathcal{F}$.

Definition 3.3.4 *(*Extension *[102])*
An extension *of* $\mathcal{F}, \Delta$ *is the set of logical consequences of a maximal (with respect to set inclusion) scenario of* $\mathcal{F}, \Delta$*.*

This notion of extension is more important for default reasoning than abduction in general. We include it here for later reference.

Example 3.3.1 *[102]*
Suppose we have the following facts and defaults:

$$
\begin{aligned}
\mathcal{F} = \{\ & \forall x \ \ \mathit{emu}(x) \rightarrow \mathit{bird}(x), \\
& \forall x \ \ \mathit{emu}(x) \rightarrow \neg \mathit{flies}(x), \\
& \mathit{emu}(\mathit{polly}), \\
& \mathit{bird}(\mathit{tweety}) \\
& \} \\
\Delta = \{\ & \mathit{bird}(x) \rightarrow \mathit{flies}(x)\}
\end{aligned}
$$

In this instance, the default set can be interpreted as saying that birds normally fly. The formula flies$(tweety)$ *can be explained by the instance* $\{$*bird*$(tweety) \rightarrow$ *flies*$(tweety)\}$. *That is, tweety is expected to fly because it is a bird. The formula flies*$(polly)$ *is not explainable because it is not consistent with the facts.* □

Poole shows that this framework is compact (i.e., any g that can be explained, can be done so with a finite scenario) and that it possesses a monotonicity property in that anything that can be explained with a set of facts and a set of hypotheses can also be explained using the same facts and a superset of the hypotheses. This does not hold in general if the set of facts is changed. A more important property follows.

Theorem 3.3.1 *[102]*
α *is explainable if and only if* α *is in some extension.*

Poole introduces a method for naming defaults which simplifies both the notation and the implementation of the system (THEORIST) on which the formalism is based. He also introduces the notion of constraints, which we shall see in the following section, that can be used to rule out the applicability of some hypotheses. Brewka [10] generalises this framework by blurring the distinction between facts and defaults and allowing any number of partitions of formulae which, for practical purposes, can be considered linearly ordered when constructing extensions. Poole [103] also extends this approach by introducing a probabilistic approach based on Bayes' rule for determining the best explanation. As with other probabilistic approaches, he makes a number of assumptions to reduce the complexity of the calculations involved.

3.3.2 Abduction and Negation as Failure

An approach essentially the same as that of THEORIST but couched within a logic programming setting, is proposed by Eshghi and Kowlaski [27]. They begin by defining an *abduction framework*.

Definition 3.3.5 *Abduction Framework [27]*

$\langle T,\ I,\ A\rangle$ *is an abduction framework iff,*

T *is a Horn clause theory (without denials)*

I *is a set of integrity constraints; and,*

A *is a set of predicate symbols, called abducible predicates*

Definition 3.3.6 *[27]*
Given the abduction framework $\langle T,\ I,\ A\rangle$ *the hypothesis set* Δ *is an abductive solution for the existentially quantified conjunction of atoms* Q *if and only if*

Δ *is a set of variable free abducible atoms*

$T \cup \Delta \models Q$

$T \cup \Delta$ *satisfies* I.

Using integrity constraints that are denials, this is the same use of integrity constraints as proposed by Poole [102]. He stipulates that $T \cup \Delta \cup I$ (or using his notation, $D \cup \mathcal{F} \cup C$) should be consistent. In this way, constraints are used to reject scenarios but they are not used to generate abductions or explanations directly. This abductive framework can be used to give an abductive semantics for negation as failure [68] in logic programming. Essentially, abductive hypotheses satisfying certain integrity constraints with respect to a logic program can be identified with negated literals. The field of *abductive logic programming* has grown out of such approaches and become a very important area of artificial intelligence. The interested reader is referred to the survey by Kakas *et al.* [56].

3.3.3 Abduction and Truth Maintenance

The link between the ATMS and abduction is made clear by Reiter and de Kleer [112]. Actually, they develop the notion of a *Clause Management System* (CMS) which is a generalisation of the ATMS. A CMS receives clauses and keeps track of them. It may also be queried with a propositional clause C in which case its task is essentially to respond with all minimal justifications. More specifically, it returns all clauses S such that $S \vee C$ follows logically from the clauses received so far but S itself does not follow logically from these clauses (i.e., $\Gamma \vdash S \vee C$ but $\Gamma \not\vdash S$ — where Γ represents the currently stored clauses). Moreover, S is required to be minimal in the sense that there is no clause S' containing a subset of the literals (of appropriate sign) in S also satisfying these conditions. Reiter and de Kleer point out that the ATMS is simply a CMS where clauses sent to the system are either Horn (corresponding to justifications) or negative (corresponding to nogoods) and queries are restricted to literals. A CMS, like a truth maintenance system, can be utilised as part of a larger problem-solving architecture and, especially, in conjunction with a domain dependent reasoner.

It turns out that what the CMS returns in response to a query can be converted into abductions. Consider a query C and suppose the CMS (characterised by Γ) returns, among other things, a clause S. The first restriction above, that $S \vee C$ logically follows from the clauses currently stored in the CMS, is equivalent to $\Gamma \vdash \neg S \rightarrow C$ (where $\neg S$ is viewed as the negation of a clause — so, if $S = s_1 \vee s_2 \vee ... \vee s_n$, then $\neg S = \neg s_1 \wedge \neg s_2 \wedge ... \wedge \neg s_n$).[9]

The second proviso, that S not be a logical consequence of the clauses in the CMS, is equivalent to saying that Γ and $\neg S$ are consistent (i.e., $\Gamma \cup \{\neg S\} \not\vdash \perp$). Clearly then, $\neg S$ is an abduction. If it were to be added to the CMS, then the query C would be a logical consequence of the clauses stored in the CMS.[10]

However, the abductions returned by the CMS, or more correctly, the abductions that can be obtained from the clauses returned, are of a special type. It can be seen that they are restricted syntactically to consist of conjunctions of literals and that these conjuncts are minimal in the sense that no conjunct with a subset of these literals (including sign) is also an abduction of the query with respect to the CMS. However, the CMS does not apply any further mechanism to select the best clause (S). Referring back to our overview of the ATMS (§ 2.1), it can be seen that the CMS is essentially returning a label for the query. When negated, the returned clauses are simply environments. The entire collection of returned (negated) clauses constitute a label satisfying the requisite criteria

[9] Alternatively, if we view a clause as a set of literals $S = \{s_1, s_2, ..., s_n\}$, then its negation can be viewed as a set of (singleton) clauses $\neg S = \{\{\neg s_1\}, \{\neg s_2\}, ..., \{\neg s_n\}\}$.

[10] It is easy to add $\neg S$ to the CMS by transmitting all the negated literals in S to it.

(i.e., consistency, completeness, soundness, and minimality). The interested reader is also referred to the survey on Abductive Logic Programming by Kakas *et al.* [56] for other relationships between abduction and truth maintenance systems.

3.3.4 Knowledge Level Approach to Abduction

Levesque [63] presents a knowledge level approach to abduction which subsumes, to a certain extent, the approaches presented thus far. Some surveys (see Paul [95] for instance) identify this as an approach separate from both set-cover and logic based approaches though it is essentially logical in nature. Levesque's approach is based on enriching the language to include a belief operator $\mathbf{B}_\lambda$.[11] By varying this notion of belief, different notions of abduction can be characterised. His motivation stems from a desire to move away from notions of abduction based on material implication. It may of course be possible to achieve the same result to a certain extent by altering the underlying logic without extending the language.[12] He cites the following example:

> If we know that Marc is 3 or 4 years old, then the fact that he is not yet 4 does not explain why he is 3 even though it does imply it.

This prompts the following definition of an abduction or explanation.

Definition 3.3.7 α expl$_\lambda$ β *with respect to epistemic state* e *if and only if* $e \models [\mathbf{B}_\lambda(\alpha \rightarrow \beta) \wedge \neg\mathbf{B}_\lambda\neg\alpha]$.

Basically, this definition says that α is an acceptable explanation for β if and only if we believe that α implies β and we have no reason to believe that α is false. Levesque also provides a notion of simplicity based on the set of (signed) literals of an explanation in order to define the concept of a minimal explanation. Ultimately, the best explanations of a formula can be considered the disjunction of all minimal explanations of that formula. We

[11] The λ is used to denote different types of belief.

[12] This would of course preclude modal logics but the point remains that Levesque does not show the necessity of introducing an operator for belief.

shall not concern ourselves with the details here but refer the interested reader to Levesque [63].

3.3.5 Computing Abduction

A variety of methods have been suggested for computing abductions. Reiter and de Kleer [112] suggest that these methods may be classified into two groups: *interpreted* approaches and *compiled* approaches. Interpreted approaches store the formulae in the domain theory in the form they have been supplied and compute abductions on the fly. That is to say, whenever abductions are required, they are computed from scratch. Compiled approaches on the other hand "pre-compile" the background theory Γ into an intermediate form which allows the generation of abductions to be performed quickly and easily. Precompilation may consume time and resources although the subsequent generation of abductions is usually faster than interpreted approaches.

Interpreted Approach

This type of approach is commonly found in the artificial intelligence literature on abduction. It is present in very early work [105], forms the basis of the procedure developed for the THEORIST system [104] and the method of Cox and Pietrzykowski [17] and is very common in abductive logic programming. More often than not, the domain theory is assumed to be a collection of Horn clauses or definite clauses. Adopting a procedural view reminiscent of Prolog [68], the basic idea behind many of these approaches is to treat the new information as a goal to be proved. Whenever there is a subgoal that cannot be proved it is simply assumed. A possible abduction can be derived by conjoining all the assumptions made. Alternative abductions can be obtained through backtracking. An algorithm of this type can be found in the description of the THEORIST system (see [104] p. 350).

Compiled Approach

The most common method of pre-compilation transforms a clausal domain theory Γ into a set of logically equivalent *prime implicates*.[13]

Definition 3.3.8 *(see [112])*

A prime implicate of a set of clauses Γ *is a clause* C *such that*

$\Gamma \vdash C$*, and*

$\Gamma \nvdash C'$ *for any* $C' \subset C$

The set of all prime implicates of a set of clauses Γ *is denoted* $PI(\Gamma)$.

Algorithms for generating prime implicates may be found in Jackson [54] and Kean [60]. Once the prime implicates of a domain theory have been generated, the task of computing abductions (in the sense of conjunctions of literals) is rather simple. There are two methods commonly employed. The first is that described in § 3.3.3 for obtaining abductions from a CMS. Given a clause O representing the explanandum, and a prime implicate $P \in PI(\Gamma)$ of the domain theory such that $P = E \vee O$ for some clause E, an abduction is given by negating the clause E. Minimal abductions in the sense of set inclusion are obtained by considering only those clauses E generated as above which are not subsumed by any other likewise generated clauses (i.e., there is no other E' such that $E' \vee O \in PI(\Gamma)$ and $E' \subset E$). More details on this method may be found in Reiter and de Kleer [112]. Another way of computing abductions using prime implicates is to use a contrapositive argument. We are looking for a set of singleton clauses, Δ, such that $\Gamma \cup \Delta \vdash O$. Using contraposition gives $\Gamma \cup \neg O \vdash \neg\Delta$. Now $\neg\Delta$ is a clause. Therefore abductions can be easily generated by negating the explanandum clause O and deriving logical consequences with the domain theory Γ (via resolution say). Since $PI(\Gamma)$ is logically equivalent to Γ, we need only consider the prime implicates of the domain theory. Minimal abductions may be obtained

[13] The term prime implicant is also used. However, we do not use it here in order to make a clear distinction between the concept we now define and a dual notion which is often called a prime implicant.

using a subsumption test. An algorithm using this method, based on an idea by Jackson [52], will be presented in the next chapter.

We notice that the majority of approaches to computing abductions confine their attention to domain theories specified in clausal or Horn clausal form and compute abductions which are (minimal) conjunctions of literals (and, in the first-order case, also ground). The problem of finding all minimal abductions in the sense of Reiter and de Kleer (i.e., minimal conjunction of literals) using the interpreted approach is, in general, NP-hard [112]. On the other hand, the generation of prime implicates may require an exponential (in terms of propositional symbols) amount of space. Empirical studies have shown that pre-compilation of the domain theory can be beneficial [8]. It is easy to see, however, that the interpreted approach will be more useful if there are likely to be more additions to the domain theory rather than requests for abductions while the compiled approach is better if a lot more requests for abductions are made. Kean [60] provides an algorithm for incrementally computing prime implicates so that, when new clauses are added to the domain theory, additional prime implicates can be determined without computing the whole lot over again.

3.4 Summary and Discussion

We have surveyed a cross-section of the work on abduction, mainly in the field of artificial intelligence. This form of inference is beginning to receive increased attention as an important form of reasoning. It has been applied to problems in a large number of areas: diagnostic tasks [98, 105], database updates [57], natural language interpretation [13, 14, 88, 123] and scientific discovery and learning [76].

Such diverse applications suggest that it would be useful to incorporate abduction into a belief revision framework which allows an inquiring agent to keep track of knowledge it has gleaned. We adopt a logical notion of abduction in this dissertation because it is well suited to the AGM framework for belief revision and present a formal study of the use of abduction for the purpose of belief revision.

3.4.1 Why (Logical) Abduction?

Why should one use abduction in preference to other existing techniques for reasoning? A popular competitor to abduction in artificial intelligence, especially for diagnostic problems, is the rule-based or production system often used in expert systems [53]. In these systems, as the name suggests, knowledge is represented by a collection of rules or productions which generally take the form:

If effects *then* causes.

These rules are then used deductively to diagnose a problem. Peng and Reggia [98] highlight a number of problems with this approach which we shall briefly point out here. Firstly, it is difficult to represent knowledge in this form because, intuitively, we often think in terms of the other direction (i.e., from causes to effects rather than the other way round). This leads to some problems in the application of the rules. Say we have two production rules in the form above leading to two different causes but one set of effects is a proper subset of the other. If the system is presented with effects only from the smaller set then only the associated cause will be suggested. Yet, the other cause is definitely a possibility since it can produce the desired effects (and usually more). Another problem occurs when the system is presented with, say, the larger set of effects. In this case both causes would be suggested by the system (because they follow deductively) yet one would likely rule out the cause associated with the smaller set of effects because it is not capable of accounting for all the effects. There may be ways around these problems by adding further rules. This solution, however, may be cumbersome and unintuitive. A further problem with this approach is that it has difficulty handling multiple causes. An abductive mechanism would have the advantage of allowing the knowledge to be represented in a more intuitive way, thus avoiding these problems.

This dissertation adopts a logical approach to abduction which will be explored in greater depth in the following chapter. Our main reason for doing so is because a logical definition fits well with the AGM framework which guides our work. However, what advantages or disadvantages does such an approach have over a set-cover based approach? Parsimonious Covering Theory surveyed here is only geared to finding explanations for conjunctions of manifestations and cannot deal with more general situations (though it may not be too difficult to suggest ways of overcoming this problem). One other disadvantage of Parsimonious Covering Theory is that it relies heavily on the causal diagram for representing problems. Thagard [129] points out that this means it cannot deal with negative evidence; the absence of a manifestation would rule out a disorder as a possible cause.

Thagard [129] also raises a number of other problems. He mentions that the notion of parsimony in terms of irredundant covers may not provide the best explanation in all cases. In some cases it may be possible that the presence of more than one disorder provides the best explanation. For example [129], someone with a cough and a runny nose may be suffering from both a cold and hay fever. This argument however also argues against certain notions of minimality in a logical notion of abduction. Thagard claims that another problem occurs when a cover cannot be found for the observations. (This is in fact not possible given the definition of a diagnostic problem with Parsimonious Covering Theory.) However, his argument is that in many cases an acceptable explanation need not cover all observed manifestations. For instance [129], Newton's particle theory of light was deemed acceptable even though it did not account for every known behaviour of light. In such a case, Thagard suggests that the property of minimum cardinality has one advantage of being applicable although a cover does not exist. The fact that every manifestation must be caused by some disorder also means that the system cannot discover disorders to explain the manifestations. This does not comply with Peirce's idea that abductive inference is concerned with discovery. Levesque [63] notes that set-cover based approaches suffer from difficulty in being able to express how a minor alteration to the domain knowledge can contribute to changing what constitutes an explanation. Logic based approaches, on the other hand, tend to confine reasoning to global properties of logic like consistency and implication.[14] Logic based approaches tend to be more general than set-cover based approaches.

[14] Levesque's [63] knowledge level approach goes some way to solving this problem.

Chapter 4

A Logical Exposition of the Notion of Abduction[1]

. . . a new fact is a grain of sand that irritates the mind and abduction forms a pearl of wisdom around it.
John F. Sowa [120] 1984[2]

Since our chosen belief revision framework, the AGM, is essentially a logical one it will be advantageous to have a clear idea of the logical nature of the notion of abduction. Therefore, we begin with an exploration of the logical aspects of abduction. After furnishing a logical definition of abduction we shall investigate, in a logical manner, various types of abduction; some already mentioned in the previous chapter. We then turn to the process of induction and attempt to understand the difference between abduction and a particular form of induction. Armed with our definition of abduction we shall investigate the role it plays in belief revision in subsequent chapters.

4.1 Defining Abduction

We start with the definition of abduction given in the previous chapter. As noted there, this definition is one of the more common ones to be found in the literature [52, 95, 102].

[1]Some of the work in this chapter has appeared in [90].
[2]Describing a metaphor by C. Lewis and R. Mack [66].

Definition 4.1.1 *An abduction of a set of formulae* Φ *with respect to a set of background formulae* Γ *is a set* Ψ *such that*

$\Gamma \cup \Psi \vdash \Phi$

$\Gamma \cup \Psi \nvdash \perp$

Let us return to our example in the first chapter and suppose our background theory Γ contains the following formulae:

$$\begin{aligned} \Gamma = \{ \quad & o_1 \wedge ph_1 \rightarrow e, \\ & o_2 \wedge ph_2 \rightarrow e, \\ & ch \rightarrow e, \\ & co_1 \rightarrow o_1, \\ & co_2 \rightarrow o_2 \\ & \} \end{aligned}$$

Suppose we wish to account for simply the data $\Phi = \{e\}$. Possible abductions include:

$$\begin{aligned} & \{o_1,\ ph_1\} \\ & \{o_1 \wedge ph_1\} \\ & \{o_1 \wedge ph_1,\ o_2 \wedge ph_2\} \\ & \{(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2)\} \\ & \{co_1 \wedge ph_1\} \\ & \{co_1 \wedge ph_1,\ o_1\} \\ & \{ch\} \end{aligned}$$

Now, instead of considering those candidate Φ which contain infinitely many formulae, let us restrict our attention to those containing only a finite number of formulae. We can justify this restriction in a number of ways. Since $\Gamma \cup \Psi \vdash \Phi$ is to be interpreted as $\Gamma \cup \Psi \vdash \phi$ for every $\phi \in \Phi$ we can think of $\Gamma \cup \Psi \vdash \Phi$ as $\Gamma \cup \Psi \vdash \bigwedge \Phi$ where $\bigwedge \Phi$ represents $\bigwedge_i \phi_i$ such that $\phi_i \in \Phi$. However, if Φ is infinite or, more precisely, not finitely axiomatisable, then $\bigwedge \Phi$ is not expressible in $\mathcal{L}$. More to the point, since we are considering $\mathcal{L}$ to be finite here (see § 1.3), then there will only be finitely many truth functional propositions and so we are always assured that $\bigwedge \Phi$ can be expressed as a formula $\phi \in \mathcal{L}$. The same argument can be applied to the abduction Ψ justifying its replacement as a formula $\psi \in \mathcal{L}$. However, the following two results show that this would follow from the compactness of Cn and the representation of Φ as a unit formulae.

Observation 4.1.1 [3] *If an abduction Ψ of a formula ϕ with respect to a domain theory Γ exists, then a finite abduction $\Psi' \subseteq \Psi$ (where Ψ' is understood to be finite) of ϕ with respect to Γ exists.*

By a finite abduction, we mean one containing a finite number of formulae from $\mathcal{L}$. The following result is also easily proved.

Lemma 4.1.2 *If a finite abduction Ψ of ϕ with respect to Γ exists, then it can be represented by a single formula ψ (i.e., $\Gamma \cup \{\psi\} \vdash \phi$, $\Gamma \cup \{\psi\} \not\vdash \bot$).*

We can strengthen this result as it holds for any finitely axiomatisable abduction Ψ.

These results motivate the following rephrasing of our definition of abduction.

Definition 4.1.2 *An abduction of a formula ϕ with respect to a domain theory Γ is a formula ψ such that:*

(i) $\Gamma \cup \{\psi\} \vdash \phi$;

(ii) $\Gamma \cup \{\psi\}$ *is consistent (i.e.,* $\Gamma \cup \{\psi\} \not\vdash \bot$*).*

We also say that ψ is abduced *from Γ and ϕ.*

This is the definition of abduction we shall adopt in this dissertation. It is also commonly found in the literature [73] and says a lot about our approach. The main reason for adopting this approach is that it will allow an easy integration with the AGM, as we shall see in the next chapter. It makes clear that we are not as much interested in the syntactic form of the abductive inferences we make (or the form of new information for that matter) as we are in the content of those inferences. That is, as with the AGM (and, in a sense, because

[3] Proofs for the results in this chapter may be found in Appendix A.

of it) we adopt the Principle of Irrelevance of Syntax. The rationale behind this is that, under our proposal, abductions in turn become objects of the agent's beliefs and, due to the adoption of this principle, beliefs are taken to be propositions. It also shows that we are only interested in abductions expressible in our language. In fact, it is not clear why an agent would need to account for infinitely many items of data in one go although this situation may well occur if all data is considered as one over the course of time. Moreover, any finiteness asumptions will have a direct relevance to any eventual implementation.

In general, the definition of abduction above still permits a number of abductions for some piece of new information. We shall now go on to look at several types of abduction discussed in the previous chapter and this definition will prove helpful in characterising them with simple definitions. These different types of abduction can be used to prune the space of abductions that need to be considered.

4.2 Minimality

As we have seen in the previous chapter, one consideration often deemed significant when determining abductions is to assume as little as possible in proving a formula ϕ. This expresses the desire to avoid superfluous abductions. A first attempt at accomplishing this within a logical setting is to use the consequence relation $\vdash$ to define a notion of one abduction being (logically) "weaker" than another.

Definition 4.2.1 *An abduction ψ of ϕ with respect to Γ is weaker than an abduction ψ' of ϕ with respect to Γ if and only if $\psi' \vdash \psi$. We write this as $\psi \leq_{\Gamma,\phi} \psi'$.*
An abduction ψ of ϕ with respect to Γ is minimal iff $\psi \leq_{\Gamma,\phi} \psi'$ for all other abductions ψ' of ϕ with respect to Γ.

That is, ψ is a weaker abduction of ϕ than ψ' with respect to Γ if it can prove no more than ψ'. The fact that restaurant 1 is open and I bought a hamburger there is a possible abduction for you meeting me while I eat a hamburger; so is the fact that restaurant 2 is open and I bought a hamburger there. Of course you may not be able to assume either fact with great certainty and may be better off assuming that either one or the other is open and I bought it there (i.e., $(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2)$ is an abduction and $(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2) \leq_{\Gamma,e} o_1 \wedge ph_1$ $(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2) \leq_{\Gamma,e} o_2 \wedge ph_2$).

Observation 4.2.1 *The "weakness" relation $\leq_{\Gamma,\phi}$ induces a partial order over the set of abductions of ϕ with respect to Γ.*[4]

Intuitively, a minimal abduction under this weakness ordering would be some form of maximal disjunction of possible abductions. Since we assume a finite language, such a proposition is expressible in our language. This would not necessarily be the case with an infinite language.

Observation 4.2.2 *If there is an expressible minimal abduction with respect to $\leq_{\Gamma,\phi}$ then it must be weaker than the new information ϕ.*

So, if a minimal abduction exists, it must be weaker than the new information ϕ under the ordering given by $\leq_{\Gamma,\phi}$. The new information ϕ is, of course, an abduction of itself.

Theorem 4.2.3 *For any abduction ψ of ϕ with respect to Γ weaker than ϕ and for any $\delta \in \mathcal{L}$, $\Gamma \cup \{\psi\} \vdash \delta$ iff $\Gamma \cup \{\phi\} \vdash \delta$.*

This result has important consequences for abductive belief dynamics. Thus far we have assumed a fixed domain theory Γ and so, in a sense, confined ourselves to *static* abduction. Any interesting theory concerning an inquiring agent however, must allow for *dynamic abduction*. One way to do this, and the way we shall do so here, is to let the chosen abduction ψ in light of new information ϕ help the domain theory to evolve to Γ'_ψ in a manner analogous to the aims of belief revision. The nature of Γ'_ψ will be partially determined by the rationality criteria adopted. In accordance with the second of our rationality criteria (see § 2.2) our domain theory should be closed under Cn. A virtue of this constraint is that it vastly simplifies the account of belief change. Given this constraint, Γ'_ψ may be identified with $Cn(\Gamma \cup \{\psi\})$. By Theorem 4.2.3 it then follows that, for every abduction ψ weaker

[4]Here we are taking identity $=_{\Gamma,\phi}$ to be logical equivalence (i.e., $\vdash \psi \leftrightarrow \phi$) rather than syntactic equivalence. Effectively, we are treating formulae as propositions rather than sentences, as dictated by the Principle of Irrelevance of Syntax.

than ϕ, the new domain theory Γ'_ψ is simply $Cn(\Gamma \cup \{\phi\})$. Thus the differences among various abductions ψ weaker than ϕ is effectively obliterated. This provides a reason for not requiring the above rationality criterion to be satisfied; the domain theory need not necessarily be closed under Cn. An analogous situation is found in the belief change literature. Many writers advocate the use of belief bases rather than belief sets [80, 85]. In this way Γ'_ψ can simply be identified with $\Gamma \cup \{\psi\}$. It must be noted, however, that as long as ψ is weaker than ϕ the deductive closure of Γ'_ψ will suffer this problem. On the other hand, one might question the rationality of abductions weaker than the new information ϕ itself. Intuitively, such an abduction would not seem to make much sense. If you observe me eating a hamburger, then assuming (or abducing) that either I am eating a hamburger or restaurant 1 is open and I bought a hamburger there is not very sensible (or rational perhaps). These observations suggest that this type of abduction is best avoided. We shall maintain the rationality criterion that epistemic states be closed under Cn in accord with the AGM. As we shall see later though, minimality when coupled with other restrictions may be useful.

4.3 Triviality

It should be fairly clear that, using our definition of abduction, we can obtain an abduction of a formula ϕ with respect to a domain theory Γ simply by taking the abduction to be ϕ itself, provided of course $\Gamma \not\vdash \neg\phi$. In doing this we are gaining no new knowledge about our domain apart from ϕ and its deductive consequences. We can extend this notion of triviality by specifying that the abduction should make use of the domain theory and not be able to prove the new information on its own (the case where $\Gamma = \emptyset$ being an exceptional circumstance).

Definition 4.3.1 *An abduction ψ of ϕ with respect to Γ is* trivial *if and only if $\psi \vdash \phi$. Otherwise, it is* non-trivial.

Such a trivial abduction can be considered to occur with respect to a theorem of the logic (viz., $\vdash \psi \rightarrow \phi$ by the deduction theorem). These types of abduction are inherent in the logic in a certain sense and may always be obtained regardless of the domain theory (up to inconsistency).

Example 4.3.1 *In the example of p. 78, simply hypothesising that e is trivial, as also, for instance, is $e \wedge o_1$.* □

However, as we have seen in the previous section, those abductions ψ of ϕ with respect to Γ that are logically weaker than ϕ, (i.e., $\psi \leq_{\Gamma,\psi} \phi$) add no more new information and are in a sense trivial. It is therefore tempting to identify trivial abductions of ϕ with respect to Γ with those weaker than ϕ. However, consider the scenario where $\Gamma = \{a\}$ and $\phi = c$. One possible abduction is $\psi = c \wedge a$. It is a trivial abduction in our earlier sense but it is not weaker than ϕ. Intuitively, this abduction is trivial because it simply assumes what we are trying to account for, so the use of logical "weakness" would not be an appropriate way of capturing this notion.

4.4 Specificity

In the previous chapter we discussed Stickel's [123] notions of most and least specific abduction. Stickel, however, does not provide a formal definition of these concepts and the setting he adopts utilises Horn clauses. We shall introduce three types of specificity. Moreover, the two extremes mentioned by Stickel can be generalised to help determine a number of *levels of specificity*.

Abduction can in a sense be viewed as an inference "backwards" over an implication; from consequent to antecedent. One way to view specificity then, is to treat propositions further "back" along an implication chain as more specific. We shall say that one abduction is more specific than another if, together with the domain theory, it can prove the latter abduction. It contains more information relative to the domain theory.

Definition 4.4.1 *(relative specificity)*
An abduction ψ of ϕ with respect to Γ is relatively more specific than an abduction ψ' of ϕ with respect to Γ iff $\Gamma \cup \{\psi\} \vdash \psi'$. We write $\psi' \preceq_{\Gamma,\phi} \psi$. We also say that ψ is a relatively

least (respectively most) specific abduction of ϕ with respect to Γ iff $\psi \preceq_{\Gamma,\phi} \psi'$ (respectively $\psi' \preceq_{\Gamma,\phi} \psi$) for all abductions ψ' of ϕ with respect to Γ such that $\not\vdash \psi \leftrightarrow \psi'$.

Example 4.4.1 *In the example that we have been using thus far, we have the following relative specificities:*

$$e \preceq_{\Gamma,e} o_1 \wedge ph_1 \preceq_{\Gamma,e} co_1 \wedge ph_1$$
$$e \preceq_{\Gamma,e} o_2 \wedge ph_2$$

□

Observation 4.4.1 *The relative specificity relation $\preceq_{\Gamma,\phi}$ is a (partial) pre-order over the set of abductions of ϕ with respect to Γ.*

Now, by the definition of abduction, $\Gamma \cup \{\psi\} \vdash \phi$ for any abduction ψ of ϕ. Consequently, every abduction of ϕ is more specific than ϕ (i.e., $\phi \preceq_{\Gamma,\phi} \psi$) and therefore ϕ is a minimal element under this ordering. Moreover, other possible least specific abductions are characterised by the following result.

Lemma 4.4.2 *If an abduction of ϕ with respect to Γ exists, then an abduction ψ of ϕ with respect to Γ is a relatively least specific abduction of ϕ with respect to Γ iff $\Gamma \vdash \phi \leftrightarrow \psi$.*

This extends our previous observation by noting that the ordering is well-founded. In fact, this ordering can be utilised as a way of arranging abductions into “levels of specificity”. The relatively least and most[5] specific abductions can be considered at opposite ends of the “spectrum” of levels. The strict part of the ordering can be used to determine a particular formula’s level of specificity.

One problem with the definition of relative specificity is that it includes abductions that prove other abductions without the aid of the domain theory ($\{\psi\} \vdash \psi'$ and therefore $\Gamma \cup \{\psi\} \vdash \psi'$ by monotonicity) in the ordering. We may be more interested in those abductions establishing a “specificity sequence” with the required aid of the domain theory. We have already seen arguments for disregarding abductions which prove the new information

[5]Which is guaranteed to exist under our current assumption of a finite language.

without the aid of the domain theory (trivial abductions), and similar arguments can be applied to this case. This leads us to introduce the notion of *strong relative specificity*.

Definition 4.4.2 *(strong relative specificity)*
An abduction ψ of ϕ with respect to Γ is strongly relatively more specific than an abduction ψ' of ϕ with respect to Γ iff ψ is relatively more specific than ψ' but $\{\psi\} \nvdash \psi'$.

Unfortunately, the resulting relation does not even satisfy transitivity (though it is irreflexive). It is therefore difficult to see how to sensibly define levels of specificity or least and most specific abductions. For completeness we introduce a further type of specificity.

Definition 4.4.3 *(absolute specificity)*
An abduction ψ of ϕ with respect to Γ is absolutely more specific than an abduction ψ' of ϕ with respect to Γ iff $\{\psi\} \vdash \psi'$.

Note, however, that this induces an ordering identical to the weakness ordering used to investigate the notion of (logical) minimality and so we shall not consider it further here.

4.5 Selection Criteria

All these types of abduction identify a subset of the potential abductions. However, even restricting our attention to those abductions satisfying a number of these criteria may not result in choosing a single "best" abduction. We could of course combine those that remain in some way — taking their disjunction for instance — or we could impose further selection criteria (in particular, using extralogical considerations) to help choose among those that remain. A number of such selection methods were detailed in the previous chapter. We shall not consider any other selection criteria, beyond those involving syntactic restrictions, in the following sections but note that in the constructions that result from our belief change operator, a selection mechanism will arise and can be viewed in a number of ways.

4.6 Syntactic Restrictions

We mentioned in the previous chapter that syntactic criteria were a common restriction placed on logic based abduction especially where computational issues are concerned. In this and the following section then, the main inspiration is a pragmatic one. As such, we shall assume that formulae are represented in clausal form (i.e., disjunctions of literals). This means that conjunctions may be represented as sets of (singleton) clauses and so we shall revert to our initial definition of abduction for ease of presentation.

In discussing how to compute abductions in the previous chapter we noted a method based on a contrapositive argument: using the negated form of the new information to obtain negated forms of abductions. This idea forms the basis of Jackson's [52] method for calculating the *abductive closure* of a set of clauses (causal axioms) Γ over a new data set Φ. An algorithm based on this might be developed as follows (we assume the more interesting situation where $\Gamma \not\models \neg\Phi$ and $\Gamma \not\models \Phi$).

Two sets are used:

HS hypothesis set containing possible abductions (i.e., a set containing sets of clauses). Initially HS contains Φ ($HS = \{\{\Phi\}\}$) since Φ is trivially an abduction of itself with respect to Γ.

WS working set containing clauses used to calculate possible abductions. Initially WS contains the clauses in the domain theory ($WS = \Gamma$).

1. Add the prime implicates $I(\Gamma)$ of the domain theory Γ to WS.
   $WS = WS \cup I(\Gamma)$.

2. Negate an element Δ from HS obtaining $\neg\Delta$.

3. Resolve an element of $\neg\Delta$ with a clause in WS to derive a new goal C.

4. If the new goal C is not a tautology and it is not subsumed by any other clause in the closure WS add it to WS, negate it to obtain $\neg C$ and add this negation to HS.

5. Repeat steps 2 - 4 until no new elements of HS can be derived.

When the algorithm terminates, the resulting abductions are contained in the hypothesis set HS.

Observation 4.6.1 *Any abduction, other than the new data, generated by the above procedure is a conjunction of literals.*

Although this result may seem restrictive, we saw in the previous chapter that it is a very common (in fact, the most common) type of abduction in the literature. In artificial intelligence this is often what is meant by abduction. We shall refer to abductions of this type as *conjunctive abductions*. If we restrict our attention to conjunctive abductions, then the minimality criterion mentioned above can be made more specific. In fact, it coincides with set inclusion[6]

- Minimality Criterion for Conjunctive Abductions[7]
  A conjunctive abduction Ψ of Φ with respect to Γ is *conjunctive minimal* iff there is no other conjunctive abduction Ψ' of Φ with respect to Γ such that $\Psi' \subset \Psi$.

Theorem 4.6.2 *If Ψ is a conjunctive minimal abduction of Φ with respect to Γ then Ψ is in the hypothesis set HS.*

The converse may also be obtained by adding a further step to the algorithm removing subsumed elements from the hypothesis set HS.

We now shift our attention to the (pragmatic) relationship between abduction and induction. It will turn out to be related in an interesting way to what we have presented in this section.

4.7 Abductive Inference Versus Inductive Inference

The title of this section is a little misleading as we are really interested in pragmatic considerations regarding these two types of inference. In particular, we shall contrast

[6]Recall that in clausal form, a conjunction of literals $l_1 \wedge \ldots \wedge l_n$ can be represented as $\{\{l_1\}, \ldots \{l_n\}\}$.

[7]Jackson [52] presents a similar definition but fails to specify that Ψ (which he denotes by E) should be a conjunction of literals. This is clearly what is intended since, without it, some of the results presented there would be incorrect. I am indebted to Peter Jackson for his discussion regarding this matter.

abductive inference, restricted as in the last section, with a particular method claimed to be inductive known as inverse resolution [77, 78, 79, 115]. But first a note on abduction and induction in general by Carettini [11] (pp. 139, 140):

> ... Induction is based on a comparative process. It is a comparison of homogeneous facts, samples of a certain class; from this comparison it enunciates general properties. Abduction on the contrary is based on a single fact, which sometimes presents itself as an enigma, something unexplainable: at this point the observer postulates a hypothesis, that is, he puts an idea into reality by asking himself if it can be demonstrated. ...
>
> Peirce insisted on induction's lack of originality, opposing to it the creative character of the hypotheses generated by abduction.

The important point to note is that induction is often thought of as deriving generalisations.

Harman [47][8] however, claims that induction[9] is a special instance of abduction (or, more precisely, what he terms "inference to the best explanation"). The generalisation that is inferred during the process of induction can be considered an abduction; if assumed as a hypothesis, then, together with the domain theory, it will prove the instances. In artificial intelligence research, one popular method for performing induction is *inverse resolution* which, as the name suggests, is based on inverting the resolution process.

4.7.1 Overview of Inverse Resolution

Resolution is a valid inference procedure which deduces a clause C from two clauses C_1 and C_2. Given a clause C_1 containing a literal l and a clause C_2 containing the literal $\neg l$, the resolved product (or *resolvent*) of C_1 and C_2 is denoted $C = C_1.C_2$, where

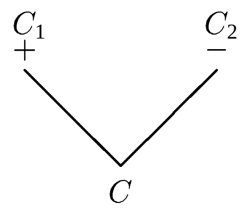
$$C = (C_1 \setminus \{l\}) \cup (C_2 \setminus \{\neg l\}) \tag{4.1}$$

This process may be visualised with the help of the following diagram:[10]

[8] See also Ennis' [26] views on Harman and Harman's reply [48].

[9] Actually, he confines his attention to *enumerative induction*. That is, inference of a generalisation from a series of instances.

[10] The plus ($+$) (respectively minus ($-$)) sign in the diagram denotes that the literal resolved upon appears positive (respectively negative) in that clause.



Inverse resolution, on the other hand, is not a valid inference procedure but is based upon the following characterisation of inductive inference [77, 78]. Given a partial domain theory Γ and a positive example E that is not a consequence of the domain theory ($\Gamma \not\vdash E$) we attempt to determine a new domain theory Γ', using Γ and E, that will account for the example ($\Gamma' \vdash E$) and also the original domain theory ($\Gamma' \vdash \Gamma$). If we think of Γ' as $\Gamma \cup I$ where I represents the result of inverse resolution, then the relationship with abduction should become much clearer. In practice, the domain theory and example are usually represented as Horn clauses. As the name suggests, this technique is based on inverting the resolution process and consists of five operators: two V-operators, two W-operators and the truncation operator.

So as not to countenance invalid inference, the notion of an *oracle* is introduced. An oracle is an entity that accepts a clause, constructed using one of the inverse resolution operators, if it is valid in the intended model. Since we are only interested in the calculation of the constructed clauses we shall not consider the oracle further here.

V-Operators

Previously, we represented a single resolution step in terms of a "V"-shaped diagram. The two V-operators can derive a clause at one of the arms of this V given the clause at the other arm and the clause at the base. The *absorption* operator constructs C_2 given C_1 and C while the *identification* operator constructs C_1 given C_2 and C.

Since the new clause is constructed by finding the inverse of a resolved product, we define the notion of a resolved quotient of C and C_1[11] as $C_2 = C/C_1$. Rearranging equation (4.1) for resolution we can obtain $C_2 = (C \setminus (C_1 \setminus \{l\})) \cup \{\neg l\}$ under the following assumption

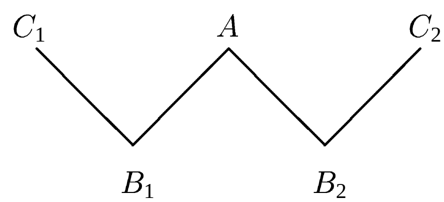
[11] We consider absorption here. Identification is similar.

- Separability Assumption — Clauses $C_1 \setminus \{l\}$ and $C_2 \setminus \{\neg l\}$ contain no common literals.

This assumption also simplifies the calculation of resolved quotients (i.e., absorptions or identifications).

W-Operators

Combining two resolution "V"s we obtain a form analogous to that for the V-operators.



In this situation a common literal l, contained in A, resolves with clauses C_1 and clauses C_2 to produce B_1 and B_2 respectively. Clauses B_1 and B_2 represent the new information and clauses A, C_1 and C_2 the constructed clauses. Interestingly, since l is resolved away, the constructed clauses A, C_1 and C_2 will contain a literal whose propositional symbol does not appear in either B_1 or B_2. If l occurs negative in A then the operator is referred to as *intra-construction* and if it occurs positive in A the operator is called *inter-construction*.

Truncation

The *truncation* operator results from the special case where the empty clause occurs at the base of a V or W schemata. In a propositional system, this corresponds to dropping negative literals from a clause. In the first-order case Muggleton and Buntine [79] show that two literals may be truncated by taking their least-general-generalisation. Rouveriol and Puget [115] generalise this to a truncation operator which replaces terms by variables and drops literals from clauses.

In the case of a propositional language, a number of schema may be used to compute the

Name	*Rule*
Absorption	$\frac{\alpha\to k,\ \alpha\wedge\beta\to j}{\beta\wedge k\to j}$
Identification	$\frac{\beta\wedge k\to j,\ \alpha\wedge\beta\to j}{\alpha\to k}$
Inter-Construction	$\frac{\alpha\wedge\beta\to j,\ \alpha\wedge\gamma\to k}{\beta\wedge l\to j,\ \gamma\wedge l\to k,\ \alpha\to l}$
Intra-Construction	$\frac{\alpha\wedge\beta\to j,\ \alpha\wedge\gamma\to j}{\alpha\wedge l\to j,\ \beta\to l,\ \gamma\to l}$
Truncation	$\frac{\alpha\wedge\beta\to j}{\alpha\to j}$

Table 4.1: Propositional inverse resolution operators.[13]

required inverse resolution. These are displayed in Table 4.1 (see [77, 78]).[12]

4.7.2 Calculating Inverse Resolution V-Operators

Returning to the V resolution schema that was used to explain the inverse resolution V-operators absorption and identification, we notice that this is equivalent to $\{C_1\}\cup\{C_2\}\vdash C$. Let us consider the absorption operator (identification is similar). Absorption attempts to construct the clause C_2 given the clauses C_1 and C. We can in fact consider the clause C_1 to be a clause from the current domain theory Γ and the clause C to be the new data. Therefore, according to the definition of abduction, $\{C_2\}$ is an abduction of C with respect to $\{C_1\}$ (or Γ since $C_1 \in \Gamma$). Applying the same analysis as that for conjunctive abduction above, we obtain $\{C_1\} \cup \neg C \vdash \neg C_2$.

This analysis is essentially provided also in [115], with emphasis on Horn clauses and together with a further analysis for absorption. However, if we apply the algorithm described in the previous section, we will derive only conjunctive abductions as we have seen. Absorption on the other hand generates clauses which are disjunctions of literals.

A resolution-based algorithm capable of performing absorption and identification is suggested by the above analysis. As in the previous algorithm, this algorithm attempts to

[12]We adopt a slight renaming of the terms to those presented in [78], having found them more amenable to study. In the case of absorption and identification, the first clause on the top line of the schemata is taken from the domain theory while the second represents the new data.

[13]Here $\alpha,\ \beta,\ \gamma$ represent conjunctions of literals while $j,\ k,\ l$ represent literals.

determine all absorptions and identifications relative to clauses appearing in the domain theory. (We do not consider those which may be obtained from consequences of the domain theory although it would not be difficult to do so at the cost of added complexity).

We require three sets:

GS generated set containing possible absorptions/identifications (i.e., a set of clauses). Initially contains the new data ($GS = \{C\}$).

WS working set containing clauses used to calculate possible disjunctive abductions (absorptions and identifications in particular). Initially WS contains the clauses in the domain theory ($WS = \Gamma$).

CS construction set used to construct the absorptions or identifications

1. $WS = WS \cup I(\Gamma)$.
2. Negate the example clause $C = \{l_1, \ldots, l_n\}$ to obtain $\neg C = \{\{\neg l_1\}, \ldots, \{\neg l_2\}\}$.
3. Select a clause $D = \{k_1, \ldots, k_m\}$ from WS such that $\{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_m\} \subseteq C$ for some i, $1 \leq i \leq m$.
4. Let $CS = \neg C \cup \{D\}$.
5. Resolve any two elements of CS, replacing the two resolvents by their resolved product $CS = ((CS \backslash \{X\}) \backslash \{Y\}) \cup \{X.Y\}$. Repeat until no resolution can be performed.
6. Negate CS and place the resulting clause in GS. $GS = GS \cup \{\neg CS\}$.
7. Repeat steps 2 - 6 until no new elements can be added to GS.

Alternatively, elements of GS could simply be added to WS (and subsumed clauses removed).

Observation 4.7.1 *Any abduction generated by the above procedure (i.e., in GS) is a disjunction of literals (i.e., a clause).*

We shall refer to such abductions as *disjunctive abductions* in analogy to the conjunctive abductions of the previous section.

The role of minimality in disjunctive abductions is slightly different from that played in conjunctive abductions. On the one hand, if we use a syntactic criterion and attempt to minimise the size of the abduction (i.e., the number of literals) we notice that, if $\{l_1, \ldots, l_n\}$ is a disjunctive abduction, then each of $l_1, \ldots, l_n$ are minimal disjunctive abductions. However, such abductions are likely to specialise the theory too much.[14] On the other hand, "logical minimality", as expressed in Definition 4.1.2, would favour longer clauses which may contain superfluous literals. Such clauses can be misleading; they are logically too "weak" and, as we have seen, not of much use. The separability assumption can be seen as an attempt to avoid such abductions at either extreme; a compromise between logical minimality on the one hand and syntactic minimality on the other. As such, we see that there are, in a sense, two opposing forces at work here — one attempting to prevent disjunctive abductions from being too specialised and the other preventing them from being too general. They are analogous to the notion of *least general generalisations* of Muggleton and Buntine [79]. The syntactic and logical aspects of minimality alluded to above are also manifest in conjunctive abductions. In that case, however, the logical and syntactic aspects coincide rather than oppose each other.[15]

Lemma 4.7.2 *Let the domain theory Γ consist of consistent and non-tautological clauses. If a clause A is in the generated set GS, then it satisfies the separability assumption with respect to the clause from the domain theory that was used to generate it.*

Theorem 4.7.3 *Let the domain theory Γ consist of consistent and non-tautological clauses. If a clause A is the result of an absorption or identification of the new data C together with a clause from the domain theory Γ then it will be in the generated set GS.*

The domain theory is restricted to be consistent as the other case is less interesting and non-tautologous elements are excluded because they are superfluous.

[14] A disjunctive abduction which is minimal in this way would in fact be considered "maximal" in the sense of Definition 4.1.2 (since, for clauses A and B, if $A \subset B$, then $A \vdash B$).

[15] Again, because considering conjunctions of literals as sets of singleton clauses, $A \subset B$ implies $A \vdash B$.

4.7.3 Extension to a First-Order Language

Probably the easiest way of extending the results in Section 4.7.2 to a first-order language is to use the "flattening" technique proposed by Rouveirol and Puget [115]. The basic idea behind this method is to transform functions (including constants) into predicates so that the inverse resolution operations can deal with formulae (clauses) that do not contain function symbols. For example, a clause such as $\{p(X,\ f(Y))\}$ is replaced by the flattened clause $\{p(X,\ Z),\ \neg new_f(Y,\ Z)\}$ and, in order to preserve the semantics, we introduce the new predicate $\{new_f(Y,\ f(Y))\}$

The propositional schema presented in Table 4.1 can in fact be used to construct new clauses from these flattened clauses. Therefore, after flattening clauses in the domain theory and the new data clause, we can, with a slight modification, apply the algorithm supplied in Section 4.7.2 to perform absorption and identification. Any variables in the construction set that are unified via a resolution step are unified with all instances of those variables in the construction set. This assumes that clauses have been standardised apart before being placed in the construction set. However, this may not guarantee that all absorptions and identifications will be calculated given the incompleteness of first order resolution.

4.7.4 W-Operators and Truncation

The W-operators would appear to be the most interesting of the inverse resolution operators for they construct a clause by introducing a new literal not present in the original domain theory. Referring back to the W figure in Section 4.7.1 used to motivate the W-operators, we can express the logical relationship between the formulae as $\{A\} \cup \{C_1\} \cup \{C_2\} \vdash B_1 \wedge B_2$. In both inter-construction and intra-construction, the two clauses B_1 and B_2 are used to construct clauses $A,\ C_1$ and C_2. This, too, conforms to the definition of abduction provided $A \wedge C_1 \wedge C_2$ is consistent with $B_1 \wedge B_2$. However, in this case the abduction proves the new data directly, without help from the domain theory. As pointed out by Console *et al.* [16], we are performing abduction with respect to an empty domain theory in this case. That is, without the help of any formulae from the domain theory.

Name	*Theorem*
Inter-Construction	$\vdash [(\beta \wedge l \rightarrow j) \wedge (\gamma \wedge l \rightarrow k) \wedge (\alpha \rightarrow l)] \rightarrow [(\alpha \wedge \beta \rightarrow j) \wedge (\alpha \wedge \gamma \rightarrow k)]$
Intra-Construction	$\vdash [(\alpha \wedge l \rightarrow j) \wedge (\beta \rightarrow l) \wedge (\gamma \rightarrow l)] \rightarrow [(\alpha \wedge \beta \rightarrow j) \wedge (\alpha \wedge \gamma \rightarrow j)]$
Truncation	$\vdash [\alpha \rightarrow j] \rightarrow [\alpha \wedge \beta \rightarrow j]$

Table 4.2: Theorems corresponding to propositional inverse resolution operators.

Using the deduction theorem on the formula above we can obtain the logical theorem $\vdash [A \wedge C_1 \wedge C_2] \rightarrow [B_1 \wedge B_2]$. Therefore, we can consider abduction in this case to occur with respect to theorems of the logic rather than the domain theory. The performing of induction in this way has been suggested previously. Morgan [75] suggests a method of induction based on inverting the inference rules of a logical system. A similar analysis can be applied to the truncation operator. The relevant theorems for the W-operators and truncation, based upon the propositional schema presented in Table 4.1 are provided in Table 4.2.

This all means that the W-operators construct abductions which are, in the sense of Definition 4.3.1, trivial abductions. The following result reveals that this is not the only apparent misgiving of this type of abduction. It suggests that, as a method of learning, the W-operators are somewhat limited in power.

Theorem 4.7.4 *Let Γ_1 be a set of propositional Horn clauses over the language $\mathcal{L}(\Gamma_1)$, Γ_2 be the result of performing Inter-construction or Intra-construction on Γ_1, and l be the newly introduced literal (i.e., $l \in \mathcal{L}(\Gamma_2)$). If $l \notin \mathcal{L}(\Gamma_1)$, then, for any formula $\phi \in \mathcal{L}(\Gamma_1)$, $\Gamma_2 \vdash \phi$ implies $\Gamma_1 \vdash \phi$.*

This result tells us that use of the two W-operators leads to a conservative extension of the original theory whenever the introduced literal does not occur in the original theory (which is invariably the case). In the case where the newly introduced literal is already in the language of Γ_1, it is possible to give counterexamples although this situation is contrary to the spirit of the W-operators. The newly introduced clauses are only capable of expressing concepts that are already expressible by the original theory. That is, the new literal may represent some new concept but that concept is simply the naming of a conjunction of existing concepts. In general, this result does not hold for a first-order language since generalisation may also occur (i.e., constant symbols may be replaced by variables).

Console *et al.* [16] point out how the inverse resolution operators are consistent with the logical definition of abduction and identify a number of relationships between abduction and inverse resolution. They point out, for instance, that in absorption and starting from the deductive schemata $\frac{\alpha \to k \ \ \beta \wedge k \to j}{\alpha \wedge \beta \to j}$, abduction swaps the second premise with the conclusion. In identification, on the other hand, abduction swaps the first premise with the conclusion. Another point they make is that abduction of atoms is equivalent to deduction in a completed theory [68] while the results of inverse resolution V-operators are less specific than those computed by abduction in a completed theory. They fail, however, to note what we consider to be a very simple difference between the two — which is the point being made in this section. In pragmatic terms (and, to be honest, inverse resolution is largely driven by pragmatic considerations) abduction computes conjunctions of literals while inverse resolution computes disjunctions of literals. Moreover, considerations of minimality are important to both. A final point: one reason why inverse resolution is considered inductive is the fact that it generates disjunctions of literals or clauses. These are easily converted into implicational form and can be (loosely) viewed as "rules". It is often considered the role of induction to generate rules from particular instances.

4.8 Background Theory Entails New Information

Before completing this chapter we briefly consider a further situation that may seem troublesome. When the domain theory Γ already accounts for the new information ϕ an interesting situation occurs.

Observation 4.8.1 *If $\Gamma \vdash \phi$ then any $\delta \in \mathcal{L}$ consistent with Γ (i.e., $\Gamma \cup \{\delta\} \nvdash \bot$) is an abduction of ϕ with respect to Γ.*

So, any formula consistent with the domain theory Γ would be a possible abduction. In contrast to trivial abductions the domain theory proves the new information independently of any abduction. One way around this problem would be to contract Γ by ϕ and perform abduction with respect to the contracted domain theory. This may be useful in other ways for, if Γ is closed under logical consequence Cn, as in the AGM, then the damage has already been done in a certain sense because $\psi \rightarrow \phi \in \Gamma$ for all $\psi \in \mathcal{L}$, and it may be difficult to determine which members of Γ are relevant (or more relevant than others). However, an operation like AGM contraction will remove those that are less epistemically entrenched. In this way, a belief change operation can be useful for the abductive process. On the other hand, it may be that the selection criteria for a particular abductive operator is discriminating enough to handle this situation, where many more abductions need to be considered.

4.9 Summary and Discussion

We have looked at various aspects of logic based abduction including a number of restrictions outlined in the previous chapter. The proposed definitions were phrased in terms of a consequence relation $\vdash$. That is because we are interested in logical characteristics of abduction as they will be helpful in what follows. In fact, by varying the underlying logic, the properties (and usefulness) of such definitions will change. This approach may be able to accomplish what Levesque [63] aimed to do — introducing an operator for belief to characterise different forms of abduction — without needing to enrich the object language. In subsequent chapters we will effectively have a selection mechanism for abduction, using extralogical structures, which will single out the best abduction(s).

We also had a look at the relationship between abduction and induction. In particular, we adopted a pragmatic view of abduction and induction (in terms of inverse resolution) as they are often considered in artificial intelligence. Inverse resolution can in fact be seen to conform to our general definition of abduction thus lending credence to Harman's [47] claim that (enumerative) induction is just a special case of abduction (inference to the best explanation) — at least in the propositional case. In practice abduction is used to calculate hypotheses that are conjunctions of literals while the inverse resolution V-operators (and W-operators for that matter) calculate disjunctions of literals (i.e., clauses). This result might lead one to suggest that a better name for the inferential process in Definition 4.1.2 is ampliative inference rather than abduction. We, however, retain the definition and the name abduction as this is the way it is defined in the literature. Moreover, notions of minimality, taking various forms, are important to both types of abduction. One reason for considering inverse resolution as inductive is the fact that clauses can be interpreted as rules (though one must be careful with this interpretation of material implication). The inverse resolution W-operators can also be considered to be performing abduction with respect to theorems of the logic (i.e., computing trivial abductions). Even though, at first, these operators seem very interesting because they are able to introduce a new literal not present in the original domain theory (often called *predicate invention* or *constructive induction*) it turns out that this leads to no additional expressive power in the propositional case. The new theory is simply a conservative extension of the original. This means that this form of learning is somewhat limited from a logical viewpoint. It may have other advantages, such as making the representation of the theory more compact.

Chapter 5

Abductive Expansion[1]

A man must be downright crazy to deny that science has made many true discoveries. But every single item of scientific theory which stands established today has been due to abduction.
Charles Sanders Peirce, [96] 5.172

One of the fundamental types of belief change employed by an inquiring agent is that of belief expansion. New information is incorporated into the agent's current epistemic state without the retraction of any existing beliefs. Most belief change frameworks — including the AGM — simply incorporate this new information "as is" together with any (logical) consequences. We shall concentrate on this form of belief change here but instead of incorporating solely the new information, we shall do so by looking for an explanation or reason — via abduction — for this new information and incorporate this into the current epistemic state together with any consequences (one of which is the new information). We do this within the setting of the AGM framework for belief change and adopting a definition of abduction as explored in the previous chapter because they complement each other nicely. Primarily, we are interested in characterising the class of all such belief expansion functions much in the way the AGM examines its contraction and revision functions. That is, we explore all ways of expanding an epistemic state by new information through the use of abduction.

There are various motivating concerns behind this study. One is that advocated by Levi

[1]Some of the work presented in this chapter has appeared in [92, 93, 94].

[65]: an agent is interested in acquiring new, error-free information. Abduction, which Levi adopts in his account of deliberate expansion, is one way of acquiring new information beyond the current epistemic state and the epistemic input alone. In fact, by just adding the new information and any consequences to the current epistemic state very little information is acquired. On the other hand, the agent must also consider that it may introduce error by making an abduction which is not true of the actual world. This may lead to further conclusions which will be subsequently ill-founded. Although this may be remedied through contraction later on, the agent would have wasted valuable resources — something which it is loathe to do. The amount abduced would therefore depend on the agent's willingness to court such error; their degree of caution or boldness.

We maintain that this abductive way of belief change is in fact more natural than the method proposed by the AGM. There is psychological evidence [108] suggesting that human agents do indeed follow this pattern of belief change through explanation. We leave aside the issue of how accurately the logical notion of abduction adopted here captures an intuitive notion of explanation, which is beyond the scope of this dissertation. It is at least arguable that, in a logical sense, the notion of abduction represents necessary conditions. Moreover, as noted elsewhere, we are interested in the normative aspects of such belief change and are not concerned with developing a psychological account. Another motivation stems from the way the Principle of Minimal Change (see p. 25) is viewed in the various AGM belief change operations. AGM expansion is the only operation where this principle is taken with respect to set inclusion. In AGM contraction, interpreting minimal change with respect to set inclusion leads to maxichoice contraction which we have seen to be undesirable in general (Theorem 2.2.5 (p. 32)). A similar situation holds for maxichoice revision arising out of maxichoice contraction via the Levi identity. This principle can, however, be captured semantically via Grove's sphere modelling. Following this example, we shall abandon the interpretation of this principle via set inclusion (embodied by postulate (K^+6)) and show how it can be interpreted semantically.

These motivations lead us to suggest changes to the rationality criteria suggested by Gärdenfors and Rott [35] (listed in § 2.2). Firstly, we add Levi's criterion that agents should seek as much new, error-free information as possible (within the constraints of their degree of caution). We also question the interpretation of the third criterion regarding the Principle of Minimal Change. This seems somewhat at odds with the principle we have just added. Noting as above that this principle can be interpreted semantically in the AGM but not with respect to set inclusion, we maintain that it is possible to retain this criterion, even for expansion, if it is interpreted uniformly in this way. However, the criterion added above, regarding the amount of new, error-free information, can be interpreted using set inclusion. We proceed by proposing a definition for this new belief change operation and capture this through rationality postulates. A number of constructions in the spirit of the AGM are presented together with the relevant representation theorems. We refer to this new operation as *abductive expansion*. Our basic framework will build on that of the AGM so we retain their modelling of epistemic states as belief sets and epistemic inputs as formulae.

5.1 Defining Abductive Expansion

Our main aim here is to model the claim that agents often seek some explanation or justification for newly acquired information and add this justification to their belief state together with the new information. We model this using the notion of abduction as presented in Definition 4.1.2 (p. 79). More explicitly, our belief change operation is based on the following idea: the agent attempts to find an abduction of the epistemic input (new information) with respect to its current epistemic state (domain theory) and adds this abduction to the current epistemic state, taking the deductive closure. If an abduction cannot be found — because the new information contradicts a current belief (i.e., is a disbelief) — the epistemic state is not changed. We shall have more to say on this case later but note that the definition of abduction would seem to support this and that it also suggests that there is little need to consider revision at this point. A revision operator can of course be constructed from abductive expansion (and a contraction operation) via the Levi identity. In accord with these remarks, and denoting the abductive expansion of belief set K by epistemic input α as $K^{\oplus}_{\alpha}$, we have the following definition.

Definition 5.1.1 *$K^{\oplus}_{\alpha}$ is an abductive expansion of K with respect to α iff*

$$K^{\oplus}_{\alpha} = \begin{cases} Cn(K \cup \{\beta\}) & \text{for some } \beta \in \mathcal{L} \text{ such that:} \\ & \quad \text{(i) } K \cup \{\beta\} \vdash \alpha \\ & \quad \text{(ii) } K \cup \{\beta\} \not\vdash \bot \\ & \\ K & \text{if no such } \beta \text{ exists} \end{cases} \qquad \text{(Def AbExp)}$$

Conditions (i) and (ii) correspond exactly to those of Definition 4.1.2 (p. 79). Since we are interested in developing a general framework characterising expansion by all types of abductions, we do not add any further restrictions at this point. In the following sections some restrictions will be considered to give a more well behaved operation (for instance, this definition allows a nondeterministic abductive procedure) allowing constructions closer in spirit to the AGM constructions for contraction and revision thus permitting a more reliable comparison of the respective approaches.

Example 5.1.1 *Let $K = Cn(\Gamma)$ where Γ is exactly as in the example of p. 78. If we receive new information e, then AGM expansion simply gives $K^{+}_{e} = Cn(K \cup \{e\})$. In abductive expansion there are many possiblities. Some possibilities for $K^{\oplus}_{e}$ include $Cn(K \cup \{ch\})$, $Cn(K \cup \{o_1 \wedge ph_1 \wedge o_2 \wedge ph_2\})$ and $Cn(K \cup \{(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2)\})$. Interestingly, suppose $K^{\oplus}_{e} = Cn(K \cup \{ch\})$, then it is possible, given the above definition, that $K^{\oplus}_{e \wedge e} = Cn(K \cup \{(o_1 \wedge ph_1) \vee (o_2 \wedge ph_2)\})$ (where the abductive expansion function is understood to be the same in both cases) even though e and $e \wedge e$ are logically equivalent. We shall soon introduce restrictions to ensure a more deterministic underlying abductive procedure.* $\square$

A Note Concerning Abducibles

Having noted already that we are interested in exploring a general framework for abductive belief change, we however make a small digression on the restriction to abducibles. Recall that abducibles are a demarcated set of propositions or predicates from which abductions may be constructed. This is a very common restriction imposed on abductive frameworks. It is very important to note that we do not make use of abducibles in Definition 4.1.2 nor do we consider their use in the constructions to follow. In adopting a coherence-based framework for belief revision, as Gärdenfors [32] claims the AGM to be, we have already rejected the (foundationalist) existence of epistemologically basic beliefs. Embracing abducibles would severely undermine this position for, having denied the existence of select beliefs on the one hand (i.e., basic beliefs), we would countenance them on the other (i.e., abducibles). One could argue that this simply corresponds to the special case where all propositions or predicates in the language are abducible. However, the fact remains, even in this situation, that there are no propositions having a special status.

5.2 Postulates for Abductive Expansion

Keeping in mind our definition of abductive expansion and the discussion above, we develop a number of rationality postulates for this operation.

We begin with the ubiquitous postulate of closure. An abductive expansion function $\oplus$, like the AGM operations, is assumed to be a function from pairs of belief sets and formulae to belief sets ($\oplus : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$). It determines the new epistemic state of the agent given that its current epistemic state is K and it has acquired new information α.

(K$^{\oplus}$1) For any sentence α and any belief set K,
$K^{\oplus}_{\alpha}$ is a belief set (closure)

If the newly acquired information does not conflict with what is currently believed (i.e., the new information is not a disbelief), then any explanation or justification accepted should be for this new information. That is, the new information should be accepted in the resulting epistemic state. This leads to a qualified version of the success postulates for AGM expansion and revision.

(K$^{\oplus}$2) If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$ (limited success)

During the process of abductive expansion an agent is augmenting its current epistemic state. It cannot lead to the retraction of currently held beliefs, and accords with our new rationality criteria stating that the agent wishes to acquire new information.

(K$^\oplus$3) $K \subseteq K^\oplus_\alpha$ (inclusion)

In the limiting case where the new information conflicts with currently held beliefs and it is not possible to find an abduction (i.e., a consistent explanation) for it, no action is taken in order to maintain consistency. This would be in keeping with the requirement to avoid error and gives priority to maintaining consistency over guaranteeing success. Our rational agent would refuse to accept the new information as it would lead to inconsistency.[2]

(K$^\oplus$4) If $\neg\alpha \in K$, then $K^\oplus_\alpha = K$ (failure)

An alternative idea is to make $K^\oplus_\alpha = K_\perp$ if no abduction can be found (i.e., no β satisfying the stated criteria exists). This would give a closer correspondence to AGM expansion in the case where $\neg\alpha \in K$. Levi [65] also allows his expansion operations the possibility of expanding into inconsistency. It does seem odd, however, that a logically omniscient agent concerned to avoid error would do so. We note that this only represents a limiting case and any such change would pose no difficulty in the constructions to follow. The postulates (K$^\oplus$2) and (K$^\oplus$4) would be altered to the following.[3]

(K$^\oplus$2′) $\alpha \in K^\oplus_\alpha$ (success)
(K$^\oplus$4′) If $\neg\alpha \in K$, then $K^\oplus_\alpha = K_\perp$ (inconsistency)

Whenever it is possible to find an explanation for new information (in which case the initial epistemic state would have been consistent), the resulting epistemic state should remain consistent. That is, the explanation process should not introduce any inconsistencies. This will be the case if the expanded belief state remains consistent with the new information.

(K$^\oplus$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^\oplus_\alpha$ (consistency)

[2] The situation is actually more complex and may be more appropriately handled by an abductive revision operator constructed from abductive expansion and some form of contraction using the Levi identity. The fact remains however, that, if we are interested in expansion, then this would be a rational choice.

[3] I am grateful to Abhaya Nayak who suggested this alternative to me. Krister Segerberg also points out that it gives the resulting sphere semantics a nice symmetry.

We note that postulate (K$^{\oplus}$5) is equivalent, given postulates (K$^{\oplus}$1) — (K$^{\oplus}$4), to either of the two following conditions.[4]

(5.1) If $\neg\alpha \notin K$, then $K^{\oplus}_{\alpha} \neq K_{\perp}$ (consistency′)

(5.2) If $K \neq K_{\perp}$, then $K^{\oplus}_{\alpha} \neq K_{\perp}$ (consistency″)

For reasons that will now become apparent, we shall refer to postulates (K$^{\oplus}$1) — (K$^{\oplus}$5) as the *basic postulates for abductive expansion over* K.

Theorem 5.2.1 *The function* $\oplus$ *satisfies* (K$^{\oplus}$1) — (K$^{\oplus}$5) *iff*

$$K^{\oplus}_{\alpha} = \begin{cases} Cn(K \cup \{\beta\}) & \textit{for some } \beta \in \mathcal{L} \textit{ such that:} \\ & \quad \textit{(i) } K \cup \{\beta\} \vdash \alpha \\ & \quad \textit{(ii) } K \cup \{\beta\} \nvdash \perp \\ & \\ K & \textit{if no such } \beta \textit{ exists} \end{cases}$$

This theorem tells us that (K$^{\oplus}$1) — (K$^{\oplus}$5) capture the basic notions of abductive expansion that we desire. It is in proving this result that our restriction to a finite language is important. This restriction eliminates the need to consider abductions which are inexpressible in a language with infinitely many propositional symbols. That is, the correct β may not be expressible. In fact, we could weaken this restriction for what we really require is the set $K^{\oplus}_{\alpha} \setminus K$ to be finitely axiomatisable. This restriction is not necessary in many of the results that follow although we shall retain it to simplify the presentation. An alternative approach may be to consider a complete language (i.e., one allowing infinite conjunctions and disjunctions — cf. Gärdenfors [31] p. 25).

We shall consider abductive expansion functions satisfying a further postulate. If, relative to the agent's beliefs, two inputs are considered to carry the same informational content, then their relevant abductive expansions will be identical.

(K$^{\oplus}$6) If $K \vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$ (strong extensionality)

This is an expression of the Principle of Irrelevance of Syntax with respect to the agent's beliefs and is justified by the fact that it is with respect to the agent's beliefs that abductions

[4] Proofs for this, and other claims made in this chapter, are given in Appendix B.

are to be found. We consider these six postulates (at least) since, if only the first five are satisfied, then the abductive inference procedure may behave non-deterministically. It may be possible to consider the following weaker condition in place of postulate ($K^{\oplus}6$) (i.e., the usual extensionality postulate of AGM contraction and revision).

(5.3) If $\vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$ (weak extensionality)

We shall not consider this postulate here but return to it briefly in a later section (§ 5.3.6).

It may also be interesting to contemplate other conditions which we do not adopt as postulates here.

(5.4) If $\alpha \in K$, then $K^{\oplus}_{\alpha} = K$ (vacuity)

The reason this postulate may seem plausible is that, if we already believe the new information α, then there is no need to explain it in order to incorporate it into our current beliefs (cf. Boutilier and Becher's [9] factual explanations). Moreover, as we saw in the previous chapter, when $\alpha \in K$, then $\beta \to \alpha \in K$ for every $\beta \in \mathcal{L}$ and therefore any formula consistent with K is a possible abduction. However, in the first instance it is possible that an explanation for the new information α is not among the current beliefs even though the new information is currently believed. In the second instance, although every formula consistent with K implies α, it may be possible that the selection mechanism is discerning enough to choose an explanation from among the many possibilities. This restriction is not absolutely necessary for a minimal set of rationality postulates characterising abductive expansion so we do not include it among our postulates. This means that the agent, depending on its cautious nature, will still attempt to acquire new, error-free information even if it believes the new information it receives. We could, however, add this condition to our set of postulates with only minor alterations to the constructions to follow as it would only represent an (additional) boundary case.

We also do not allow the monotonicity postulate of AGM expansion ($K^{+}5$) since the selection mechanism may differ as the epistemic state changes.[5] Clearly we do not accept

[5]How, exactly, this selection mechanism varies would be determined by an account of iterated belief change which is beyond the scope of this dissertation.

the AGM minimality postulate (K^+6) having rejected it above as not permitting abduction (beyond the trivial).

One thing that will not work is the following condition.

(5.5) If $\neg\alpha \notin K$ then $K \subset K^{\oplus}_{\alpha}$ (proper inclusion)

The important case to consider here is when $\alpha \in K$.[6] We have already dismissed vacuity above but that does not mean that the agent is always forced to expand when it believes the new information. A cautious agent may choose not to court error by maintaining its current epistemic state. In general, vacuity does not hold, though it may in some instances, thus this condition will not hold in general either. The alternative, if $\neg\alpha \notin K$, then $K^{+}_{\alpha} \subset K^{\oplus}_{\alpha}$ does not hold for similar reasons.

5.3 Constructions

In the AGM framework for belief revision three major constructions are considered for contraction and revision operators (see § 2.2.2): selection functions (on maximally consistent subsets of K failing to imply α), Grove's system of spheres and epistemic entrenchment.[7] We now consider analogous constructions for abductive expansion operators. This analysis will help to place abductive expansion in the context of the AGM belief change operators thus elucidating its relationship with them.

5.3.1 Selection Functions

If the epistemic input α is consistent with the current epistemic state, the resulting abductively expanded belief set will be a superset of K containing α. Therefore, a choice has to be made from among the supersets of K implying α. However, it is easily noted that every such superset closed under logical consequence Cn (i.e., a belief set) can be represented as the intersection of a set of maximally consistent supersets of K that imply α.

[6] Otherwise this postulate holds by success and inclusion.

[7] We do not consider safe contraction here as it appears less prominently in the literature than these other constructions. Peppas and Williams [100] also present a construction in terms of nice preorders on models.

Definition 5.3.1 *(Maximally Consistent Superset of K that Implies α)*
A set K' is a maximally consistent superset of a belief set K that implies α *if and only if*

(i) $K \subseteq K'$

(ii) $\alpha \in Cn(K')$

(iii) $Cn(K') \neq K_{\perp}$

(iv) There is no $K'' \supset K'$ satisfying (i), (ii) and (iii).[8]

We denote the set of all sets K' that are maximally consistent supersets of K that imply α by $K\top\alpha$ (in analogy to $K\perp\alpha$).

Definition 5.3.2 *The set of maximally consistent supersets that imply α is denoted $K\top\alpha$.*

Note that if it were the case that $\neg\alpha \in K$, we obtain $K\top\alpha = \emptyset$. It is straightforward to show that $K\top\alpha$ is composed of belief sets.

Observation 5.3.1 *Let K be a belief set and $\alpha \in \mathcal{L}$. Any $K' \in K\top\alpha$ is a belief set.*

All elements of $K\top\alpha$ are, in fact, consistent complete theories.

Various constructions can now be provided using the notion of a *selection function* γ. The most flexible idea would be for the selection function to choose a subset of the maximal consistent supersets of K implying α. Intuitively, γ selects the "best" elements from $K\top\alpha$. This may be contrasted with Levi's [65] deliberate expansion where potential expansions of K by α (not necessarily maximal) are identified and some chosen after assessing their expected epistemic utility. Note also that a selection function γ is relative to a given belief set K (this also holds for the analogous construction in the AGM — see Alchourrón *et al.* [1] p. 512). It may be more appropriate to denote it by γ_K though we omit the subscript unless it is required, to avoid confusion. Before considering the proposal where γ selects a subset of $K\top\alpha$ we examine two special cases.

[8]That is, K' can be identified with a $K \cup \{\alpha\}$ *"world"*. See also § 5.3.3.

Maxichoice Abductive Expansion

The first idea is to consider, as in the AGM, a selection function choosing a single "best" element of $K \top \alpha$. That is, we apply a maxichoice selection function. If $K^{\oplus}_{\alpha}$ is determined by such a selection function, γ, ensuring that $\gamma(K \top \alpha)$ is always a singleton for any K and α whenever $K \top \alpha \neq \emptyset$, then we call $\oplus$ a *maxichoice abductive expansion function* and define it as follows.

Definition 5.3.3 *(Def Max)*

$$K^{\oplus}_{\alpha} = \begin{cases} \gamma(K \top \alpha) & \textit{whenever } K \top \alpha \textit{ is nonempty} \\ K & \textit{otherwise} \end{cases}$$

It can be shown that maxichoice abductive expansion functions satisfy the first six postulates for abductive expansion over K.

Lemma 5.3.2 *Any maxichoice abductive expansion function satisfies* (K$^{\oplus}$1) — (K$^{\oplus}$6).

Of course, such a function results in a maximally consistent belief set (i.e., a complete theory). It corresponds to selecting an abduction which is maximally specific in the sense of the previous chapter. An agent expanding in this way can be considered highly credulous. As such, the agent is not very cautious and the risk of error is maximal. Therefore, this strategy is not advisable in general.

Full Meet Abductive Expansion

At the other end of the spectrum we can consider a selection function choosing all elements of $K \top \alpha$ (i.e., a full meet selection function). If $K^{\oplus}_{\alpha}$ is determined by this type of selection function — one that ensures $\gamma(K \top \alpha) = K \top \alpha$ for any K and α whenever $K \top \alpha \neq \emptyset$ — then $\oplus$ is referred to as a *full meet abductive expansion function*.

Definition 5.3.4 *(Def Meet)*

$$K^{\oplus}_{\alpha} = \begin{cases} \bigcap(K \top \alpha) & \textit{whenever } K \top \alpha \textit{ is nonempty} \\ K & \textit{otherwise} \end{cases}$$

Such a selection function will result in a belief set as attested by the following observation.

Observation 5.3.3 $\bigcap(K\top\alpha)$ *is a belief set whenever* $K\top\alpha$ *is nonempty.*

This type of abductive expansion also satisfies the first six postulates for abductive expansion over K.

Lemma 5.3.4 *Any full meet abductive expansion function satisfies* ($K^{\oplus}1$) — ($K^{\oplus}6$).

Moreover, it can be shown that, in general, this type of abductive expansion corresponds to AGM expansion.

Theorem 5.3.5 *Let* $\oplus$ *be an abductive expansion function. For any formula* $\alpha \in \mathcal{L}$ *and belief sets* K *and* H *such that* $\neg\alpha \notin K$ *and* $\neg\alpha \notin H$*, the operation* $\oplus$ *is a full meet abductive expansion for* K *with respect to* α iff $\oplus$ *satisfies postulates* ($K^{+}1$) — ($K^{+}6$) *for AGM expansion over* K.

This theorem requires some explanation as it may appear a bit confusing at first sight. The reference to a belief set H corresponds to that mentioned in the AGM monotonicity postulate for expansion ($K^{+}5$). Without the proviso that $\neg\alpha \notin H$ it may occur that AGM expansion will expand into inconsistency while we have already seen that that is not possible with abductive expansion (unless $K = K_{\perp}$). In such a case the full meet abductive expansion postulates would fail to satisfy the AGM postulates for expansion. This theorem shows that AGM expansion represents a special case of abductive expansion (i.e., a full meet abductive expansion function).

A full meet abductive expansion function corresponds to choosing the least specific (or trivial) abduction α for new information α with respect to K. Using results from the previous chapter we can see that it also corresponds to choosing an abduction logically weaker than α. This means that the agent has minimised the acquisition of new information. Such a strategy would be adopted by a skeptical agent or hypochondriacal agent fearing possible "contamination" of its beliefs. It will have avoided the introduction of error but at the cost of no informational gain besides the new information and any deductive consequences.

Partial Meet Abductive Expansion

The obvious strategy, outlined above and to which we now return, is to adopt a partial meet selection function (i.e., one selecting some subset of $K \top \alpha$) — a compromise between the two extremes just discussed. If $K^{\oplus}_{\alpha}$ is determined by this type of selection function γ, always returning some subset of $K \top \alpha$ for any K and α whenever $K \top \alpha \neq \emptyset$, we call $\oplus$ a partial meet abductive expansion function.

Definition 5.3.5 *(Def Part)*

$$K^{\oplus}_{\alpha} = \begin{cases} \bigcap \gamma(K \top \alpha) & \textit{whenever } K \top \alpha \textit{ is nonempty} \\ K & \textit{otherwise} \end{cases}$$

Taking the intersection of the "best" elements selected by this type of function will also lead to a belief set.

Observation 5.3.6 *Let K be a belief set and $\alpha \in \mathcal{L}$. Then $\bigcap \gamma(K \top \alpha)$ is a belief set whenever $K \top \alpha$ is nonempty.*

Moreover, such a function does indeed characterise the first six postulates for abductive expansion over K.

Theorem 5.3.7 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a partial meet abductive expansion function* if and only if $\oplus$ *satisfies postulates* ($\mathrm{K}^{\oplus}1$) — ($\mathrm{K}^{\oplus}6$) *for abductive expansion over K.*

This type of abductive expansion function does not possess a bias towards any of the particular types of abduction explored in the last chapter and generalises both maxichoice and full meet abductive expansion functions. The selection function γ represents the abductive selection mechanism in this case.

Example 5.3.1 *Continuing example 5.1.1, we see that the same behaviour as described there is obtained, except that postulate ($\mathrm{K}^{\oplus}6$) dictates that $K^{\oplus}_{e} = K^{\oplus}_{e \wedge e}$ since $\Gamma \vdash e \leftrightarrow (e \wedge e)$ (in fact, $\vdash e \leftrightarrow (e \wedge e)$).* □

5.3.2 Supplementary Postulates

Analogously to the AGM, a selection function γ can be defined by imposing an ordering $\preceq$ over the elements of $K\top\alpha$. This will allow us to examine more closely how the selection function γ determines the "best" elements of $K\top\alpha$.

$$\text{(Def } \gamma) \qquad \gamma(K\top\alpha) = \{K' \in K\top\alpha : K'' \preceq K' \; for\ all\ K'' \in K\top\alpha\}$$

If the selection function γ picks out the elements of $K\top\alpha$ which are best according to some preference relation $\preceq$, then we say γ is *relational* over K. The resulting (partial meet) abductive expansion function is referred to as a *relational partial meet abductive expansion function*. We now examine supplementary postulates related to relational selection functions.

(K$^{\oplus}$7) $\quad K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ (Supplementary 1)

This postulate says that any belief resulting from the abductive expansion of K by α will also result through adding α to the abductive expansion of K by $\alpha \vee \beta$. Its purpose will become clearer upon examining some of its consequences (particularly the factoring condition below). The first consequence of this postulate states that any adopted explanation that α and β have in common with respect to K will also be adopted in explaining $\alpha \vee \beta$ with respect to K.

(5.6) $\quad K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta} \subseteq K^{\oplus}_{\alpha\vee\beta}$

Another consequence of postulate (K$^{\oplus}$7) states that, whenever α is included in explaining $\alpha \vee \beta$ then an explanation adopted in explaining α with respect to K is also adopted in explaining $\alpha \vee \beta$ with respect to K.

(5.7) $\quad$ If $\alpha \in K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha} \subseteq K^{\oplus}_{\alpha\vee\beta}$

As desired, any relational partial meet abductive expansion function satisfies postulate (K$^{\oplus}$7).

Lemma 5.3.8 *Any relational partial meet abductive expansion function satisfies* ($K^{\oplus}7$).

The most interesting case occurs when $\preceq$ is transitive (which is a minimal requirement of an ordering). In this case γ is called *transitively relational* over K and the resulting $\oplus$ a *transitively relational partial meet abductive expansion function*. A supplementary postulate beneficial in this instance is:

($K^{\oplus}8$) If $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ (Supplementary 2)

It states that whenever $\neg\alpha$ is not included in an explanation of $\alpha \vee \beta$ — so any explanation of α would suffice as an explanation of $\alpha \vee \beta$ — then any explanation chosen for $\alpha \vee \beta$ with respect to K is also chosen as an explanation for α with respect to K. Again, the situation where a more specific explanation of α is also chosen is not precluded. The postulate just sets a lower bound on the type of explanation chosen for α with respect to K. One consequence of this postulate (together with postulates ($K^{\oplus}1$) — ($K^{\oplus}6$)) is that any explanation chosen for $\alpha \vee \beta$ are also either chosen for α or chosen for β.[9]

(5.8) Either $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$

Another consequence of this postulate is that, if α is not included in explaining $\alpha \vee \beta$ — so neither is any explanation of α — then any explanation of $\alpha \vee \beta$ is also adopted in explaining β.

(5.9) If $\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$

Lemma 5.3.9 *Any transitively relational partial meet abductive expansion function satisfies* ($K^{\oplus}8$).

The following postulate is the conditional converse of postulate ($K^{\oplus}7$) and can be shown to be equivalent to ($K^{\oplus}8$) (in the presence of the other postulates).

[9]We omit the qualification that an explanation be with respect to the epistemic state K with the understanding that all explanations (i.e., abductions, since we are using abduction as our notion of explanation) are relative to the current epistemic state.

(5.10) If $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\}) \subseteq K^{\oplus}_{\alpha}$

Together, postulates (K$^{\oplus}$7) and (K$^{\oplus}$8) imply the following important factoring condition which can be used to give them a clear motivation.

(5.11) Either $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\beta}$ or $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$

It states that in explaining $\alpha \vee \beta$ we can either adopt the best explanation for α or the best explanation for β or we can take an explanation that they have in common. This condition also holds for AGM revision (cf. Gärdenfors [31] p. 57).

Another consequence of these two postulates is that, if one were to accept the weaker form of extensionality (i.e., condition (5.3)), then their presence would guarantee that the stronger form (postulate (K$^{\oplus}$6)) holds.

Lemma 5.3.10 *Postulates* (K$^{\oplus}$7) *and* (K$^{\oplus}$8) *together with (5.3) imply postulate* (K$^{\oplus}$6) *in the presence of the other postulates ((K$^{\oplus}$1) — (K$^{\oplus}$5)) for abductive expansion over K.*

More importantly, these two supplementary postulates exactly characterise transitively relational partial meet abductive expansion functions. This is the central result of this section.

Theorem 5.3.11 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a transitively relational partial meet abductive expansion function* if and only if $\oplus$ *satisfies postulates* (K$^{\oplus}$1) — (K$^{\oplus}$8) *for abductive expansion over K.*

5.3.3 Systems of Spheres

We now turn to a semantic construction inspired by Grove's [39] sphere semantics. This will be important in giving a clear picture of what occurs in abductive expansion. Recall that Grove defines a sphere to be a set of possible worlds (i.e., a set of maximally consistent sets of formulae) and a system of spheres centred on K to be an ordering over worlds in which $[K]$ (the set of worlds consistent with K) comprises the innermost sphere and

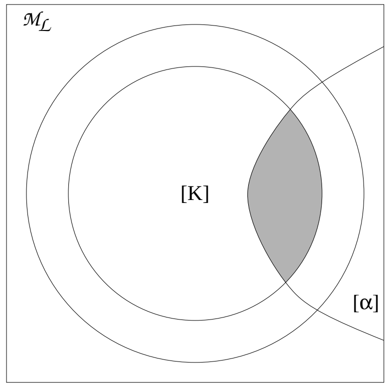


Figure 5.1: System of spheres with $[K^+_\alpha]$ shaded.

$\mathcal{M}_{\mathcal{L}}$ (all possible worlds) the outermost sphere. That is, he essentially orders those worlds inconsistent with K. Concentrating on AGM expansion for a moment, bring to mind the overview (§ 2.2.2) where we pointed out that the worlds consistent with K^+_α are exactly those consistent with both K and α. That is to say, $[K^+_\alpha] = [K] \cap [\alpha]$ as shown in Figure 5.1. However, we have already seen that AGM expansion corresponds to the case where the agent is acquiring a minimal amount of information. In abductive expansion however, the agent is attempting to acquire as much new, error-free information as possible and so, in general, adds more information to its epistemic state.

Adding more information corresponds to eliminating more worlds since each world represents certain possibilities and by acquiring more information the agent is ruling out possibilities. This means that we need to select some subset of the worlds identified by AGM expansion (i.e., some subset of $[K] \cap [\alpha]$). To achieve this, we introduce a new system of spheres within $[K]$ itself. That is, the agent's current epistemic state, in terms of worlds, is partitioned. This partitioning can be thought of as an ordering over the worlds consistent with K (cf. our remark about Grove's original ordering being over worlds consistent with K). This ordering can be seen as expressing a preference over K-worlds. We can consider $[K]$ to be an "internal" system of spheres and define such a system, centred within a set of worlds X, as follows.

Definition 5.3.6 *Let $\mathcal{IS}$ be any collection of subsets of $\mathcal{M}_{\mathcal{L}}$ ($\mathcal{IS} \subseteq 2^{\mathcal{M}_{\mathcal{L}}}$) and X some subset of $\mathcal{M}_{\mathcal{L}}$ ($X \subseteq \mathcal{M}_{\mathcal{L}}$). We say $\mathcal{IS}$ is a system of spheres centred within (or by) X if it* *satisfies the following conditions:*

(IS1) *$\mathcal{IS}$ is totally ordered by $\subseteq$*
(i.e., for any $U,\ V \in \mathcal{IS}$, $U \subseteq V$ or $V \subseteq U$)

(IS2) *X is in $\mathcal{IS}$ and, moreover, it is the $\subseteq$-maximal element of $\mathcal{IS}$*
(i.e., $X \in \mathcal{IS}$ and, for any $U \in \mathcal{IS}$, $X \subseteq U$)

(IS3) *If $\alpha \in \mathcal{L}$ and there is a sphere in $\mathcal{IS}$ intersecting $[\alpha]$, then there is a $\subseteq$-smallest sphere in $\mathcal{IS}$ intersecting $[\alpha]$*
(i.e., if $[\alpha] \cap X \neq \emptyset$, then there is a $U \in \mathcal{IS}$ such that $U \cap [\alpha] \neq \emptyset$ and for any $V \in \mathcal{IS}$, $V \cap [\alpha] \neq \emptyset$ implies $U \subseteq V$)

Note that, for any language, it immediately follows that $\mathcal{IS}$ has a $\subseteq$-minimal element (i.e., there is some $V \in \mathcal{IS}$ such that $V \subseteq U$ for all $U \in \mathcal{S}$). This is obvious for a finite language. To see that it also holds in the general case notice that for any tautology α (in fact, for any belief $\alpha \in K$) we have that $[K] \subseteq [\alpha]$. Therefore, the smallest sphere intersecting α, whose existence is guaranteed by (IS3), will be the "innermost" sphere – in other words, the $\subseteq$-minimal element.

Condition (IS1) states that the internal spheres are nested one within the other. (IS2) says that the $\subseteq$-maximal internal sphere contains all (and only) the worlds in X (which will be identified with the current epistemic state $[K]$). (IS3) ensures that, when X and $[\alpha]$ have worlds in common (i.e., they are consistent), for any $\alpha \in \mathcal{L}$, then there is some $\subseteq$-minimal (innermost) sphere in $\mathcal{IS}$ intersecting $[\alpha]$. This sphere is denoted $c_{\mathcal{IS}}(\alpha)$. This last condition, as in the Grove modelling, corresponds to Lewis' [67] Limit Assumption.

Having defined an internal system of spheres, $\mathcal{IS}$ (centred within X) we associate a function $f_{\mathcal{IS}} : \mathcal{L} \rightarrow 2^{\mathcal{M}_{\mathcal{L}}}$ with it, defined in the following manner.

$$f_{\mathcal{IS}}(\alpha) = \begin{cases} [\alpha] \cap c_{\mathcal{IS}}(\alpha) & \text{if } [\alpha] \cap X \neq \emptyset \\ X & \text{otherwise} \end{cases} \qquad \text{(Def } f_{\mathcal{IS}}\text{)}$$

Intuitively, when X and α are consistent, $f_{\mathcal{IS}}(\alpha)$ returns the most preferred worlds in X where α holds. If α does not hold in any X world, $f_{\mathcal{IS}}(\alpha)$ returns all X worlds.

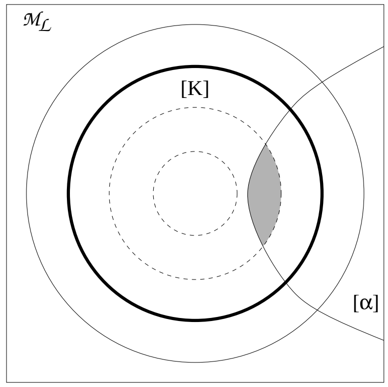


Figure 5.2: System of spheres with $[K^{\oplus}_{\alpha}]$ shaded.

This internal system of spheres, together with the function $f_{\mathcal{IS}}$, can now furnish a sphere semantics for abductive expansion over K which is similar in spirit to that for AGM revision. Referring to Figure 5.2, the structure inside the bold boundary marked $[K]$ represents an internal system of spheres $\mathcal{IS}$ centred within $[K]$ (we can ignore any spheres outside $[K]$). The innermost internal sphere (dashed ring) represents the $\subseteq$-minimal element of $\mathcal{IS}$. It contains the most preferred worlds. That is, the agent has some predisposition towards these worlds over the other worlds consistent with K. The next (outer) internal sphere contains the worlds within the next outermost dashed ring. This continues until the outermost ($\subseteq$-maximal element of $\mathcal{IS}$) internal sphere, represented by the bold boundary, containing exactly all the worlds in $[K]$. The worlds that belong exclusively to this internal sphere (i.e., to this and no other internal sphere) are the least preferred worlds consistent with K. This means we adopt the subset minimal change (AGM expansion) only when there is no preference over worlds consistent with K and α.

Overall, the internal system of spheres can be viewed as follows. The agent believes its actual world can be identified with one of the worlds in $[K]$ but does not have enough information to decide which one. However, it has a predisposition to certain worlds consistent with its beliefs over other worlds also consistent with its beliefs. When new information α arises the agent would like to rule out as many worlds as possible but would like to avoid, as much as possible, eliminating the actual world from further consideration. Therefore, depending on the agent's degree of boldness or caution — which specifies, in a sense, the way in which the tension between these two desiderata is to be resolved — the agent uses this predisposition to select those α-worlds in which it has most confidence. The agent's degree of boldness or caution can, in fact, be identified with the granularity of the internal system of spheres as we shall soon see.

The possible worlds consistent with the abductively expanded epistemic state $K^{\oplus}_{\alpha}$ are those determined by $f_{\mathcal{IS}}(\alpha)$ (i.e., $[K^{\oplus}_{\alpha}] = f_{\mathcal{IS}}(\alpha)$). That is, those in the intersection of $[\alpha]$ and the innermost sphere intersecting $[\alpha]$ (i.e., $[\alpha] \cap c_{\mathcal{IS}}(\alpha)$) if K and α are consistent. If K and α are inconsistent, then $[K^{\oplus}_{\alpha}]$ is simply $[K]$. The formulae in the new epistemic state can be determined by $K^{\oplus}_{\alpha} = th([K^{\oplus}_{\alpha}]) = th(f_{\mathcal{IS}}(\alpha))$.[10]

We can now interpret the Principle of Minimal Change with respect to this semantics. In fact, we do so in a manner that integrates nicely with the sphere semantics for AGM revision. We consider ourselves to have a system of spheres centred on the most preferred worlds (i.e., those within the innermost dashed ring). When we perform abductive expansion of K with respect to α, we take those α-worlds closest to the most preferable worlds. We are, however, limited by the boundary $[K]$ and cannot move beyond this for expansion. Minimal change, then, is taken from the most preferred worlds. For AGM revision, in the principal case where the new information α is inconsistent with the current epistemic state K, we go beyond $[K]$ in search of the closest α-worlds. The agent's desire to avoid error would preclude it from identifying $[K]$ with the most preferred worlds initially. It is unsure which of the worlds in $[K]$ is the actual world but does have some preference or affinity for certain worlds over others consistent with its beliefs.

The following theorems are the central results of this section and show that the sphere semantics suggested here exactly characterises transitively relational partial meet abductive expansion functions.

Theorem 5.3.12 *Let $K \in \mathcal{K}$ be some belief set and $\mathcal{IS}$ any internal system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred within $[K]$. If for any $\alpha \in \mathcal{L}$ we define $K^{\oplus}_{\alpha}$ to be $th(f_{\mathcal{IS}}(\alpha))$, then postulates* (K$^{\oplus}$1) — (K$^{\oplus}$8) *are satisfied.*

Theorem 5.3.13 *Let $\oplus : \mathcal{K} \times \mathcal{L} \to \mathcal{K}$ be any function satisfying postulates* (K$^{\oplus}$1) — (K$^{\oplus}$8).

[10] Properties of the function $th : 2^{\mathcal{M}_{\mathcal{L}}} \to \mathcal{K}$ are surveyed in Lemma 2.2.14 (p. 35).

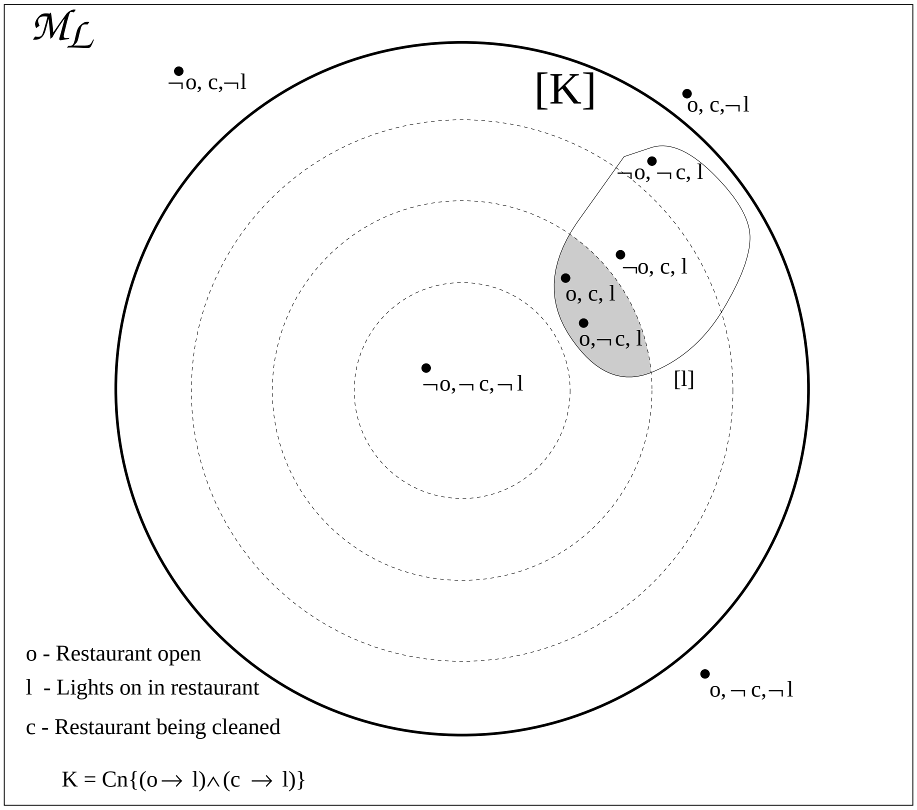


Figure 5.3: System of spheres for example.

Then for any belief set $K \in \mathcal{K}$ there is an internal system of spheres $\mathcal{IS}$ on $\mathcal{M}_{\mathcal{L}}$ centred within $[K]$ which, for all $\alpha \in \mathcal{L}$, satisfies $K^{\oplus}_{\alpha} = th(f_{\mathcal{S}}(\alpha))$.

We continue with our example on the restaurant theme which is illustrated in Figure 5.3. In order to keep the number of possible worlds at a manageable level, we restrict our language to one consisting of three propositions:

o	*The restaurant is open*
c	*The restaurant is being cleaned*
l	*The lights are on in the restaurant*

Suppose our agent currently believes that, if the restaurant is open, then its lights are on and, if it is being cleaned, its lights are also on. The manner in which it is predisposed to the various possible worlds can be gleaned from Figure 5.3. The agent is presented with new information that the lights of the restaurant are on (l). According to Figure 5.3 two

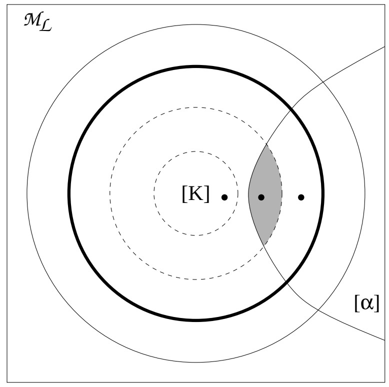


Figure 5.4: Maxichoice abductive expansion — one world per internal “band”.

worlds are preferred (shaded section), leading to the belief set $K^{\oplus}_{l} = Cn(\{o\} \cup \{l\})$. The restaurant is open (and its lights are on).

Having determined sphere semantics for transitively relational partial meet abductive expansion functions we shall briefly look at those for full meet and maxichoice abductive expansion functions. In the case of full meet abductive expansion we have already noted that it corresponds to AGM expansion (refer to Figure 5.1 p. 115) with the proviso that the negation of the new information does not appear in the current epistemic state. In this case, $K^{\oplus}_{\alpha} = K$ while $K^{+}_{\alpha} = K_{\perp}$. It is clear that full meet abductive expansion corresponds to the situation in which there is only one internal sphere, namely $[K]$. The agent does not have any preference among the worlds consistent with K. This corresponds to a cautious or skeptical agent who refuses to guess and takes hard facts only. In the case of maxichoice abductive expansion one world (maximally consistent theory) is selected by the abductive process. In this situation, the innermost sphere contains one world and each subsequent outer sphere contains one more world than the previous smaller sphere. That is, there is one possible world per “band” or “level” as illustrated in Figure 5.4. This corresponds to a maximally credulous agent who makes very bold conjectures.

This analysis also suggests a way of assessing how cautious or bold an agent is in a given epistemic state. If an epistemic state is divided — recall that spheres are nested one within the other so we mean “divided” here in the sense of how many spheres there are — into many spheres compared to the number of worlds, then the agent will be, on average, very bold (i.e., have a high degree of boldness; a low degree of caution). On the other hand, if there are few spheres compared to the number of worlds, then the agent will be, in general, very conservative (i.e., have a high degree of caution). This gives a qualitative view of the agent's degree of boldness (or caution, depending on how it is measured). There would be many ways of assessing the agent's degree of boldness or caution quantitatively. One could simply count the number of spheres. This would have the advantage of giving a comparative measure not only when there are a finite number of worlds in $[K]$ but also in certain cases where there are infinitely many worlds consistent with K (infinite language) but contained in a finite number of spheres. Another idea would be to divide the number of spheres by the number of worlds consistent with K. This would work if $\mid [K] \mid$, the number of worlds consistent with K, is finite. In the case of a maxichoice abductive expansion function $\frac{|\mathcal{IS}|}{|[K]|} = 1$ where $\mid \mathcal{IS} \mid$ is the number of internal spheres and, for full meet abductive expansion, $\frac{|\mathcal{IS}|}{|[K]|} \ll 1$ (closer to 0). Many other combinations are possible. A similar idea can be used to determine the specificity of an abduction in abductive expansion. This is not unsurprising given the clear link between boldness and specificity. A bold agent would seek a very specific abduction and the choice of a maximally specific abduction corresponds to a maxichoice abductive expansion. A cautious agent would choose a less specific abduction. Choosing a least specific abduction leads to full meet abductive expansion (i.e., AGM expansion). The number of levels of specificity allowed by an abductive expansion function will correspond to the number of internal spheres.

5.3.4 Epistemic Entrenchment

It has been shown that Grove's ordering on possible worlds is equivalent to an ordering over the formulae of $\mathcal{L}$ [39] (see also Gärdenfors [31] p. 95). Similarly, we can show that the internal system of spheres presented in the previous section also leads to an ordering over formulae. In the AGM framework, an epistemic entrenchment ordering is an ordering over the formulae of $\mathcal{L}$ (refer to the overview in § 2.2.2). Intuitively, it represents a preference over current beliefs. In contraction, less entrenched formulae are given up in preference to more entrenched formulae; they are easier to give up. The (standard) epistemic entrenchment ordering conditions (SEE1) — (SEE5) [33] specify an ordering in which tautologies are maximally entrenched and non-beliefs are minimally entrenched. Therefore, the ordering is essentially over formulae in the current epistemic state K. Gärdenfors and Makinson [34] also consider orderings satisfying conditions (SEE1) — (SEE3) which they refer to as expectations orderings. They are mainly used to unify the area of belief revision with that of nonmonotonic inference.

In a like manner, we can specify an ordering over formulae to help determine which formulae to include after abductive expansion.

Definition 5.3.7 *An ordering $\leq$ over $\mathcal{L}$ is an abductive entrenchment ordering if it satisfies (SEE1) — (SEE3) and condition (AE4):*

(SEE1)	*For any $\alpha,\ \beta,\ \gamma \in \mathcal{L}$, if $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$*	*(transitivity)*
(SEE2)	*For any $\alpha,\ \beta \in \mathcal{L}$, if $\{\alpha\} \vdash \beta$ then $\alpha \leq \beta$*	*(dominance)*
(SEE3)	*For any $\alpha,\ \beta \in \mathcal{L}$, $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$*	*(conjunctiveness)*
(AE4)	*When $K \neq K_\perp$, $\alpha \in K$ iff $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$*	*(maximality)*

Condition (AE4) specifies an ordering $\leq$ that is essentially over formulae that are not believed. An abductive entrenchment ordering is, then, an expectations ordering in which all beliefs are maximally entrenched. Like an expectations ordering, abductive entrenchment is a total preorder. We note some properties of this type of ordering.

Lemma 5.3.14 *If $\leq$ satisfies postulates* (SEE1) — (SEE3) *and* (AE4) *then it also has the following properties:*

(*i*) *If $\alpha \in K$ and $\beta \notin K$ then $\beta < \alpha$*

(*ii*) *When $K \neq K_\perp$, if $\neg\alpha \in K$ then $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$*

(*iii*) *When $K \neq K_\perp$, if $K \cup \{\alpha\} \vdash \beta$, then $\alpha \leq \beta$.*

Condition (*i*) tells us that beliefs, and only beliefs, are strictly maximally abductively entrenched. They are maximal elements of the ordering and all are equally entrenched. This gives us a way of extracting the current epistemic state K from an abductive entrenchment ordering; just look at the maximally abductively entrenched formulae. The second condition says that disbeliefs are minimally abductively entrenched although they need not be the only minimally abductively entrenched formulae. Condition (*iii*) shows that a strengthened version of the dominance condition for expectations orderings (SEE2) results from the addition of (AE4). Dominance holds with respect to the current epistemic state, not only with respect to the logic. Condition (*ii*) can also be seen to result from the following interesting property of expectations orderings and condition (*i*).

Lemma 5.3.15 *Let $\leq$ be an SEE relation satisfying* (SEE1)—(SEE3)*. For any $\alpha \in \mathcal{L}$, either $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$ or $\neg\alpha \leq \beta$ for all $\beta \in \mathcal{L}$.*

It says that, in an expectations ordering, either a formula or its negation (or both) is minimally ordered, for any formula of the language.

Intuitively, the inequality $\alpha \leq \beta$ can be thought of as expressing that it is no more difficult to assume β than it is to assume α in an abduction. We now consider conditions specifying how to move backwards and forwards between an abductive entrenchment ordering $\leq$ and an abductive expansion function $\oplus$ over an epistemic state K. The following condition determines an abductive entrenchment ordering $\leq_\oplus$ given an abductive expansion function $\oplus$ and a belief set K.

$$(C \leq) \qquad \alpha \leq_\oplus \beta \textit{ iff either } \alpha \notin K^\oplus_{\neg\alpha\vee\neg\beta} \textit{ or } K \vdash \alpha \wedge \beta$$

The subscript $\oplus$ serves to emphasise that the abductive entrenchment ordering $\leq_\oplus$ is derived from an abductive expansion function $\oplus$ (assumed relative to an epistemic state K). We omit it in future discussions and in the proofs unless this connection needs to be made explicit. This condition can be motivated as follows. Going from right to left and concentrating first on the latter part of the right-hand side, if both α and β are currently believed, then they will be both maximally abductive entrenched by postulate (AE4) and therefore equally entrenched, so trivially $\alpha \leq \beta$. Shifting to the former part of the condition on the right-hand side, there are two possibilities to consider: either $\beta \in K$ or $\beta \notin K$. Clearly, α is not believed initially by postulate ($K^\oplus 3$) and since $\alpha \notin K^\oplus_{\neg\alpha\vee\neg\beta}$. If β is believed initially, then certainly $\alpha \leq \beta$ by postulate (AE4). However, if β is not believed, then all we know is that α does not occur in the abductively expanded belief state and therefore α cannot be strictly more abductively entrenched than β, hence $\alpha \leq \beta$. Going from left to right can best be motivated by considering the contrapositive (i.e., if $\alpha \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$ and $K \not\vdash \alpha \wedge \beta$, then $\beta < \alpha$). If α is in the abductively expanded belief state either it was initially believed ($\alpha \in K$) or it was neither believed nor disbelieved initially ($\alpha, \neg\alpha \notin K$). If α was believed, then by the latter part of the condition, β was not believed and therefore β is strictly less entrenched than α. On the other hand, if α was neither believed nor disbelieved initially, then the fact that it is now currently believed (in the abductively expanded belief state) means that β cannot be believed in the abductively expanded belief state (nor was it initially believed) and therefore, again, β is strictly less entrenched than α since α was assumed in preference to β.

The next condition allows us to determine an abductive expansion function $\oplus_{\leq}$ for a particular epistemic state K given an abductive entrenchment ordering $\leq$.

$$(C\oplus) \qquad \beta \in K^{\oplus_{\leq}}_{\alpha} \textit{ iff either } \beta \in K \textit{ or both } \neg\alpha \notin K \textit{ and } \alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$$

Again, we omit the subscript $\leq$ unless necessary. We can motivate this condition as follows. Going first from right to left, the first part of the condition says that, if β is currently believed, then it will also be believed in the abductively expanded epistemic state (this follows from postulate ($K^{\oplus}3$)). The latter part of the condition says that, if abduction is possible, then β will be included in the abductively expanded epistemic state if the information in β relative to α is (strictly) easier to assume than that in $\neg\beta$ relative to α. Going from left to right can be motivated by considering the contrapositive (i.e., if $\beta \notin K$ and either $\neg\alpha \in K$ or $\alpha \rightarrow \neg\beta \not< \alpha \rightarrow \beta$, then $\beta \notin K^{\oplus}_{\alpha}$). It says that, if β is not already believed and, either abduction is not possible (i.e., α is not "explainable") or the information in β is no more assumable than the information in $\neg\beta$ with respect to α, then β should not be assumed in explaining α.

The following theorems show the adequacy of these definitions. The first shows that an abductive expansion function, for a particular K, defined from an abductive entrenchment relation using condition (C$\oplus$) has the desired properties.

Theorem 5.3.16 *Let $K \in \mathcal{K}$ be some belief set and $\leq$ an abductive entrenchment for K.*

If for any $\alpha \in \mathcal{L}$, we define $K^{\oplus}_{\alpha}$ using $(C\oplus)$, then the operation $\oplus$ so defined satisfies postulates (K$^{\oplus}$1) — (K$^{\oplus}$8) *as well as the condition* (C$\leq$).

The next result shows the converse. We can use an abductive expansion function $\oplus$ and condition (C$\leq$) to construct an abductive entrenchment relation with the requisite properties.

Theorem 5.3.17 *Let $\oplus : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$ be any function satisfying* (K$^{\oplus}$1) — (K$^{\oplus}$8)*. Then, for any belief set $K \in \mathcal{K}$, if we define $\leq$ using* (C$\leq$)*, then the relation $\leq$ so defined is an abductive entrenchment relation (i.e., it satisfies* (SEE1) — (SEE3) *and* (AE4)*) and also satisfies condition* ($C\oplus$)*.*

The following two results are very important and, as far as we know, the corresponding results have not been shown for epistemic entrenchment.[11] They further highlight the appropriateness of conditions (C$\oplus$) and (C$\leq$). It is important to note that these results are relative to a belief set K. This is because an abductive entrenchment relation $\leq$ is relative to a belief set K and, for different belief sets, we have different abductive entrenchment relations. The first result states that, if relative to a belief set K we start with an abductive expansion function $\oplus$ and determine the related abductive entrenchment ordering $\leq_{\oplus}$ using condition (C$\leq$) before applying condition (C$\oplus$) to obtain a new abductive entrenchment function $\oplus_{\leq_{\oplus}}$, then the resulting function is exactly the same as the original (i.e., $\oplus = \oplus_{\leq_{\oplus}}$) relative to K.

Theorem 5.3.18 *Let $K \in \mathcal{K}$ be any belief set. For any $\alpha,\ \beta \in \mathcal{L}$, $K^{\oplus}_{\alpha} = K^{\oplus_{\leq_{\oplus}}}_{\alpha}$.*

The next theorem shows the analogous result, having started from an abductive entrenchment ordering and applying condition (C$\oplus$) followed by condition (C$\leq$) to obtain the same abductive entrenchment ordering (i.e., $\leq = \leq_{\oplus_{\leq}}$) relative to K.

Theorem 5.3.19 *Let $K \in \mathcal{K}$ be any belief set. For any $\alpha,\ \beta \in \mathcal{L}$, $\alpha \leq \beta$ iff $\alpha \leq_{\oplus_{\leq}} \beta$.*

In light of these results, we can view abductive entrenchment as an extension of epistemic entrenchment. In fact, both are different types of the more general expectations ordering

[11] I am indebted to Abhaya Nayak for stressing the importance of being able to prove these results.

with epistemic entrenchment essentially ordering beliefs and abductive entrenchment essentially ordering non-beliefs. Therefore, given a full expectations ordering over a belief set K (i.e., one not just ordering beliefs or ordering only the non-beliefs), we can use that part of it which orders the elements of K to determine AGM contraction (and revision) and that part ordering the non-beliefs to perform abductive expansion. This is quite an interesting result in view of the suggested links between expectations orderings and nonmonotonic inference [72] which we shall return to shortly. It shows what use may be made of two different "parts" of an expectations ordering for the purpose of belief change. Moreover, the extension of the Grove sphere semantics into the internal part of $[K]$ has removed an asymmetry while imbuing these extended spheres with a natural interpretation.

Example 5.3.2 *We continue the example used to illustrate the sphere semantics. The agent's predisposition would be reflected in an abductive entrenchment including the following relative orderings:*

$$l \rightarrow \neg o < l \rightarrow o$$

$$l \rightarrow \neg c = l \rightarrow c$$

where $\alpha = \beta$ means $\alpha \leq \beta$ and $\beta \leq \alpha$. Using condition (C⊕) and supposing $\neg l,\ \neg o,\ c \notin K$ initially, it is easy to determine that $o \in K_l^\oplus$ and $c \notin K_l^\oplus$.

Suppose furthermore, that if the lights are on and there is a sign on the door saying the restaurant is being cleaned (s), then we cannot abduce that the restaurant is open. This can be expressed by the following inequalities in the same ordering as above:

$$l \wedge s \rightarrow \neg o = l \wedge s \rightarrow o$$

$$l \wedge s \rightarrow \neg c < l \wedge s \rightarrow c$$

In this case, using the same K, we have $c \in K_{l \wedge s}^\oplus$ but $o \notin K_{l \wedge s}^\oplus$. □

In the section on sphere semantics we indicated how it is possible to assess the degree of caution or boldness of an agent in a particular epistemic state. These observations can be carried over to abductive entrenchment orderings. However, instead of examining the number of spheres, it is important to examine the number of "ranks" in the abductive entrenchment preordering. Similar remarks to those in the previous section can be made on quantitatively or qualitatively assessing the degree of caution or boldness.

Abductive entrenchment can also be considered an embodiment of the notion of positive coherence. It tells us how formulae positively cohere with one another. An inequality $\alpha \to \neg\beta < \alpha \to \beta$ can be thought of as saying that β coheres better with α than does $\neg\beta$ (i.e., α and β positively cohere). That would explain why β should be embraced along with α when new information α is acquired. On the other hand, epistemic entrenchment is an expression of negative coherence. In an epistemic entrenchment ordering, $\alpha \leq \beta$ is taken to mean that, if a choice needs to be made in giving up α or β, then prefer to give up α. In the (C$-$) condition, the inequality $\alpha < \alpha \vee \beta$ expresses that α negatively coheres with β. This would explain why belief in β won't be affected when the belief in α is abandoned. Therefore, we can view abductive entrenchment as providing an element of positive coherence and epistemic entrenchment an element of negative coherence. In the AGM, expansion does provide an element of positive coherence but it is very weak. Here positive coherence is used to determine which are the new beliefs and negative coherence which beliefs to abandon (new non-beliefs). They complement each other as do expansion and contraction. These remarks can also be carried over to the sphere semantics to some extent. The internal spheres can be thought of as expressing which worlds cohere better with each other. Then, when new information is received, those cohering best with each other and the new information are retained.

5.3.5 The Relationship Between Internal Spheres and Abductive Entrenchment

We highlighted the fact that the internal sphere modelling for abductive expansion is essentially an ordering over those worlds consistent with the current beliefs while abductive entrenchment essentially orders formulae representing non-beliefs. The following condition specifies how to translate between the two modellings.

(AE$\mathcal{IS}$) For every $\alpha,\ \beta \in \mathcal{L}$, $\alpha \leq \beta$ if and only if either $\beta \in K$ or both $\alpha \notin K$ and $c_{\mathcal{IS}}(\neg\alpha) \subseteq c_{\mathcal{IS}}(\neg\beta)$

where $c_{\mathcal{IS}}(\neg\alpha)$ denotes the innermost internal sphere containing $\neg\alpha$-worlds if $\neg\alpha$ is consistent with K and $[K]$ otherwise. The appropriateness of this condition is shown by the following result.

Theorem 5.3.20 *Let $K \in \mathcal{K}$ be a consistent belief set. If $\leq$ is an abductive entrenchment ordering for K and $\mathcal{IS}$ an internal system of spheres centred within $[K]$, then an abductive expansion function determined from $\leq$ by condition $(C\oplus)$ and one from $\mathcal{IS}$ via $f_{\mathcal{IS}}$ are the same if and only if condition* (AE$\mathcal{IS}$) *is satisfied.*

A similar condition holds for Grove's sphere modelling and epistemic entrenchment (see [39] pp. 164 – 167[12], [31] p. 95 and [100]).

5.3.6 Weakening Extensionality

Of the basic abductive expansion postulates presented in this chapter perhaps strong extensionality can be singled out as the most contentious. When considering epistemic states as belief sets this postulate actually makes good sense due to the Principle of Irrelevance of Syntax. Two formulae that are logically equivalent with respect to the current epistemic state will have the same potential abductions. So it is only fair, if syntax is to be considered irrelevant, to choose the same abduction(s) in both cases. The Principle of Irrelevance of Syntax, however, can be imposed on a weaker level — with respect to the underlying logic itself as is the case in AGM contraction and revision. This would especially seem to make more sense if we are representing epistemic states as belief bases rather than the deductively closed belief sets since more "syntactic relevance" is given to those formulae explicitly present in the base over any implicit consequences.

Adopting weak extensionality (condition (5.3) (p. 106)) in place of strong extensionality clearly requires a modification to the existing constructions. Confining our attention to the construction based on selection functions applied to maximal consistent supersets of a belief set K implying new information α, we see that a single selection function for each belief set K applied to $K \top \alpha$ does not suffice. This coincides with strong extensionality (see the proof to Theorem 5.3.7). What we require is a different selection function $\gamma_{K,\alpha}$ for each belief set K and logically distinct α. That is, given a belief set K there will

[12]Note however, that the relationship in Grove's paper is not with epistemic entrenchment but an alternative ordering over formulae proposed by Grove [39].

be a different selection function for each different truth functional proposition α applied to $K \top \alpha$. There is in fact some similarity to Levi's deliberate expansion. The elements chosen by the selection function $\gamma_{K,\alpha}$ can be considered an ultimate partition. However, under the current proposal these elements are not subjected to scrutiny through evaluation of their epistemic utility as in deliberate expansion. In this light, our original proposal makes more sense. The elements of $K \top \alpha$ can be considered the ultimate partition[13] and the selection function $\gamma_{K,\alpha}$ returns those elements with highest epistemic utility. In terms of the internal sphere modelling, we can consider each distinct truth functional proposition α as partitioning $[K]$ into internal spheres. There is no one fixed internal system of spheres for each K. There would be many sets of internal system of spheres for K and the one to be adopted depends on the new information. Note, however, that by Lemma 5.3.10 if the supplementary postulates ($K^{\oplus}7$) and ($K^{\oplus}8$) for abductive expansion over K hold, then strong extensionality results.

5.4 Default Reasoning

The AGM postulates are intended to expound in epistemic terms how an agent's beliefs change when the agent is confronted with new information. That is, they are expressed in terms of a current epistemic state, an epistemic input and the modified epistemic state. Their motivation is entirely with epistemic considerations in mind. Makinson and Gärdenfors [34, 72] link belief revision to non-monotonic inference and, in particular, full meet AGM revision to a THEORIST style default logic [104, 102]. We consider this type of default logic but without the notion of constraints. Using this connection they motivate this type of default logic in epistemic terms. It is our contention here that the epistemic interpretation given to defaults can be better handled using abductive expansion. This is not entirely surprising given the similar nature of this type of default logic and abduction expansion. We start by reviewing the development from AGM belief change operations to non-monotonic inference (and default logic) and back again.

In § 2.2.2 we saw how the AGM operators, described via postulates, can be modelled in

[13] Levi's ultimate partition is usually smaller than $K \top \alpha$.

various ways. One important construction was that of epistemic entrenchment; a total preordering of the formulae in the language in which tautologies (or logical truths) are maximally entrenched and non-beliefs minimally entrenched. Essentially, it is an ordering of the agent's current beliefs. Such an ordering satisfies the five postulates (SEE1) — (SEE5) (refer to p. 39). Intuitively, less entrenched formulae would be removed in favour of more entrenched formulae in belief contraction. Epistemic entrenchment can be considered an expression of the epistemic importance of different formulae. It is possible to provide conditions specifying which formulae are to be included in a contracted or revised epistemic state given the original epistemic state and the epistemic input. Such a condition was given for belief contraction by Gärdenfors and Makinson [33] and the corresponding condition for belief revision may be determined using the Levi Identity ($K^*_\alpha = (K^-_{\neg\alpha})^+_\alpha$).

The next development was to notice that a belief revision operator $*$ could be used to determine conditions on a nonmonotonic consequence relation $\mid\!\sim$[14] (and vice versa) [34, 72]. This was achieved via the following definition.

Definition 5.4.1 $\alpha \mid\!\sim \beta$ *if and only if* $\beta \in K^*_\alpha$

The resulting translations turn out to be quite natural conditions on nonmonotonic consequence relations. Many in fact have analogues in common nonmonotonic logics. A natural progression was to apply the epistemic entrenchment construction to nonmonotonic consequence. Gärdenfors and Makinson [34] noted that while the restriction that tautologies be maximally entrenched and non-beliefs minimally entrenched could be justified in terms of belief contraction and revision, they were not necessary when considering nonmonotonic consequences. This led them to drop the postulates of minimality and maximality for epistemic entrenchment and define the more general expectations ordering. Expectations orderings need satisfy only postulates (SEE1) — (SEE3).

Intuitively, an expectations ordering can be interpreted as "degrees of firmness" [34] (p. 209) (or "degrees of defeasibility" [34] (p. 209), as we shall see). It can be used in conjunction

[14] We also write $C(\Gamma)$ to denote $\{\alpha : \Gamma \mid\!\sim \alpha\}$.

with the following condition [34] to determine nonmonotonic consequences.[15]

$$\alpha \mid\!\sim \gamma \text{ iff either } \alpha \vdash \gamma \text{ or } \alpha \to \neg\gamma < \alpha \to \gamma$$

Observe that the latter part of this condition corresponds to the inequality in condition (C$\oplus$). More importantly, Gärdenfors and Makinson [34] suggested a way of encoding defaults using expectations orderings and the latter part of this condition. We shall illustrate with the following simple example.

Example 5.4.1 *Suppose our language has the following predicates:*

$G(x)$**:** *The grass was wet on day x*

$S(x)$**:** *The sprinkler was on during day x*

$W(x)$**:** *Water restrictions were in place on day x*

Now suppose we have two default rules:

(i) "The grass is wet normally due to the sprinkler", and

(ii) "Sprinklers cannot normally be used when water restrictions are in place"

In an expectations ordering, these two defaults could be expressed by the following inequalities (using the inequality given in the condition above)

$$G(a) \to \neg S(a) < G(a) \to S(a)$$

$$W(x) \to S(a) < W(a) \to \neg S(a)$$

for certain constants a in our object language representing days.

Now suppose on one day we notice that the grass is wet despite the fact that water restrictions are in place. What might we conclude? Evidently, this depends on the relative ordering of certain formulae in the expectations ordering. The following possibility may seem appropriate in this case.

$$(G(a) \wedge W(a)) \to \neg S(a) < (G(a) \wedge W(a)) \to S(a)$$

[15] Gärdenfors and Makinson [34] provide a number of other, equivalent conditions, but this one is more suitable for our purpose here.

That is, the sprinkler could still be used despite the fact that water restrictions were in place. □

This example shows how nonmonotonicity can be achieved and defaults expressed through expectations orderings. Actually, Williams [131] notes: "[u]sing results from Gärdenfors and Makinson [34] together with the observation that setting normal defaults to supernormal defaults by way of $\frac{\alpha:\beta}{\beta} \Longrightarrow \frac{\top:\alpha\to\beta}{\alpha\to\beta}$, it can be seen that expectation orderings are as expressive as normal defaults with linear priorities." We can have a default $\frac{\top:\alpha\to\beta}{\alpha\to\beta}$ by ensuring the inequality $\alpha \to \neg\beta < \alpha \to \beta$ in an expectations ordering. Williams [131] achieves this by using what is known as a *cut* (see Rott [113]).

This brings us to the crux of the problem we wish to address here. Makinson and Gärdenfors [72] give an interpretation of THEORIST's default logic [104, 102] (see § 3.3.1 p. 65) in terms of full meet revision [2, 31] based on the above ideas.[16] They concentrate on the *skeptical inference operation*

$$C(\mathcal{F}) = \bigcap\{Cn(\mathcal{F} \cup D) : D \text{ is a maximal subset of } \Delta \text{ consistent with } \mathcal{F}\}$$

They suggest a correspondence between full meet revision K^*_α and the skeptical inference operation $C(\alpha)$.[17] They note that $K^*_\alpha \subseteq C(\alpha)$ although the converse is not true in general.

We first note the following property of AGM revision.

$$\text{If } \beta \in K^*_\alpha, \text{ then } \alpha \to \beta \in K$$

Now suppose we would like the presence of β in K^*_α to be due to some default rule $\frac{\top:\alpha\to\beta}{\alpha\to\beta}$. This property tells us that we would have initially believed $\alpha \to \beta$. Returning to our example above, if we would like to believe $S(a)$ on the basis of $G(a)$ through our first default rule, then we would need to believe

$$G(a) \to S(a).$$

But this is essentially our default rule (expressed with the help of material implication). It is our contention that this is somewhat unintuitive. It suggests that defaults must, at some stage,

[16]Recall that we do not consider Poole's [102] extension concerning constraints.
[17]That is, Γ restricted to the singleton $\{\alpha\}$.

possess the status of full beliefs in order to be used. Thus, defaults are elevated to the same status as all other beliefs. In fact, as far as the epistemic state of the agent is concerned, these "defaults" are indistinguishable from other beliefs. We maintain that defaults and beliefs should not possess the same epistemic status — that some distinction should exist between them — and suggest abductive expansion and its construction abductive entrenchment as a way of making this distinction clear.

In Gärdenfors and Makinson's proposal, the fact that defaults are at some time beliefs is clearly borne out. There, as evidenced by the definitions above, K, in a sense, represents the default set.[18] They justify their stance: "[f]or so long as we are *using* a belief set K, its elements function as full beliefs. But as soon as we seek to *revise* K thus putting its elements into question, they lose the status of full belief and become merely expectations, . . .". On this account, there appears to be no difference between defaults and beliefs up until the point where beliefs are altered. Intuitively, it would seem useful for an agent to better distinguish between beliefs or facts on the one hand and defaults on the other. Beliefs/facts would have a privileged epistemic status; only being retracted when absolutely necessary. It is doubtful whether one would afford defaults the same epistemic status. Therefore, we shall provide an epistemic interpretation of defaults in terms of an abductive expansion operator which we feel is more appropriate.

Note also that THEORIST type default reasoning is better modelled by abductive expansion in the sense that facts (i.e., the agent's firm beliefs) are never given up, which is also the case for the agent's beliefs in expansion. In fact, THEORIST's defaults sit separately in Δ and are not conflated with the facts $\mathcal{F}$. Gärdenfors and Makinson are conflating facts and defaults in the belief set K. Abductive expansion makes this distinction quite clear.

Our goal is to find an epistemic model for defaults which distinguishes them from beliefs thus, in our opinion, more accurately reflecting their epistemic status. Williams [131] makes an interesting distinction between the formulae in an expectations ordering.[19] She essentially claims that those formulae more deeply entrenched are beliefs and the formulae less entrenched than the beliefs but not least entrenched are defaults. The least entrenched

[18] Presumably, the facts are also part of the default set.

[19] Actually, she deals with *expectation rankings* — a particular modelling of expectations orderings.

formulae are neither beliefs nor defaults. This distinction can be easily modelled using abductive entrenchment.

Using the same encoding for defaults as before but adopting an abductive entrenchment provides a nice correspondence with Williams' [131] claim, noted above, that defaults are less entrenched than beliefs. Moreover, it allows a more direct rendering of Poole's default logic in epistemic terms. It can be seen that the facts $\mathcal{F}$ correspond to the current beliefs K held by the agent.[20] The defaults in Δ correspond to non-beliefs and can be encoded in abductive entrenchment as noted above.

Since an abductive expansion function maintains consistency, it is clear that it represents some choice over defaults (for defaults may conflict). In fact, $K^{\oplus}_{\alpha}$ determines an extension. Actually, the situation is slightly more complicated because different abductive expansion operators are capable of expressing situations in which no defaults are applied[21] through to situations in which a maximum number of consistent defaults are applied. Restricting our attention to abductive entrenchments where a maximum number of consistent defaults from Δ are encoded[22] however, results in abductive expansion operators which calculate extensions. This lends credence to Williams' [131] claim that expectations orderings are as expressive as normal defaults with linear priorities.

We see the contents of K as those things the agent really believes. What the agent considers to be "normally the case" or "usually true", are not regarded in the same light but reside outside K and are ordered among the non-beliefs.

Makinson and Gärdenfors [72], on the other hand, essentially consider K to represent the defaults (and supposedly the facts too), while the interpretation given here associates K with the facts and appropriately ordered non-beliefs with defaults. In this way, the epistemic status of facts and that of defaults are not conflated and their role in belief change, and nonmonotonic inference for that matter, becomes more explicit. This allows a modelling of more credulous forms of nonmonotonic inference than the skeptical inference operations considered by Makinson and Gärdenfors. We also avoid the full meet property of revision

[20] We note however that while K is deductively closed, $\mathcal{F}$ need not be. This creates no real problems.

[21] Apart, of course, from trivial ones like $\frac{\top:\alpha\to\alpha}{\alpha\to\alpha}$ — if one considers them defaults also — and deductive consequences.

[22] Because such situations more accurately reflect what is actually intended by Δ.

which is known to have undesirable consequences.

Perhaps, more importantly, this link between abductive expansion and default reasoning can be used to provide a "semantics" for default reasoning. We can use the sphere semantics for abductive expansion discussed in Section 5.3.3 to model THEORIST style default reasoning. A default $\frac{\top : \alpha \rightarrow \beta}{\alpha \rightarrow \beta}$ (or $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$) exists under this particular modelling provided that $([\alpha] \cap c_{\mathcal{IS}}(\alpha)) \cap [\neg\beta] = \emptyset)$. That is, provided no $\neg\beta$-worlds coincide with the closest α-worlds. Intuitively, all the preferred α-worlds are β-worlds; we do not want $\neg\beta$ among our preferred α-worlds. Of course, it is also necessary that $\neg\beta$ is not initially believed.[23]

5.5 Extracting Abductions from Abductive Expansion

Our main concern in belief revision is determining the nature of epistemic states as they undergo change. In abductive expansion, as we have seen, the concern is to determine which beliefs should be incorporated into the current epistemic state using an abductive strategy to identify the appropriate expansion given new information. In so doing however, the process of abduction has become "internalised" in the belief expansion process and therefore, the actual abduction(s) made to effect a change in epistemic state is, in a sense, lost. That is, it may not be possible to determine the abduction selected for a belief set K and epistemic input α in the sense of Definition 5.1.1 (i.e., it may not be possible to identify β). It is possible however, to determine, to a certain extent, an abduction capable of doing the job. Examining the proof to Theorem 5.2.1 gives us an idea of how this can be done. All we need do is examine the set $K_\alpha^\oplus \setminus K$ and determine a finite axiomatisation of it.[24] The conjunction of the elements of this finite axiomatisation will suffice as an appropriate abduction. In fact, whenever $\neg\alpha \notin K$ and abductive expansion actually occurs, it might make more sense to examine the set $K_\alpha^\oplus \setminus K_\alpha^+$ provided the two do not coincide (i.e., $K_\alpha^\oplus = K_\alpha^+$ gives that $K_\alpha^\oplus \setminus K_\alpha^+ = \emptyset$) as this set will be smaller. In the case where

[23]This can be obtained using the result of Section 5.3.5 relating epistemic entrenchment to sphere semantics.

[24]One is sure to exist given our finite language assumption which ensures all abductions are expressible in the object language.

$K_\alpha^+ \subset K_\alpha^\oplus$ we know that an abduction beyond the trivial α was selected and that it will occur in $K_\alpha^\oplus$ but not in K_α^+. There may of course be many ways to (finitely) axiomatise $K_\alpha^\oplus \setminus K$ (or $K_\alpha^\oplus \setminus K_\alpha^+$) and here we may impose restrictions like those discussed in the previous chapter to obtain abductions which are, for instance, minimal relative to some syntactic restriction, or of some level of specificity etc.

Example 5.5.1 *Suppose we extend Example 5.3.2 (p. 126) slightly so that* $K = Cn(\{o \to l, c \to l, o \wedge ph \to e, co \to o\})$ *where* co *means that the cook is present in the restaurant and* ph *that I purchased a hamburger, at the restaurant. Now imagine the receipt of new information* e*, that I am eating a hamburger, to result in the following abductive expansion*

$$K_e^\oplus = Cn(K \cup \{o, ph, co, e\}).$$

A possible finite axiomatisation of $K_e^\oplus \setminus K$ *is* $o \wedge ph \wedge co \wedge e$.
However, if we specify that the abduction should be conjunctive minimal, then $co \wedge ph$ *will do because* $Cn(K \cup \{co \wedge ph\}) = Cn(K \cup \{o, ph, co, e\})$. *One must keep in mind however that it may not always be possible to find an abduction of the desired type.* □

An interesting question at this point is whether, due to the postulate of weak extensionality, we should identify the same abduction for two syntactically different epistemic inputs which are equivalent relative to the current epistemic state. Weak extensionality does not specify that the abductions should be the same because it is concerned with the nature of epistemic states. One could argue that they should be, through the Principle of Irrelevance of Syntax but we do not pursue the issue further here.

5.6 Restricting Possible Abductions

In this chapter our aim has been to furnish a general framework for abductive expansion. That is, to characterise expansion of an epistemic state by new information through all manner of abductions. As such, we have only considered the very basic definition of abduction (see Definition 4.1.2) and only added postulates that give a more well-behaved abductive process. It is possible however, to impose further restrictions on potential abductions. One could specify that they be minimal, non-trivial or of a certain specificity for instance. This would have the effect of further reducing the possible abductive expansion functions. This would manifest itself in the form of additional postulates and, more importantly, restrictions on the constructions. In fact, we have already seen some examples. If we specify that the abductions adopted should always be (logically) minimal, or least specific, then a full meet abductive expansion function results (i.e., AGM expansion). If, on the other hand, we specify that abductions should be maximally specific, then we have maxichoice abductive expansions (i.e., there is one world per internal "band" and a similar restriction results for abductive entrenchment). We can also envisage other interesting restrictions. Although we discounted abducibles at the outset (p. 102), it is possible to enforce such a restriction if one so desires.[25] That is, in order to introduce new information α, we must find an abduction β consisting entirely of abducibles and then take the deductive closure of K and β. In terms of abductive entrenchment we can see that, for a belief set K and new information α, we must guarantee a formula β composed entirely of abducible propositions (or predicates) such that, in the ordering, (a) $\alpha \to \neg\beta < \alpha \to \beta$ and (b) for any newly introduced belief γ, it must be a consequence of β (i.e., $\beta \leq \gamma$ via Lemma 5.3.14 (iii) for any $\gamma \in \mathcal{L}$ where $\alpha \to \neg\gamma < \alpha \to \gamma$). Other cases can be handled in a similar manner and restrictions can be translated into the various constructions. The important point to note however is that, if one wishes to restrict the notion of abduction, to capture a more intuitive notion of explanation for instance, then this translates into restrictions on the admissible abductive entrenchments, internal spheres etc. (i.e., essentially a restriction on the admissible abductive expansion functions). Of course the same epistemic state may be achieved even though alternative abductions were selected.

5.7 Summary and Discussion

This chapter has focussed on modelling the claim that it is more natural for agents to seek an explanation or reason for new information and incorporate that explanation together with the new information when modifying their epistemic state. We concentrated on the process of expansion for a number of reasons. Firstly, in adopting Levi's commensurability

[25]This can also be considered a restriction to Stickel's [123] predicate specific abductions.

thesis, we consider expansion and contraction to be the fundamental forms of belief change and that any revision can be achieved through a series of expansions and contractions. Moreover, when adding new information through an abductive process it makes more sense to consider expansion than revision because it is close in spirit to abduction. Determining an explanation by abduction must be done relative to some domain theory and the natural choice in belief revision is to identify this with the agent's epistemic state. If the new information is consistent with the epistemic state, then there is no need to resort to revision to effect the change. On the other hand, if the new information is inconsistent with the epistemic state, then no abduction is to be found and so there is no explanation to suggest itself for either expansion or revision. This situation is better handled through contraction to remove impediments to the abductive process and subsequently applying abductive expansion. We shall come back to this issue in Chapter 7. Abductive expansion is also an excellent way of capturing the idea that an agent is interested in acquiring new, error-free information in a manner dictated by the agent's degree of boldness or caution. The account of expansion offered by AGM results in the acquisition of little new information and is in fact a special case of abductive expansion. Moreover, it is the only AGM belief change operation where the Principal of Minimal Change can be interpreted with respect to set inclusion. In this chapter we have provided a normative account of an abductive expansion operation in the spirit of the AGM.

Our account began with a definition of the belief change operation in question. This was followed by rationality postulates including some supplementary postulates that represented a more well behaved abductive process. Three constructions, guided by those for AGM contraction and revision, were presented: selection functions over maximally consistent supersets of K implying α, systems of spheres centred within $[K]$, and abductive entrenchment orderings. It was shown how the Principle of Minimal Change could be interpreted with respect to the sphere modelling in a like manner to AGM contraction and revision. Moreover, the agent's degree of boldness or caution can be reflected in the sphere modelling or other constructions. It can also be seen that abductive expansion embodies an aspect of positive coherence (explanatory coherence) whereas contraction embodies that of negative coherence. In fact, the constructions detail which propositions cohere (positively or negatively). Note however, that one must be very careful with this interpretation. Coherentism (and foundationalism for that matter) is concerned with the nature of epistemic states and not their dynamics. However, belief sets, as models of epistemic states, are largely devoid of structure and constructions such as the sphere modelling or entrenchment can be viewed as adding structure to the belief set (albeit for the purpose of epistemic dynamics). This view, of using a system of spheres or entrenchment to add structure to belief sets, is adopted by some approaches to iterated belief change [81, 83].

The process of abductive expansion can also be applied to the problem of nonmonotonic reasoning. In particular, it can be used to give an epistemic interpretation to default reasoning. This clearly reflects the epistemic status of defaults. It in fact allows a more direct rendering of the default logic underlying THEORIST (without constraints) than that proposed by Makinson and Gärdenfors [72]. This is not altogether surprising given the similar motivation behind abductive expansion and THEORIST.

We presented a general framework for abductive expansion but further restrictions can be considered. These lead to restrictions on the admissible abductive expansion functions which manifest themselves as restrictions on the admissible internal systems of spheres or abductive entrenchments. One might consider restrictions in an attempt to capture a more intuitive notion of explanation than that afforded by abduction. Another topic considered was that, in abductive expansion, the abductive process has become internalised to a certain extent. It is possible however to identify abductions capable of effecting the resulting change especially if one restricts their attention to certain particular types of abduction.

Chapter 6

Abduction and Contraction

The scientific spirit requires a man to be at all times ready to dump his whole cartload of beliefs, the moment experience is against them.
Charles Sanders Peirce, [96] 1.55

The process of contraction is essentially concerned with the removal of a belief in which the agent no longer has confidence. Achieving this may involve the removal of further beliefs that, together with others, logically entail the formula to be removed. That is, certain reasons for holding a belief may need to be removed in order to retract the belief. Failure to do so would mean that the belief which the agent wishes to expunge is maintained.

In the introduction we noted that abduction could be used to identify culprits to remove in contraction. We need to determine the contracted state Γ and set of culprits Ψ for α, the information to be removed. If we can find a Γ such that $\Gamma \not\vdash \alpha$ (and $\Gamma \subseteq K$) then we can identify $Cn(\Gamma)$ with the contracted epistemic state K^-_α and Ψ represents those beliefs ("culprits") to be removed from K (i.e. $K = K^-_\alpha \cup \Psi$ or, alternatively, $K^-_\alpha = Cn(K \setminus \Psi)$). Abduction can be used to identify these culprits provided $K \neq K_\perp$. A method based on a related idea is investigated by Aravindan and Dung [6] although they concentrate on epistemic states represented as belief bases rather than belief sets. They perform abduction with respect to an immutable set $K' \subset K$ which is assumed not to entail α. Abduction of α with respect to K' determines which beliefs to remove from the belief base K. We review this work later in the chapter. We do not concentrate on this aspect of abduction here because existing techniques are capable of determining Γ and Ψ. This simply represents

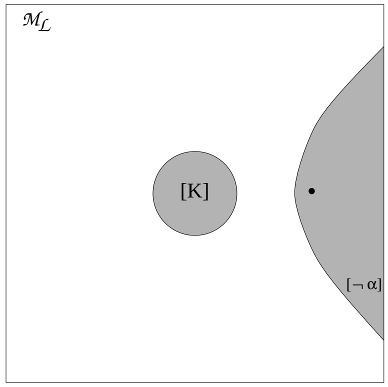


Figure 6.1: Sphere semantics for full meet AGM contraction.

a way of interpreting them abductively when K is consistent. Instead, we concentrate on an alternative form of contraction to that proposed by the AGM before investigating some other issues.

In the previous chapter we noted that abductive entrenchment used for abductive expansion and epistemic entrenchment used for contraction complement each other. Expansion and contraction would be expected to complement each other in a number of ways. In fact, we could even go so far as to expect them to be duals of one another. After all, we are taking them to be our two primitive belief change operations through the commensurability thesis. However, considering our account of abductive expansion and AGM contraction, there is at least one area in which this duality does not appear to be borne out. Casting our minds back to maxichoice abductive expansion § 5.3.1 we noted that the resulting expansion consists of a consistent complete theory containing α (i.e, a single α-world, $m \in [\alpha]$ see also Figure 5.4, p. 120). This corresponds to an abduction which is maximally specific, (i.e., hypothesising as much as possible). One would expect the dual operation, full meet contraction, to result in giving up as much as possible — whenever something is to be given up — that is, all but the logical truths to which an agent is always committed (i.e., $K_\alpha^- = Cn(\emptyset)$). For AGM contraction this is not the case in general (see Figure 6.1 — we consider, of course, the principal case where $\alpha \in K$ and $\not\vdash \alpha$). However, there is a form of contraction suggested by Levi [65] that possesses this property and we shall take a look at it now (particularly its semantics). In this sense at least it would appear to be an appropriate dual to abductive expansion.

6.1 Levi Contraction

Levi's [65] contraction provides an interesting alternative to AGM contraction. In particular, because it does not satisfy the contentious property of Recovery $(K^-_\alpha)^+_\alpha = K$ (where $\alpha \in K$). Hansson and Olsson [46] have attempted a formalisation of this contraction. We concentrate on this formal version here.

The motivation behind Levi's proposal stems from the construction of an AGM contraction function based on maximal consistent subsets of K failing to imply α (i.e., $K\bot\alpha$). Elements of $K\bot\alpha$ have the following property [2]

$Cn(K' \cup \{\neg\alpha\})$ is a complete consistent theory for any $K' \in K\bot\alpha$.

These are not the only belief sets having this property and Levi claims we should consider the wider class of belief sets satisfying this property. He refers to such belief sets as *saturatable contractions of K by removing α*.

Definition 6.1.1 *[46]*
A set K' is a saturatable contraction of K by removing α *if and only if*
(i) $K' = Cn(K')$
(ii) $K' \subseteq K$
(iii) $Cn(K' \cup \{\neg\alpha\})$ *is maximally consistent in $\mathcal{L}$.*

We let $K\perp\!\!\!\perp\alpha$ denote the set of saturatable contractions of K by removing α. Of course, all maximal consistent subsets of K failing to imply α are also saturatable contractions of K by removing α although the converse does not hold in general.

Lemma 6.1.1 *[46] Let K be a logically closed subset of $\mathcal{L}$, and let $\alpha \in \mathcal{L}$. Then $K\bot\alpha \subseteq K\perp\!\!\!\perp\alpha$.*

Hansson and Olsson then apply a selection function γ to $K\perp\!\!\!\perp\alpha$ and investigate constructions in the spirit of AGM's partial meet, full meet and maxichoice contractions. Before reviewing some of these results we briefly examine some postulates that such a contraction function would be expected to satisfy. In the following we denote the Levi contraction of a belief set K by epistemic input α as $K^\ominus_\alpha$.

6.1.1 Postulates

It turns out that the various constructions will satisfy a variety of the following postulates.

(K$^{\ominus}$1)	For any sentence α and any belief set K,	
	$K^{\ominus}_{\alpha}$ is a belief set	(closure)
(K$^{\ominus}$2)	$K^{\ominus}_{\alpha} \subseteq K$	(inclusion)
(K$^{\ominus}$3)	If $\nvdash \alpha$, then $\alpha \notin K^{\ominus}_{\alpha}$	(success)
(K$^{\ominus}$4)	If $\vdash \alpha \leftrightarrow \beta$, then $K^{\ominus}_{\alpha} = K^{\ominus}_{\beta}$	(extensionality)
(K$^{\ominus}$5)	If $\alpha \notin K$, then $K^{\ominus}_{\alpha} = K$	(vacuity)
(K$^{\ominus}$6)	If $\vdash \alpha$, then $K^{\ominus}_{\alpha} = K$	(failure)
(K$^{\ominus}$7)	$K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha \wedge \beta}$	(intersection)
(K$^{\ominus}$8)	If $\alpha \notin K^{\ominus}_{\alpha \wedge \beta}$, then $K^{\ominus}_{\alpha \wedge \beta} \subseteq K^{\ominus}_{\alpha}$	(conjunction)

All these postulates except that of failure (K$^{\ominus}$6) should be familiar from our survey of AGM postulates for contraction over K (§ 2.2.1 p. 28). The failure postulate says that there is no contraction of logical truths; the agent remains in its initial epistemic state. This postulate also holds of AGM contraction being a consequence of postulates (K^{-}1) — (K^{-}6). Note also that the postulate of recovery (K^{-}5), which is satisfied by AGM contraction, is missing. This is one of the main attractions of this type of contraction. The first six postulates can be thought of as the basic postulates for Levi contraction over K while postulates (K$^{\ominus}$7) and (K$^{\ominus}$8) assume a supplementary role again.

6.1.2 Results

We briefly survey some results by Hansson and Olsson [46] before adding some of our own. In analogy to the AGM framework, Hansson and Olsson apply maxichoice, full meet and partial meet selection functions γ setting the relevant contraction function $K^{\ominus}_{\alpha} = \bigcap \gamma(K \perp\!\!\!\perp \alpha)$ whenever $\alpha \in K$ and $K^{\ominus}_{\alpha} = K$ otherwise. One of their main results is that partial meet Levi contraction satisfies the basic postulates for Levi contraction over K.

Theorem 6.1.2 *[46] Let K be a belief set. $\ominus$ is a partial meet Levi-contraction operator over K if and only if it satisfies postulates* ($\mathrm{K}^{\ominus}1$)—($\mathrm{K}^{\ominus}6$).

In the case of full meet Levi contraction they derive the following result which was our original motivation.

Theorem 6.1.3 *[46] If $\ominus$ is a full meet Levi-contraction operator for K, then $K_\alpha^\ominus = Cn(\emptyset)$ for all non-tautological $\alpha \in K$.*

Based on Levi's argument, Hansson and Olsson introduce a measure of informational value $\mathcal{V}$ on belief sets satisfying the following weak monotonicity condition.[1]

$$\text{If } K \subset H, \text{ then } \mathcal{V}(K) \preceq \mathcal{V}(H)$$

This is used to define a selection function in the manner of (Def γ) (see § 2.2.2 p. 34).

$$\gamma(K \perp\!\!\!\perp \alpha) = \{K' \in K \perp\!\!\!\perp \alpha : \mathcal{V}(K'') \preceq \mathcal{V}(K') \text{ for all } K'' \in K \perp\!\!\!\perp \alpha\}$$

Levi advocates a probabilistic measure of informational value but Hansson and Olsson consider a real-valued measure $\mathcal{V}$. Using a selection function γ defined in this way leads to a *value-based Levi-contraction function*.[2] Value-based contraction functions satisfy the supplementary postulates ($\mathrm{K}^{\ominus}7$) and ($\mathrm{K}^{\ominus}8$).

Theorem 6.1.4 *[46] Value-based Levi-contraction satisfies postulates* ($\mathrm{K}^{\ominus}7$) *and* ($\mathrm{K}^{\ominus}8$).

Another interesting result concerning saturatable contractions is the following result.

Lemma 6.1.5 *[46]* $K \perp\!\!\!\perp \alpha \wedge \beta \subseteq K \perp\!\!\!\perp \alpha \cup K \perp\!\!\!\perp \beta$

A similar result holds for maximal subsets of K but interestingly, other results regarding the interaction of $K \perp \alpha$, $K \perp \beta$ and $K \perp \alpha \wedge \beta$ (or $K \perp \alpha \vee \beta$ for that matter) do not carry over to saturatable contractions.

[1] They also consider a strong monotonicity condition

$$\text{If } K \subset H, \text{ then } \mathcal{V}(K) \prec \mathcal{V}(H).$$

It turns out that constructions based on $K \perp \alpha$ and $K \perp\!\!\!\perp \alpha$ coincide if this measure is imposed.

[2] In the AGM, a transitively relational selection function γ can be defined using a real-valued measure $\mathcal{V}$.

Hansson and Olsson do not provide a completeness result for value-based contractions. Such a result would say whether a Levi-contraction function $\ominus$ satisfying postulates ($K^{\ominus}1$) — ($K^{\ominus}8$) is a value-based contraction function. Our impression is that this will not hold in general. We do not consider this further here for saturatable contractions but turn our attention to the more intuitively appealing sphere modelling. The sphere modelling will allow us a much better comparison with AGM contraction (through its sphere construction) and is arguably better motivated than saturatable contractions. Before doing so, however, we make one final remark about saturatable contractions.

It turns out that we can consider the saturatable set $K \perp\!\!\!\perp \alpha$ to be partitioned into lattices relative to set inclusion ($\subseteq$). This gives a nice way of viewing the elements of $K \perp\!\!\!\perp \alpha$ and the internal structure of $K \perp\!\!\!\perp \alpha$ when a measure of informational value satisfying weak monotonicity is imposed. We begin by introducing some notation that will be helpful in clarifying the results that follow.

Definition 6.1.2 *Let K be a belief set and $\alpha \in K$ non-tautological. Define the $\Delta_{K,\alpha}$-restriction of $K \perp\!\!\!\perp \alpha$ for a maximally consistent set Δ in $\mathcal{L}$ containing $\neg\alpha$ to be the set $K \perp\!\!\!\perp \alpha \mid \Delta = \{K' \in K \perp\!\!\!\perp \alpha : K' \subseteq \Delta\}$.*

This idea of $\Delta_{K,\alpha}$-restriction will be used to partition $K \perp\!\!\!\perp \alpha$.

Lemma 6.1.6 [3] *Let K be a belief set and $\alpha \in K$ non-tautological. The $\Delta_{K,\alpha}$-restriction sets for all maximally consistent sets Δ in $\mathcal{L}$ containing $\neg\alpha$ partition $K \perp\!\!\!\perp \alpha$.*

The following result establishes connections between the elements of each $\Delta_{K,\alpha}$-restricted partitions with respect to set inclusion.

Lemma 6.1.7 *Let K be a belief set, $\alpha \in K$ non-tautological and Δ a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. If $K', K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$, then:*

(*i*) *$K' \cap K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$ and moreover $K' \cap K''$ is the greatest lower bound of $\{K', K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.*

[3] The proofs to this and the following results in this chapter are to be found in Appendix C.

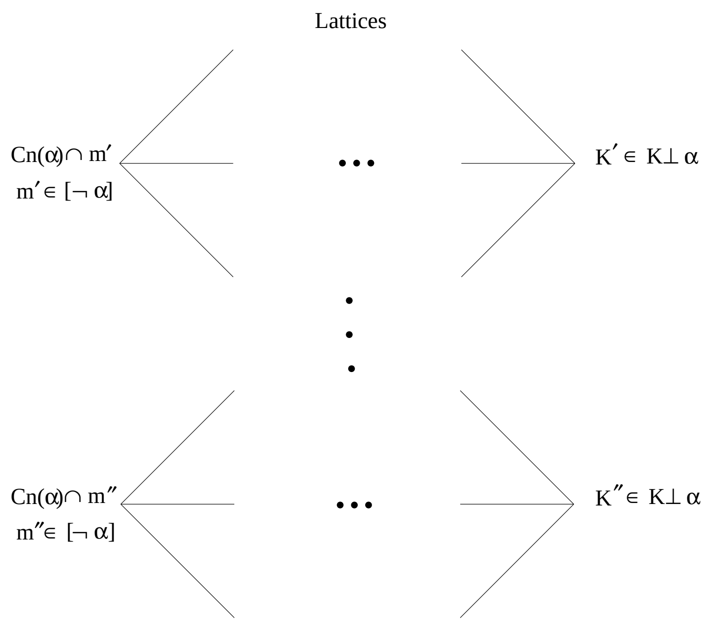


Figure 6.2: Elements of $K \perp\!\!\!\perp \alpha$ partitioned into lattices relative to set inclusion.

(*ii*) *$Cn(K' \cup K'') \in K \perp\!\!\!\perp \alpha \mid \Delta$ and moreover $Cn(K' \cup K'')$ is the least upper bound of $\{K',\ K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.*

The preceding results allow us to prove the following important theorem.

Theorem 6.1.8 *Let K be a belief set, $\alpha \in K$ non-tautological and Δ a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. Then $K \perp\!\!\!\perp \alpha \mid \Delta$ is a lattice relative to $\subseteq$.*

This result establishes that the elements of a $\Delta_{K,\alpha}$-restricted partition form a lattice relative to set inclusion. Together with Lemma 6.1.6 it shows that $K \perp\!\!\!\perp \alpha$ can be thought of as a set of lattices relative to set inclusion. Incidentally, the maximal element of each lattice will of course be an element of $K \perp \alpha$ and there will be as many lattices as elements of $K \perp \alpha$. The structure of $K \perp\!\!\!\perp \alpha$ is illustrated in Figure 6.2 (maximal elements — those in $K \perp \alpha$ — are displayed to the right). One problem with value-based contractions as they currently stand is that the following property does not hold

$$\text{If } \gamma_1(K \perp\!\!\!\perp \alpha) \neq \gamma_2(K \perp\!\!\!\perp \alpha), \text{ then } \bigcap \gamma_1(K \perp\!\!\!\perp \alpha) \neq \bigcap \gamma_2(K \perp\!\!\!\perp \alpha).$$

This suggests that more restrictions may need to be placed on value-based Levi-contraction functions in order to obtain completeness. We now take a look at sphere semantics for

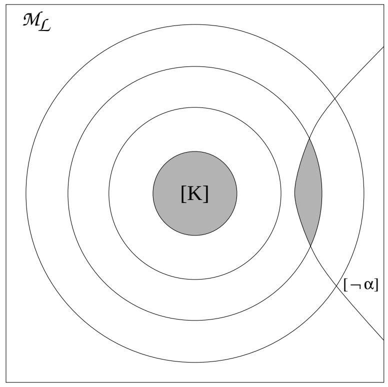


Figure 6.3: Sphere semantics for belief contraction showing $[K^-_\alpha]$ shaded.

Levi-contraction.

6.1.3 Sphere Semantics

A sphere semantics is not provided by Hansson and Olsson but we investigate it here. We require some of the notions set out in § 2.2.2 surveying Grove's sphere modelling for the AGM framework. We do not repeat them here but note that, in the case of AGM contraction, the possible worlds consistent with K^-_α are obtained by adding the closest $\neg\alpha$-worlds to the worlds consistent with K (i.e. $[K^-_\alpha] = [K] \cup f_{\mathcal{S}}(\neg\alpha)$ given a system of spheres $\mathcal{S}$ centred on $[K]$). This is illustrated in Figure 6.3 (we have reproduced Figure 2.4 for convenience).

In this section we shall again consider a system of spheres centred on $[K]$. There is no need to consider any internal spheres since they only play a part in abductive expansion and will have no role in contraction. Recall also that $c_{\mathcal{S}}(\alpha)$ denotes the smallest sphere in the system of spheres $\mathcal{S}$ intersecting $[\alpha]$ (but not the intersection itself). Let us first consider elements of $K \bot \alpha$ and $K \perp\!\!\!\perp \alpha$ in terms of a possible worlds picture in order to gain a clearer insight into the two types of constructions. We consider only the principal case where $\alpha \in K$, the other being trivial. In the case of $K \bot \alpha$ each element is essentially the result of a maxichoice selection function which results from adding exactly one $\neg\alpha$-world to $[K]$. This is illustrated in Figure 6.4. This contrasts with $K \perp\!\!\!\perp \alpha$ where we require $Cn(K' \cup \{\neg\alpha\})$ to be a consistent complete theory for each $K' \subseteq K$ (i.e. $[K] \subseteq [K']$). The worlds consistent with $K' \in K \perp\!\!\!\perp \alpha$ are any superset of K-worlds with the proviso that

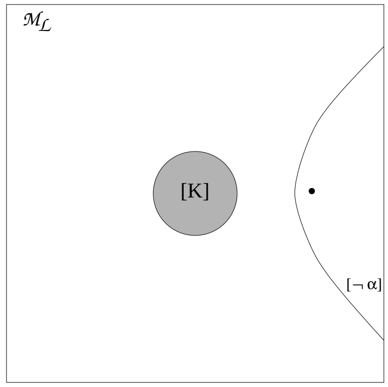


Figure 6.4: Representation of $K' \in K \perp \alpha$. $[K'] = [K] \cup m$ for some $m \in [\neg\alpha]$.

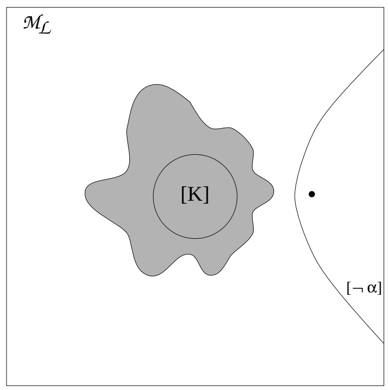


Figure 6.5: $[K']$ such that $K' \in K \perp\!\!\perp \alpha$.

one and only one of them is a $\neg\alpha$-world[4]. We illustrate this in Figure 6.5.

Now, when a selection function is applied to $K \perp \alpha$ or $K \perp\!\!\perp \alpha$ the elements are intersected to obtain the contracted belief set. In possible world terms this corresponds to taking the union of all worlds consistent with the selected elements (i.e., $[\gamma(K \perp\!\!\perp \alpha)] = \bigcup\{[K'] \mid K' \in \gamma(K \perp\!\!\perp \alpha)\}$ and similarly for $K \perp \alpha$).[5] The difference in possible world terms, between the two constructions is now clear. In AGM contraction only $\neg\alpha$-worlds are added to those consistent with K (as evidenced by Figure 6.3) while in Levi contraction not only $\neg\alpha$-worlds but also some α-worlds, previously inconsistent with K, may be added to the worlds consistent with K.

[4]Recall, by Lemma 2.2.14 (p. 35), that $Cn(K' \cup \{\neg\alpha\})$ can be rendered $[K'] \cap [\neg\alpha]$. Therefore, the requirement that $Cn(K' \cup \{\neg\alpha\})$ be a complete consistent theory translates to the stipulation that $[K'] \cap [\neg\alpha]$ be a single world.

[5]We are considering a partial meet selection function γ here.

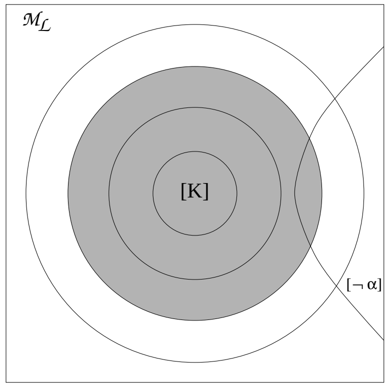


Figure 6.6: Sphere semantics for Levi contraction — Proposal 1

It turns out, however that more than one semantics is consistent with postulates ($K^{\ominus}1$) — ($K^{\ominus}8$).

Sphere Semantics — Proposal 1

In order to guarantee success we need to ensure that there are some $\neg\alpha$-worlds in the contracted epistemic state. Taking into account the Principle of Minimal Change which is to be interpreted with respect to the semantics we should at least consider those in the innermost sphere containing $\neg\alpha$-worlds (i.e., $c_{\mathcal{S}}(\neg\alpha)$). However, in this sphere there are also some α-worlds inconsistent with K. One possibility then is to take all the worlds inside the sphere $c_{\mathcal{S}}(\neg\alpha)$ as depicted in Figure 6.6.

The rationale behind this choice is that the agent has already determined a preference over worlds (inconsistent with K). The agent must include the "best" $\neg\alpha$-worlds but may have already stated a preference for certain α-worlds also. The agent does not prefer the $\neg\alpha$-worlds over these (closer) α-worlds just because it is giving up belief in α. Its preferences are determined prior to belief change taking place and there is no reason to change them in light of the new information. The new information only determines the amount of change required in order to suspend belief in it.

With any system of spheres $\mathcal{S}$ centred on $[K]$ we associate a function $g_{\mathcal{S}} : \mathcal{L} \to 2^{\mathcal{M}_{\mathcal{L}}}$

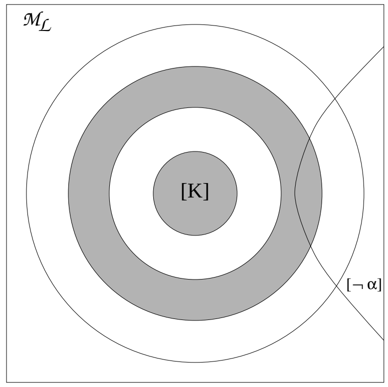


Figure 6.7: Sphere semantics for Levi contraction — Proposal 2

selecting the best worlds.

$$\text{(Def } g_\mathcal{S}\text{)} \qquad g_\mathcal{S}(\alpha) = \begin{cases} c_\mathcal{S}(\neg\alpha) & \text{whenever } [\alpha] \neq \mathcal{M}_\mathcal{L} \\ \text{[K]} & \text{otherwise} \end{cases}$$

That is, it selects all worlds in the closest sphere intersecting worlds consistent with $[\neg\alpha]$ whenever α is non-tautological and returns all K-worlds otherwise. The following theorem shows that this semantics is consistent with the postulates ($\text{K}^\ominus 1$) — ($\text{K}^\ominus 8$) for Levi-contraction over K.

Theorem 6.1.9 *Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_\mathcal{L}$ centred on $[K]$ for some belief set $K \in \mathcal{K}$. If, for any $\alpha \in \mathcal{L}$, we define $K^\ominus_\alpha$ to be $th(g_\mathcal{S}(\alpha))$, then postulates* ($\text{K}^\ominus 1$) — ($\text{K}^\ominus 8$) *are satisfied.*

Sphere Semantics — Proposal 2

An alternative idea is not to take all worlds in $c_\mathcal{S}(\neg\alpha)$ but only those solely in $c_\mathcal{S}(\neg\alpha)$ and no other sphere smaller than $c_\mathcal{S}(\neg\alpha)$ except $[K]$. This scenario is shown in Figure 6.7.

This can be seen as giving up fewer beliefs to effect contraction than the previous scheme, with respect to the same system of spheres. It can be motivated in the following way. Each band — worlds in a sphere but not inside any smaller sphere - can be seen as consisting of equally preferred worlds or in Lewis' terms, worlds of equal similarity. The agent must include at least the closest $\neg\alpha$-worlds. Since they are equally similar to other worlds in the

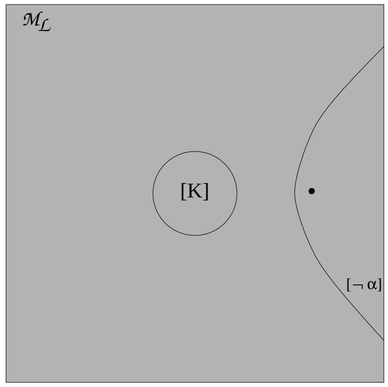


Figure 6.8: Sphere semantics for full meet Levi contraction under both proposals.

same band, the agent is unable to discriminate between them. Unable to discriminate, the agent adds all the worlds in the band to those consistent with K.

The function selecting the best worlds $g'_{\mathcal{S}} : \mathcal{L} \to 2^{\mathcal{M}_{\mathcal{L}}}$ which is associated with any system of spheres $\mathcal{L}$ centred on $[K]$ is defined as follows

$$(\text{Def } g'_{\mathcal{S}}) \quad g'_{\mathcal{S}}(\alpha) = \begin{cases} [K] \cup \{m \in c_{\mathcal{S}}(\neg\alpha) \mid m \notin U \text{ for any } U \in \mathcal{S} \\ \qquad\qquad\qquad\qquad \text{such that } U \subset c_{\mathcal{S}}(\neg\alpha)\} & \text{whenever } [\alpha] \neq \mathcal{M}_{\mathcal{L}} \\ [\mathrm{K}] & \text{otherwise} \end{cases}$$

It selects those worlds in $[K]$ and the innermost band intersecting $[\neg\alpha]$ whenever α is non-tautological and all the K-worlds otherwise. This proposal is also consistent with postulates ($\mathrm{K}^{\ominus}1$) — ($\mathrm{K}^{\ominus}8$) for Levi-contraction over K.

Theorem 6.1.10 *Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred on $[K]$ for some belief set $K \in \mathcal{K}$. If, for any $\alpha \in \mathcal{L}$, we define $K^{\ominus}_{\alpha}$ to be $th(g'_{\mathcal{S}}(\alpha))$, then postulates* ($\mathrm{K}^{\ominus}1$) — ($\mathrm{K}^{\ominus}8$) *are satisfied.*

It is now clear why both proposals give full meet contraction to be $Cn(\emptyset)$ and we illustrate the situation in Figure 6.8 (considering, of course, the situation where $\alpha \in K$ and $\nvdash \alpha$).

The sphere semantics for AGM contraction also satisfies these postulates, as one might expect, because AGM contraction also satisfies postulates ($\mathrm{K}^{\ominus}1$) — ($\mathrm{K}^{\ominus}8$). The only difference being that AGM contraction also satisfies the postulate of recovery. Postulate ($\mathrm{K}^{\ominus}6$) as we have noted earlier is a consequence of postulates ($\mathrm{K}^{-}1$) — ($\mathrm{K}^{-}6$) for AGM contraction over K.

Completeness

We now have three different sphere modellings consistent with postulates ($K^{\ominus}1$) — ($K^{\ominus}8$). The question we would now like to ask is whether any one modelling completely characterises these postulates, and, if not, what postulates need to be added to obtain this "completeness". We can easily rule out AGM contraction because it requires the postulate of recovery which is not satisfied by these postulates and, moreover, it does not exhibit the full meet property we desire, unlike the other modellings. It also turns out that neither of the other two modellings exactly characterise ($K^{\ominus}1$) — ($K^{\ominus}8$) either. The following property appears to be satisfied by both modellings but does not follow from the postulates:

$$\text{If } \alpha \notin K^{\ominus}_{\beta} \text{ and } \beta \notin K^{\ominus}_{\alpha}, \text{ then } K^{\ominus}_{\alpha} = K^{\ominus}_{\beta}.$$ [6]

This means that further postulates need to be added in order to obtain a complete characterisation in either case. It turns out, however, that such a "completeness" result is attainable for the first proposal through the addition of the following two postulates.[7]

($K^{\ominus}9$) If $\alpha \notin K^{\ominus}_{\beta}$, then $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$

($K^{\ominus}10$) If $\not\vdash \alpha$ and $\alpha \in K^{\ominus}_{\beta}$, then $K^{\ominus}_{\alpha} \subseteq K^{\ominus}_{\beta}$

In postulate ($K^{\ominus}9$), if α was not originally a belief ($\alpha \notin K$), then $K^{\ominus}_{\alpha} = K$ by postulate ($K^{\ominus}5$) and the result $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$ would follow trivially by postulate ($K^{\ominus}2$). In the principal case where $\alpha \in K$, postulate ($K^{\ominus}9$) states that, if α were to be removed when removing β from K, then removing α from K can be achieved by removing no more than was required to remove β (removing less is a possibility). That is, no more effort is required to remove α than β. Postulate ($K^{\ominus}10$) states that, if it is possible to remove a belief α when removing β from K but it is retained instead, then more would need to be done (i.e., more beliefs would

[6] One way to easily see that this postulate does not follow from ($K^{\ominus}1$) — ($K^{\ominus}8$) is that if it were to follow it would be satisfied by AGM contraction. However, it is easy to find an example where this condition is not satisfied by sphere semantics for AGM contraction.

[7] Since the initial version of this dissertation, Hans Rott has communicated to me that he has an equivalent axiomatisation achieved through the replacement of postulate ($K^{\ominus}7$) by the stronger postulate

($K^{\ominus}7a$) If $\not\vdash \alpha$, then $K^{\ominus}_{\alpha} \subseteq K^{\ominus}_{\alpha\wedge\beta}$,

intended for an epistemic entrenchment construction. Hansson [44] contains a proof of the equivalence of the sphere modelling of proposal 2 and Rott's epistemic entrenchment modelling. Moreover, Hans Rott has since shown me that postulates ($K^{\ominus}1$), ($K^{\ominus}3$) and ($K^{\ominus}9$) imply postulate ($K^{\ominus}10$). I am indebted to Hans Rott for sharing his insights and for many interesting and thought provoking discussions on this topic.

have to be removed) in order to remove α from K than was done to remove β. In light of our rationality criteria (p. 24) we might view these as follows. Postulate (K$^{\ominus}$9), in the principal case, says that if α is no more epistemically important than β, then fewer beliefs need to be given up in order to remove α. Postulate (K$^{\ominus}$10), on the other hand, says that if α is more epistemically important than β, then more needs to be given up to remove α.

It can be quickly seen that postulate (K$^{\ominus}$9) implies postulate (K$^{\ominus}$8) in the presence of postulates (K$^{\ominus}$1) — (K$^{\ominus}$6) by substituting $\alpha \wedge \beta$ for β in postulate (K$^{\ominus}$9). Postulate (K$^{\ominus}$9) also implies postulate (K$^{\ominus}$7).

Observation 6.1.11 *Postulate* (K$^{\ominus}$9) *implies postulate* (K$^{\ominus}$7) *in the presence of postulates* (K$^{\ominus}$1) — (K$^{\ominus}$6).

Therefore, we could in fact deal simply with postulates (K$^{\ominus}$1) — (K$^{\ominus}$6), (K$^{\ominus}$9) and (K$^{\ominus}$10). Another interesting consequence of these two postulates (in the presence of the basic postulates for Levi-contraction over K) is the following condition.

(5.3) Either $K^{\ominus}_{\alpha} \subseteq K^{\ominus}_{\beta}$ or $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$

This condition should be quite evident from the sphere modelling of proposal 1. These newly introduced postulates are certainly consistent with this proposal.

Lemma 6.1.12 *Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred on $[K]$ for some belief set $K \in \mathcal{K}$. If we define, for any $\alpha \in \mathcal{L}$, $K^{\ominus}_{\alpha}$ to be $th(g_{\mathcal{S}}(\alpha))$, the postulates* (K$^{\ominus}$9) *and* (K$^{\ominus}$10) *are satisfied.*

It is quite easy to find examples where the sphere modelling for AGM contraction and that of proposal 2 are not consistent with postulates (K$^{\ominus}$9) and (K$^{\ominus}$10).

The main result of this section shows that these postulates are sufficient to characterise the sphere modelling presented in proposal 1.

Theorem 6.1.13 *Let $\ominus : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ be any function satisfying postulates* (K$^{\ominus}$1) — (K$^{\ominus}$10). *Then for any belief set $K \in \mathcal{K}$ there is a system of spheres on $\mathcal{M}_{\mathcal{L}}$, say $\mathcal{S}$, centred on $[K]$ and satisfying $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha))$ for any $\alpha \in \mathcal{L}$.*

This result is important because, initially, our two sphere modellings were motivated by our desire to capture value-based Levi-contractions. We have achieved a characterisation of proposal 1 through the addition of two postulates. In fact, with respect to the same system of spheres $\mathcal{S}$ centred on $[K]$, it is easily seen that AGM contraction represents the smallest change (with respect to set inclusion) while proposal 1 represents the greatest change. Proposal 2, for which we do not have a complete characterisation, is intermediate between the two. Proposal 1 exhibits the dual behaviour in terms of full meet contraction that we initially sought and now we have a set of postulates that exactly characterise this particular proposal.

6.2 Recovery via Abduction[8]

The recovery property of AGM is arguably the most contentious of the AGM postulates for contraction over K (see [41, 71]). It states that, for any belief $\alpha \in K$, if one were to contract K by α and then (AGM) expand the resulting belief set by α, one would retrieve K.

$$K = (K^-_\alpha)^+_\alpha \text{ for any } \alpha \in K$$

It is a consequence of the recovery postulate (K^-5) together with postulates (K^-1) — (K^-4) for contraction over K and the postulates for AGM expansion over K.

We have just seen, however, a type of contraction for which the recovery property is not satisfied. Consider the first semantics we proposed for the Levi-contraction of a belief set K by α (refer back to Figure 6.6). Now, in order to abductively expand the resulting belief set $K^\ominus_\alpha$ we require a system of spheres. This is in a sense the problem of iterated belief change [38, 62, 81, 84, 114, 132] and is beyond the scope of this dissertation. However, suppose that as much of the previous sphere structure is maintained as possible[9] — so that the spheres in the shaded portion of Figure 6.6 become internal spheres (i.e., those in $[K^\ominus_\alpha]$). One need only apply abductive expansion to retrieve the original epistemic state. We illustrate this in Figure 6.9. If we were to consider the alternative sphere semantics

[8]Some of the ideas in this section have appeared in [91].

[9]The Principle of Minimal Change in another incarnation!

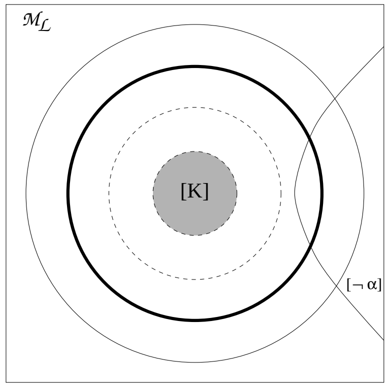


Figure 6.9: Using abductive expansion to recover from Levi contraction Proposal 1.

proposed it is easy to see that the same situation holds except there are only two internal spheres composed of the shaded portions of Figure 6.7 and the unshaded part forming the usual (external) system of spheres.

The contention then is that abduction, through abductive expansion, can be utilised to obtain the recovery property for (value-based) Levi-contraction. This claim, however, relies on a number of assumptions. Firstly there is the issue of iterated change and the fact that as much of the original sphere structure is maintained as possible. In doing so we are restricting which abductive expansion belief change operation is to be applied next, in the following sense. For a particular K, a particular belief change function, be it $\oplus$, $\ominus$, *, or whatever, can be modelled by a particular system of spheres and vice versa. By specifying the nature of the system of spheres we are restricting the agent's choice of belief change function. This is usually not considered much of a drawback in the literature on iterated belief change. A more problematic concern is that, in order for our proposal to work, we must assume that there is initially only one internal sphere — $[K]$ itself. However, it does appear that abduction can be of much use in obtaining recovery if one were to desire it. One use of recovery is suggested in the following problem. Normally, we think of the belief revision process as taking some epistemic state K and some epistemic input α to a new state K'_α (refer to Figure 1.1). Consider, however, the situation where the resulting epistemic state K'_α and epistemic input α are already known. Is it possible to determine the initial epistemic state K or at least something about it? Given the contracted belief set K^-_α or $K^\ominus_\alpha$ and epistemic input α we can go backwards, as it were, and say something about K (see

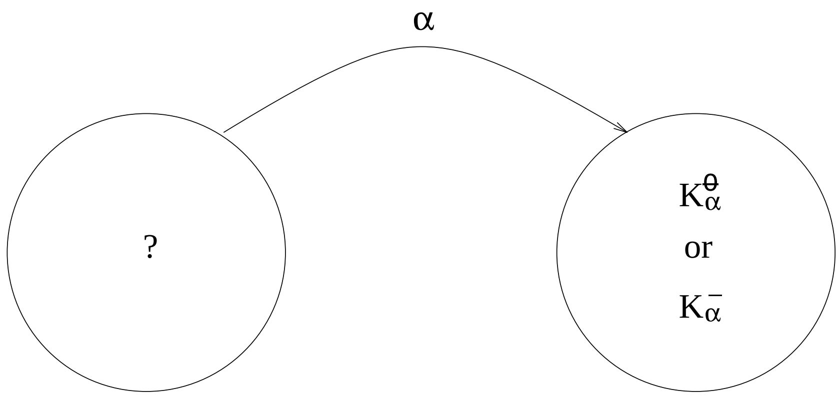


Figure 6.10: A use for recovery in belief change.

Figure 6.10). Taking a different perspective, the original epistemic state can be obtained through recovery via abductive expansion; identifying $K^{\ominus}_{\alpha}$ (or K^{-}_{α}) as our domain theory and α with the new information. It is our claim that abduction can be used to determine the missing elements and view the epistemic state K as the restoration of these missing elements. We of course assume that α occurred in the original epistemic state which is reasonable under the given circumstances.

Given the discussion above we can see that, in the case of contraction, we can use the recovery property as a means of achieving this purpose.

6.3 Related Work

An interesting method for performing contraction in a belief base setting using abduction has been proposed by Aravindan and Dung [6]. As in the approach to computing database updates by Kakas and Mancarella [57] (see § 1.1.3) they divide the belief base into two parts: an immutable theory KB_I and an updatable theory KB_U.[10] The immutable theory is supposed to represent those beliefs which the agent does not want to change over time ("the laws of science" is given as example). They then provide an algorithm for contracting a sentence α from belief (or knowledge) base $KB = KB_I \cup KB_U$. The basic idea is that all

[10]Despite KB_I and KB_U being referred to as theories, they are considered to be finite sets of sentences from the object language $\mathcal{L}$.

kernel minimal abductions for α with respect to the immutable theory KB_I are calculated.[11] Then all elements of KB_U containing a kernel minimal abduction for α with respect to KB_I are deleted. This results in the (kernel restricted) contraction of KB by α. Aravindan and Dung note that the division of the belief base into immutable and updatable parts is purely for practical rather than philosophical purposes. The division of the epistemic state in this way adds more structure to the notion of epistemic state and, as we have noted previously, would need to have a clear epistemic motivation (beyond practical concerns). It is not entirely clear what this motivation would be. To posit that an agent has a select part of its epistemic state which is not open to change seems contrary to the idea of belief change. In fact, this notion reeks of foundationalism as, of course, does their use of abducibles. Note also that the approach of Kakas and Mancarella [57] to database updates is capable of handling deletion of information by translation of the database to an alternative form to deal with negation.

6.4 Summary and Discussion

In light of the commensurability thesis, expansion and contraction can be viewed as two sides of the same coin. The expansion operation is responsible for assimilating new information in a coherent way while contraction is responsible for deleting information (in a coherent way). In either case, abduction can play an important role. In the one, to identify beliefs to add to the current epistemic state and, in the other, to determine "culprits" to remove (and thus those that remain in the new epistemic state). In this chapter we have investigated another aspect of the duality between expansion and contraction.

It was noted that maxichoice abductive expansion and full meet AGM contraction do not exhibit a certain dual behaviour that one might expect. Maxichoice abductive expansion, where possible, expands into a consistent complete theory — expansion by a maximally specific abduction. We might expect then, that full meet AGM contraction would, where possible, remove all beliefs except the logical truths — contraction by a maximally specific

[11] Abductions are sets of abducibles. An abduction is kernel minimal with respect to a set Γ if and only if there is a set $\Gamma' \subseteq \Gamma$ such that the abduction is minimal (in terms of set inclusion) with respect to Γ' but not with respect to any proper subset of Γ'.

abduction. However, this is not always the case when contraction is possible. Levi [65] has proposed a form of contraction which does possess the desired property. Hansson and Olsson [46] have subjected this form of contraction to a more formal analysis which we considered here. The idea is based on a construction involving selection functions applied to saturatable contractions of an epistemic state K by removing new information α. The set of saturatable contractions are a superset of the maximally consistent subsets of K implying α. An important feature of this form of contraction is that it does not, in general, satisfy the contentious recovery postulate satisfied by AGM contraction (of course, every contraction function satisfying the basic postulates for AGM contraction over K is also a Levi-contraction function over K). However, no completeness result is provided by Hansson and Olsson for value-based Levi-contractions where a real-valued measure satisfying weak monotonicity is used to define a selection mechanism.

We noticed that set inclusion can be used to partition the set of saturatable contractions into lattices, the greatest elements of which are maximally consistent subsets of K failing to imply α. This gives a clearer insight into the nature of saturatable contractions of K by removing α especially in contrast with maximally consistent subsets of K failing to imply α. We then switched our attention to sphere semantics for value-based Levi-contractions due to their potential for providing a clear insight into the underlying process and for effecting a comparison with AGM contraction. Two modellings were presented that are consistent with the postulates for value-based Levi-contraction over K. In the first, the ordering provided by the spheres is taken to be very important and all worlds at least as close to $[K]$ as the closest $\neg\alpha$-worlds are included in the contracted epistemic state. In the second, each "band" is taken to consist of worlds which are indistinguishable, hence all worlds in the same "band" as close as the closest $\neg\alpha$-worlds are included in the contracted epistemic state. With respect to the same system of spheres, then, we can see that AGM contraction represents a smaller change than either of these alternatives. This observation would be supported by our observation above regarding the lattice-like structure of saturatable contractions. In the case of the first proposal we also supplied two extra postulates and obtained a complete characterisation of this semantics.

One use for abduction in this setting is to furnish the opportunity to regain recovery through abductive expansion applied to (value-based) Levi-contraction. However, to do this some assumptions are needed regarding the nature of the internal system of spheres after contraction (i.e., the nature of the abductive expansion function applied). If as much structure as possible is retained after contraction, then the resulting abductive expansion can guarantee recovery. One practical use of this idea is to allow one to determine the nature of a certain epistemic state given its contracted form and the information that was removed from it.

Chapter 7

Abductive Revision

I have been hovering for some time between the exquisite sense of the luxurious and a love for philosophy — were I calculated for the former I should be glad — but as I am not I shall turn all my soul to the latter.
John Keats, *to John Taylor, April 24, 1818*

We mentioned earlier that we have adopted Levi's commensurability thesis which takes the operations of expansion and contraction to be basic and states that any revision can be achieved through a sequence of expansions and contractions. However, although we are placing more emphasis on expansion and contraction, this does not mean that revision is uninteresting. In the AGM, revision can also be achieved through expansion and contraction but still attracts a lot of attention. The difference is though, that AGM expansion is a very simplistic operation whereas the expansion operation developed here, abductive expansion, is much more powerful.

The obvious way to construct an abductive revision operator is to apply the following variant of the Levi identity

$$K^{\otimes}_{\alpha} = (K^{\ominus}_{\neg\alpha})^{\oplus}_{\alpha}$$

Here, $K^{\otimes}_{\alpha}$ represents the abductive revision of epistemic state K by new information α. The operation $\ominus$ represents some form of belief contraction — AGM contraction or Levi-contraction for instance — and $\oplus$ an abductive expansion operator. It can be thought of as functioning in the following manner. In order to revise epistemic state K by new information

α we must first remove any impediment to the explanatory or abductive process hence we contract by the negation of the new information ($K^{\ominus}_{\neg\alpha}$). We then find an explanation for the new information with respect to this contracted epistemic state using abductive expansion ($(K^{\ominus}_{\alpha})^{\oplus}_{\alpha}$). It can be seen that, in contracting by $\neg\alpha$, certain explanations will become admissible while others will not because they are inconsistent with $K^{\ominus}_{\neg\alpha}$ and α. We shall now continue our look at abductive revision through the sphere modelling with its intuitive appeal.

Before going on to the sphere modelling we make a brief digression on whether we should adopt AGM contraction or Levi-contraction. Levi-contraction would appear to be the obvious choice since, in the sense of the previous chapter, it appears to be a more appropriate dual of abductive expansion. It turns out however, that, for the purpose of revision via the Levi identity, there is very little difference between the two. Let us first consider the Levi identity using AGM expansion. Because both AGM contraction and Levi-contraction are withdrawal functions (i.e., satisfy postulates (K^{-}1) — (K^{-}4) and (K^{-}6) for AGM contraction over K — see Makinson [71] for details).[1] Therefore, we have the following observation.

Observation 7.0.1 *Let $K \in \mathcal{K}$ be a belief set. If $\ominus$ is a Levi-contraction function over K satisfying postulates (K$^{\ominus}$1) — (K$^{\ominus}$6), then there is a revision equivalent AGM contraction function $-$ over K satisfying postulates (K^{-}1) — (K^{-}6) and vice versa.*

By revision equivalent, we mean that $(K^{-}_{\neg\alpha})^{+}_{\alpha} = (K^{\ominus}_{\neg\alpha})^{+}_{\alpha}$ (i.e., equivalent under the Levi identity) where $+$ is AGM expansion. The result is a straightforward consequence of an observation by Makinson [71] regarding withdrawal functions. This fact is also clearly evident from the sphere semantics for AGM contraction and that for Levi-contraction proposed in the previous chapter (either proposal will suffice) if the same system of spheres centred on $[K]$ is adopted. In AGM contraction of K by $\neg\alpha$, the closest α-worlds are added to the K-worlds. The subsequent AGM expansion by α simply results in the closest α-worlds being retained. In Levi-contraction of K by $\neg\alpha$, the closest α-worlds are added to the K-worlds along with any $\neg\alpha$-worlds which are just as close or closer.[2] Again, the

[1] In the list of Levi-contraction postulates (p. 144) these have been renumbered (K$^{\ominus}$1) — (K$^{\ominus}$5).
[2] We consider only the first proposed sphere semantics for Levi-contraction, the other is similar.

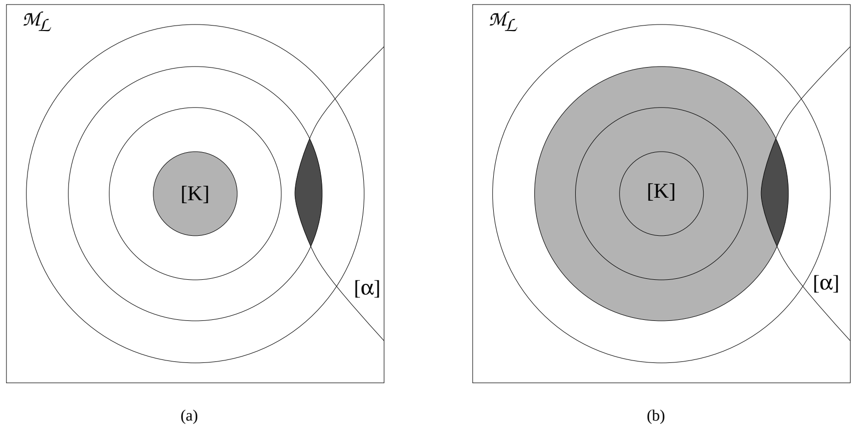


Figure 7.1: Sphere semantics for (a) AGM contraction showing $[K^-_{\neg\alpha}]$ shaded, and (b) Levi-contraction (proposal 1) showing $[K^\ominus_{\neg\alpha}]$ shaded. The part shaded darker corresponds to the resulting revision.

subsequent AGM expansion by α simply retains the closest α-worlds. This holds whether $\neg\alpha \in K$ or not. To better understand the situation, the two types of contraction are contrasted in Figure 7.1 showing the principal case where $\neg\alpha \in K$. Note also that, in Observation 7.0.1, right to left is obvious because, as we have noted previously, any AGM contraction function is also a Levi-contraction function.

With this in mind, let us return to abductive revision. We would like to use the suitably modified version of the Levi identity adopting abductive expansion in the place of AGM expansion. Referring to Figure 7.1 it is easy to see, as in AGM expansion, that we need to identify a subset of the α-worlds retained in either picture. Since, as above, there is going to be no real difference in whether we apply AGM contraction or Levi-contraction before abductive expansion in the Levi identity to obtain abductive revision, we shall use Levi-contraction (proposal 1). What we require is some internal structure in $[K^\ominus_{\neg\alpha}]$ in order to discriminate between the α-worlds. How the structure of $[K^\ominus_{\neg\alpha}]$ (or structure outside $[K^\ominus_{\neg\alpha}]$ for that matter) evolves is the problem of iterated revision which, as we have noted previously, is beyond the scope of this dissertation. The structure inside $[K^\ominus_{\neg\alpha}]$ could evolve in all manner of ways. This would allow a vast array of different abductive expansions and hence a large variety of abductive revision functions. This situation does not occur when using AGM expansion due to its simplistic nature (all α-worlds in $[K^-_{\neg\alpha}]$ or $[K^\ominus_{\neg\alpha}]$ are chosen). This is, of course, clearly evident from the nature of the Levi identity itself. An abductive revision function is obtained from a contraction function and an abductive

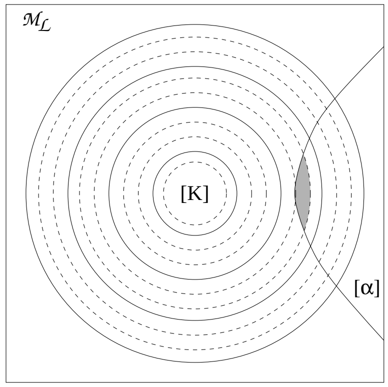


Figure 7.2: A more discerning system of spheres for abductive revision.

expansion function. There are a large number of possible combinations of contraction functions and abductive expansion functions to be had. However, there is only one AGM expansion function (viz., $Cn(K \cup \{\alpha\})$ by Theorem 2.2.1) and so, when AGM expansion is used in the Levi identity, each contraction function gives rise to one revision function.

One observation we can make is that the strategy adopted in Section 6.2, when discussing the use of abductive expansion for recovery, will not work here. The idea was to maintain as much of the present structure after contraction for the purpose of expansion. However, it is easily noted that, if this strategy is used, all the α-worlds are confined to a single sphere and we end up with the same result as if AGM expansion had been applied. An alternative idea is to impose a finer structure on a system of spheres which is then to be used during the abductive expansion stage but not for the initial contraction. This could take the form of a “finer”or more discerning system of spheres as illustrated in Figure 7.2. The solid lines are used for performing contraction and the dashed lines are then used to impose internal structure over $K^{\ominus}_{\neg\alpha}$, which can be used for the abductive expansion. The resulting abductive revision can be seen as given by the closest α-worlds relative to the dashed and solid spheres, taken together, as in the shaded section of Figure 7.2. In contrast, AGM revision is taken solely with respect to the solid spheres.

There are, however, two problems with this suggestion. The first concerns the origin of the spheres; where do they come from? However, the same question could be asked of the original system of spheres and the answer would be that they are determined, in some way, from the belief change operation to be applied (the abductive expansion operator in this case). The other problem is that, if such “finer” detail was present originally, why wasn’t use made of it all along? That is, the finer detail just gives another AGM revision function (this is evident if one considers the dashed and solid system of spheres in Figure 7.2 to be a single system of spheres centred on $[K]$). Abductive revision determined this way just gives the same class of revision functions as AGM revision. Interestingly enough, a similar statement regarding inference relations generated from epistemic entrenchment and those generated from expectations orderings is made by Gärdenfors and Makinson [34] (p. 223). The reason is due to the fact that this more discerning structure is the same as that induced by another AGM revision function. Therefore, if one wants to investigate the notion of abductive revision as an entity along with abductive expansion and contraction, this idea might not prove as fruitful as other methods for identifying internal structure. There is no doubt however, that the way this structure is to be determined is important. One can of course stick with abductive expansion and contraction as the ways of incorporating and removing information and be just as content. Nevertheless, it is still possible to consider postulates for abductive revision.

7.1 Postulates

Although, as we have seen, it is the method of iteration or, viewed differently, the interaction of different abductive expansion and contraction functions, that determines abductive revision, some postulates will hold regardless of what combination is employed. We briefly consider which rationality postulates an abductive revision function will necessarily satisfy no matter which abductive expansion and contraction functions are used to generate it via the Levi identity. It turns out that the following postulates are satisfied (we retain a numbering which facilitates comparison with the AGM revision postulates — see p. 29).

(K$^{\otimes}$1) For any sentence α and any belief set K,
$K^{\otimes}_{\alpha}$ is a belief set (closure)

(K$^{\otimes}$2) If $\nvdash \neg\alpha$, then $\alpha \in K^{\otimes}_{\alpha}$ (limited success)

(K$^{\otimes}$4) If $\neg\alpha \notin K$, then $K^{\oplus}_{\alpha} = K^{\otimes}_{\alpha}$ (preservation′)

(K$^{\otimes}$5) $K^{\otimes}_{\alpha} = K_{\perp}$ if and only if $\vdash \neg\alpha$ and $K = K_{\perp}$ (vacuity′)

($K^{\otimes}6$) If $\vdash \alpha \leftrightarrow \beta$, then $K^{\otimes}_{\alpha} = K^{\otimes}_{\beta}$ (extensionality)

The first postulate is the familiar postulate of closure. The second postulate is a conditional form of success which states that, as long as there is a possibility of finding an explanation for the new information, it will be included in the abductively revised epistemic state. The only time there will never be a possibility of finding an explanation through revision is if the new information is a logical falsehood. The reason for this conditionalised version goes back to our preference of maintaining consistency over success. Postulate ($K^{\otimes}4$) is just the postulate of preservation for AGM revision ($K^{*}4$) with a strengthened consequent. It says that, if it is possible to find an explanation for the new information with respect to the current epistemic state, then abductive revision reduces to abductive expansion. In other words, there is no need to perform contraction in this case since abduction is already permissible. The postulate of vacuity is also a slightly altered version of the corresponding postulate for AGM revision ($K^{*}5$) with an additional constraint on the right hand side. This additional constraint is also the result of our preference for consistency over success. The AGM versions of this postulate and that of success would hold if we preferred success to consistency and adopted postulates ($K^{\oplus}2'$) and ($K^{\oplus}4'$) for abductive expansion over K. The extensionality postulate is the familiar embodiment of the Principle of Irrelevance of Syntax which also holds for AGM revision. As the following result shows, these postulates are satisfied by an abductive revision function obtained via the Levi identity using any withdrawal function satisfying the postulate of failure ($K^{\ominus}6$). That is, by both AGM and Levi-contraction functions.

Theorem 7.1.1 *Let $K \in \mathcal{K}$ be a belief set. Let $\ominus$ be a contraction function satisfying postulates* ($K^{\ominus}1$) — ($K^{\ominus}6$) *over K and $\oplus$ an abductive expansion function satisfying postulates* ($K^{\oplus}1$) — ($K^{\oplus}6$) *over K. Then the abductive revision function $\otimes$ obtained through (Def $*$) satisfies postulates* ($K^{\otimes}1$), ($K^{\otimes}2$) *and* ($K^{\otimes}4$) — ($K^{\otimes}6$) *over K.*

This is quite significant because, apart from the slight modifications to postulates ($K^{*}2$) and ($K^{*}4$) due to our preference for maintaining consistency over success, the only basic postulate for AGM revision over K that is not satisfied is that of inclusion ($K^{*}3$) $K^{\otimes}_{\alpha} \subseteq K^{\oplus}_{\alpha}$.

By one half of postulate (K$^{\otimes}$4), a conditional version of it — if $\neg\alpha \notin K$, then $K^{\otimes}_{\alpha} \subseteq K^{\oplus}_{\alpha}$ — is satisfied. The reason that the unrestricted version is not satisfied is basically due to the fact that the postulate of monotonicity (K^{+}5) does not hold for abductive expansion (as one would hope). Therefore, when one applies an abductive expansion function $\oplus$ to an epistemic state K and to the epistemic state $K^{\ominus}_{\neg\alpha}$ (as one does in determining the result of $K^{\otimes}_{\alpha}$ using the Levi identity) explanations of different specificity may be chosen, thus it is not possible to determine the relationship between $K^{\oplus}_{\alpha}$ and $K^{\otimes}_{\alpha}$ without further information. Of course, we are assuming that the same $\oplus$ is applied to the epistemic state K on the one hand and to the epistemic state $K^{\ominus}_{\neg\alpha}$ using the Levi identity (to determine $K^{\otimes}_{\alpha}$) on the other. The fact that one cannot say anything about the relationship between $K^{\oplus}_{\alpha}$ and $H^{\oplus}_{\alpha}$ for two different epistemic states K and H also means that the supplementary postulates (K*7) and (K*8) for AGM revision over K are not satisfied in general.

7.2 Summary and Discussion

The commensurability thesis leads us to place less emphasis on revision than expansion and contraction. There are however still some interesting observations to be made regarding abductive revision. The abductive revision operators we discussed were constructed via the Levi identity $K^{\otimes}_{\alpha} = (K^{\ominus}_{\neg\alpha})^{\oplus}_{\alpha}$ which is an expression of the commensurability thesis. The main problem with this approach stems from the fact that there are many possible contraction operators (be they AGM or Levi) and many possible abductive expansion operators. In contrast, the AGM permits only one possible expansion operator — corresponding to a special type of abductive expansion at one extreme (i.e., set inclusion minimal change) — and therefore, there is, in a sense, a one to one relationship between AGM contraction and AGM revision operators. In abductive revision this will be a many to many relationship. This problem can also be identified with that of iterated revision where one essentially identifies a more complex structure with epistemic states (often systems of spheres or entrenchment) and attempts to give recipes as to how this structure is modified when belief change takes place. An alternative idea might be to fix contraction to one extreme; maxichoice contraction (the set inclusion minimal change) or full meet contraction, for instance. In that way, there would be, in a sense, a one to one correspondence between abductive expansion functions and abductive revision functions through the Levi identity. Even without these considerations though, it is interesting to note that many of the basic postulates for AGM revision over K are still satisfied by abductive revision over K.

There are, of course, other ways of obtaining revision functions besides using the Levi identity. Hansson [43], for instance "reverses" the Levi identity.[3] That is, one first expands by α and then performs contraction[4] to obtain consistency. This too makes a lot of sense in terms of abductive revision. One first finds the explanation that is desired and then determines what to remove to achieve consistency. There is a problem, however, if one uses abduction as the notion of explanation because, as we have seen, we will never find an abduction that will lead to inconsistency. This might suggest dropping the consistency requirement for abduction at the risk of allowing many more potential abductions. Another idea would be to perform abduction with respect to a subset of the current beliefs. However, as we have seen with Aravindan and Dung's [6] method for contraction using abduction (§ 6.3), this would require a modification of our notion of epistemic state as purely a belief set (or belief base) and, moreover, one needs to have a clear epistemic justification for demarcating this select set of beliefs which is not immediately evident (especially, if one wants to avoid a foundationalist approach).

[3] This idea, like the Levi identity, is still consistent with Levi's commensurability thesis (see Levi [65] p. 170 n. 23 and pp. 179 – 180 n. 8).

[4] When expansion leads to an inconsistent epistemic state, Levi [65] refers to this contraction as *coerced contraction* as its use is necessary to regain consistency.

Chapter 8

Conclusion

When we have found all the mysteries and lost all the meaning, we will be alone, on an empty shore.
Tom Stoppard, *Arcadia* II:7[1]

8.1 Summary

In this dissertation we have looked at the role that abduction can play within the belief change process. Abduction, in a logical sense, can be viewed as expressing necessary conditions for the process of explanation. Its utility has also been demonstrated in many areas of artificial intelligence.

We began with a logical look at abduction and various important types of abduction. In particular we studied notions of minimality, triviality and specificity. We also compared abduction with a particular type of induction, popular in artificial intelligence, known as inverse resolution. This comparison was based on pragmatic grounds where it turns out that abductions are calculated as conjunctions of literals while inverse resolution returns disjunctions of literals (i.e., clauses). The fact that inverse resolution is consistent with the definition of abduction lends credence to Harman's claim that (enumerative) induction is a special case of abduction (inference to the best explanation).

Having looked at some logical aspects of abduction we proceeded to investigate our first

[1] Stoppard [124] p. 94.

abductive belief change operator, abductive expansion. The idea behind this operator was to use the logical notion of abduction to determine an explanation or justification of the epistemic input with respect to the current epistemic state and then incorporate this explanation into the current epistemic state together with any deductive consequences. This operator can be considered a basic modelling of an inquiring agent who is interested in acquiring as much new, error-free information as possible. Such a modelling is particularly effective when the agent's situation makes it difficult for it to receive new epistemic inputs. Using the AGM framework as a guide, we investigated three constructive modellings for abductive expansion: selection functions over maximally consistent supersets of K implying α, a Grove-like sphere semantics, and an abductive entrenchment ordering. These modellings help place abductive expansion in clear perspective with regard to the AGM framework. In fact, they extend the current AGM framework in a way that permits greater flexibility and scope. It was also pointed out how the agent's degree of boldness or caution could be reflected in the sphere modelling. Moreover, we indicated how the abductive entrenchment ordering embodied the notion of positive coherence while epistemic entrenchment, when used for contraction, embodies the notion of negative coherence.

We proceeded to show how abductive expansion, through abductive entrenchments, could be used for the purpose of nonmonotonic inference, in particular, default reasoning. In fact, abductive expansion provides a more accurate modelling of the default logic underlying the THEORIST system than a proposal by Gärdenfors and Makinson involving full meet revision. It clearly distinguishes, in an epistemic sense, between facts and defaults as does THEORIST.

The belief change process is primarily concerned with the nature of epistemic states. As such, the process of abduction in abductive expansion becomes internalised and the best explanation is, in a sense, lost. We demonstrated how such explanations could be retrieved by comparing the initial and final epistemic states. In this dissertation we have been interested in the normative aspects of a general framework for abductive expansion and abductive belief change. We indicated, however, that further restrictions could be placed on the notion of abduction which would manifest themselves as restrictions on the admissible constructions — that is, on the admissible internal systems of spheres or abductive entrenchments etc. — and therefore, as restrictions on the admissible abductive expansion functions.

Shifting our attention to contraction, we noted that it can already be viewed as functioning in an abductive manner. Abduction could be used to single out elements to remove (and therefore to retain) in order to effect the removal of a belief from an epistemic state. However, we focussed our attention on another aspect of contraction. Noting that maxichoice abductive expansion and full meet AGM contraction do not exhibit a certain dual behaviour that one might expect, we concentrated on an alternative form of contraction known as Levi-contraction which exhibits the desired duality. Levi-contraction can be viewed as a generalisation of AGM contraction whose motivation stems from a construction based on saturatable contractions of K by removing α. After pointing out that the set of saturatable contractions can be viewed as being partitioned into a set of lattices relative to set inclusion, we concentrated on the more intuitively appealing sphere semantics for Levi-contraction. We demonstrated two "competing" sphere modellings for Levi-contraction. Relative to the same system of spheres centred on $[K]$ it turns out that AGM contraction represents a smaller change (in terms of set inclusion) than either of these modellings. Through the addition of two further postulates, we obtained a completeness result for one of the two modellings — that representing the greater change. We also showed how, under certain assumptions, abductive expansion could be used to attain recovery for Levi contraction based on the two sphere modellings presented. This can also be achieved for AGM contraction, however, in this case one need only use full meet abductive expansion (i.e., AGM expansion). It is possible to think of this as the agent attempting to determine what it might have believed prior to the removal of some information, given its current epistemic state.

Adopting Levi's commensurability thesis which states that any revision can be achieved through a series of expansions and contractions, we placed more emphasis on (abductive) expansion and (Levi) contraction. In fact, given the greater flexibility that can be achieved through abductive expansion and contraction (be it AGM or Levi), one could argue that the AGM framework would benefit greatly from a more general expansion operator such as the one suggested here. It would then have a nice symmetry: abductive expansion for acquiring new (error-free) information and contraction for relinquishing information. However, this does not mean that revision becomes totally devoid of interest. We suggested that one way of obtaining an abductive revision operator is to adapt the Levi identity to use some form of contraction and abductive expansion instead of AGM expansion. In fact, it turns out that little difference may occur if one uses AGM contraction or Levi-contraction. Determining the exact properties of an abductive revision operator turns out to rely on the problems facing iterated revision. However, there are still some properties an abductive revision operator, constructed using our variant of the Levi identity, must satisfy; these turn out, with one exception, to be essentially the basic postulates for AGM revision. Therefore, such a revision operator is clearly more general than the AGM revision operator.

In conclusion, adopting a logical notion of abduction has allowed us to investigate the utility of this mode of inference in a general setting reminiscent of the AGM framework and thus gain a clear insight into the belief change process. This investigation has indicated that the process of abduction can play a very important role in belief change and possesses great potential.

8.2 Future Work

There are many interesting avenues for further research arising from the work described in this dissertation. Some have been alluded to in the text. We shall briefly outline some of the more interesting of them.

One of the more glaring areas requiring investigation is that of iterated belief change — particularly as it effects, say, the development of the internal system of spheres or abductive entrenchment ordering. We have seen that this would allow a much more detailed investigation of abductive revision operators than that provided here. One problem is that it is not at first evident where a new internal system of spheres or abductive entrenchment would come from. One idea in this regard has been suggested by Nayak *et al.* [82] and involves abandoning the current idea of selecting the "best" α-worlds inside $[K]$ and embracing a strategy which rejects the "worst" α-worlds inside $[K]$. Such a strategy could even be useful in the context of AGM contraction and revision. There is also the problem of how a normal system of spheres centred on $[K]$ or epistemic entrenchment evolves after abductive expansion. An obvious suggestion would be to have the epistemic input more deeply entrenched than an abduction. This would have the effect that, if some error or suspicion happened to arise, the agent could reject the abduction in favour of the epistemic input.

We have investigated the relationship between abductive expansion and default reasoning in the style of THEORIST but we have left open the connection with nonmonotonic inference in general. The postulates could in fact be translated into properties of a nonmonotonic consequence relation $|\!\sim$ in the way Makinson and Gärdenfors [72] have done for AGM revision ($\alpha |\!\sim \beta$ iff $\beta \in K^*_\alpha$). These properties could then be analysed in the manner of Lehmann *et al.* [61]. Such a translation has actually been done in Pagnucco *et al.* [94] ($\alpha |\!\sim \beta$ iff $\beta \in K^\oplus_\alpha$) but no analysis of the resulting properties was performed. One very interesting fact did arise however. For AGM revision, conditions like $\alpha \in K$ are translated as $\top |\!\sim \alpha$ since $K^*_\top = K$. In the case of abductive expansion however, $K^\oplus_\top \neq K$ in general (recall our discussion of the property of vacuity) but $K^\oplus_\perp = K$, so these conditions were translated as $\perp |\!\sim \alpha$ instead. Having established links with nonmonotonic inference, one could take advantage of the range of constructions for abductive expansion. For instance, one could supply a sphere semantics for nonmonotonic inference. Such a modelling was discussed briefly for default reasoning.

Another possibility is the adoption of belief bases rather than belief sets as the modelling of epistemic states. Apart from the obvious pragmatic advantages, it would also mean that one could begin to consider syntactic factors in determining an abduction. This would have the effect of weakening our interpretation of the Principle of Irrelevance of Syntax. A further advantage is that the abduction adopted would be much easier to determine since it would, presumably, explicitly occur in the abductively expanded belief base and therefore $K^\oplus_\alpha \setminus K$ would be a more manageable set. One problem with such an approach is that it may be difficult to determine exactly which beliefs should occur in the abductively expanded belief base. It would be difficult to justify the addition of only the abduction and the epistemic input.

An alternative proposal in a similar vein is to adopt a more complex notion of epistemic state than the belief sets embraced here. This could have the advantage of providing extra information to guide abduction. However, one must be wary that so doing can lead away from a coherentist approach, towards a foundational approach. We have already seen the proposals of Kakas and Mancarella [57] (one of the original motivations for the theoretical work here) and Aravindan and Dung [6] heading in this direction. In this regard too, the work of Ghose [37] must be noted. Ghose adopts a more complex representation of the agent's epistemic state than that used by the AGM and in this dissertation. Epistemic states are modelled as default theories [104, 102] in the manner of THEORIST. This allows a clear distinction to be made between facts and defaults as advocated in Section 5.4 although both facts and defaults are included in the agent's epistemic state. Moreover, it allows for the explicit recording of contractions through the use of (logic programming) constraints. We allow only facts to be included in an epistemic state while defaults reside outside it and are determined by the selection mechanism (i.e., abductive expansion operator) imposed. Given the abductive interpretation that underlies THEORIST (see § 3.3.1) it is arguable that Ghose's approach also uses a deliberative procedure to perform belief change. Another recent work in this regard and much closer to the aims of our work is that of Lobo and Uzcátegui [69]. Using the Katsuno and Mendelzon [58] style of presenting belief change operators (which inherently assumes a finitary language) they present postulates for abductive belief change. However, their work is based on the adoption of abducibles which we reject here as being against the coherentist spirit of the AGM. They also concentrate more on revision (the theory of expansion not being as detailed as the one in this dissertation), contraction and update (see Katsuno and Mendelzon [59]) and thus do not adhere to Levi's commensurability thesis in the manner we do so here. A model-based construction, along the lines of Katsuno and Mendelzon [58, 59] is provided.

In discussing (value-based) Levi-contraction, we only provided a completeness result for one of our suggested sphere modellings. It would be interesting to determine the nature of the postulates that need to be added to obtain a completeness result for the second proposal. This is of particular interest because, as we have noted, the amount of change made by this proposal is, in general, intermediate between that made by AGM contraction and our first proposal.

In Chapter 7 we discussed how an abductive revision operator could be determined through the (modified) Levi identity. We also mentioned that an alternative idea is to “reverse” the Levi identity as in Hansson [43]. This is interesting because the reversed version does possess some intuitive appeal. We would be determining the explanation first and then deciding what would need to be done to incorporate it in a consistent manner. On the other hand, using the Levi identity, the result of removing information determines what explanations are admissible. One problem with the reversed Levi identity is that, for it to work, one would need to abandon the consistency requirement for abductions.

The use of methods of reasoning like abduction and induction have often been linked with the problems of scientific discovery and theory formation. Considering the general nature of the AGM framework and the operators discussed here, it would seem natural to suggest that they could be used for this purpose. One popular framework in artificial intelligence is the *Model Inference System* developed by Shapiro[118]. In this case, abductive expansion could be used to suggest new theories as information is acquired and contraction could take the place of the Contradiction Backtracking Algorithm whose job it is to identify false hypotheses.

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Appendix A

Proofs for Chapter 4

Note: This chapter contains the proofs for claims made in Chapter 4.

Observation 4.1.1 *If an abduction Ψ of a formula ϕ with respect to a domain theory Γ exists, then a finite abduction $\Psi' \subseteq \Psi$ (where Ψ' is understood to be finite) of ϕ with respect to Γ exists.*

Proof:

By definition of abduction, $\Gamma \cup \Psi \vdash \phi$. So, by compactness, there is some finite $\Psi' \subseteq \Psi$ such that $\Gamma \cup \Psi' \vdash \phi$. Furthermore, since $\Gamma \cup \Psi \nvdash \perp$ then $\Gamma \cup \Psi' \nvdash \perp$. ■

Lemma 4.1.2 *If a finite abduction Ψ of ϕ with respect to Γ exists, then it can be represented by a single formula ψ (i.e., $\Gamma \cup \{\psi\} \vdash \phi$, $\Gamma \cup \{\psi\} \nvdash \perp$).*

Proof:

We shall show the syntactic equivalence of Ψ and $\psi_1 \wedge \ldots \wedge \psi_n$ (i.e., $\psi = \psi_1 \wedge \ldots \wedge \psi_n$).

Suppose a finite abduction $\Psi = \{\psi_1, \ldots, \psi_n\}$ exists.
Proof by induction on size of the abduction Ψ (i.e., n).

(If $\{\psi_1, \ldots, \psi_n\} \vdash \delta$, then $\{\psi_1 \wedge \ldots \wedge \psi_n\} \vdash \delta$)

Base Case, $n = 1$: $\{\psi_1\} \vdash \delta \Rightarrow \{\psi_1\} \vdash \delta$.

Inductive Hypothesis, $n = k$: If $\{\psi_1, \ldots, \psi_k\} \vdash \delta$, then $\{\psi_1 \wedge \ldots \wedge \psi_k\} \vdash \delta$

Inductive Case, $n = k + 1$: $\{\psi_1, \ldots, \psi_{k+1}\} \vdash \delta$.

$\{\psi_1, \ldots, \psi_k\} \cup \{\psi_{k+1}\}\} \vdash \delta$.

$\{\psi_1, \ldots, \psi_k\} \vdash \psi_{k+1} \rightarrow \delta$ (Deduction).

$\{\psi_1 \wedge \ldots \wedge \psi_k\} \vdash \psi_{k+1} \rightarrow \delta$ (Induction Hypothesis).

$\vdash (\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow (\psi_{k+1} \rightarrow \delta)$ (Deduction).

$\vdash [(\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow (\psi_{k+1} \rightarrow \delta)] \leftrightarrow [(\psi_1 \wedge \ldots \wedge \psi_{k+1}) \rightarrow \delta]$

$\vdash (\psi_1 \wedge \ldots \wedge \psi_{k+1}) \rightarrow \delta$

$\{\psi_1 \wedge \ldots \wedge \psi_{k+1}\} \vdash \delta$ (Deduction).

(If $\{\psi_1 \wedge \ldots \wedge \psi_n\} \vdash \delta$, then $\{\psi_1, \ldots, \psi_n\} \vdash \delta$)

Base Case, $n = 1$: $\{\psi_1\} \vdash \delta \Rightarrow \{\psi_1\} \vdash \delta$.

Inductive Hypothesis, $n = k$: If $\{\psi_1 \wedge \ldots \wedge \psi_k\} \vdash \delta$, then $\{\psi_1, \ldots, \psi_k\} \vdash \delta$.

Inductive Case, $n = k + 1$: $\{\psi_1 \wedge \ldots \wedge \psi_{k+1}\} \vdash \delta$

$\vdash (\psi_1 \wedge \ldots \wedge \psi_{k+1}) \rightarrow \delta$ (Deduction).

$\vdash [(\psi_1 \wedge \ldots \wedge \psi_{k+1}) \rightarrow \delta] \leftrightarrow [(\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow (\psi_{k+1} \rightarrow \delta)]$

$\vdash (\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow (\psi_{k+1} \rightarrow \delta)$

$\{\psi_1 \wedge \ldots \wedge \psi_k\} \vdash (\psi_{k+1} \rightarrow \delta)$ (Deduction).

$\{\psi_1, \ldots, \psi_k\} \vdash \psi_{k+1} \rightarrow \delta$ (Induction Hypothesis).

$\{\psi_1, \ldots, \psi_k\} \cup \{\psi_{k+1}\} \vdash \delta$ $\{\psi_1, \ldots, \psi_k\} \cup \{\psi_{k+1}\} \vdash \delta$ (Deduction).

$\{\psi_1, \ldots, \psi_{k+1}\} \vdash \delta$. ■

Observation 4.2.1 *The "weakness" relation $\leq_{\Gamma,\phi}$ induces a partial ordering over the set of abductions of ϕ with respect to Γ.*

Proof:

Reflexivity: $\psi \vdash \psi$ for any $\psi \in \mathcal{L}$. So, if ψ is an abduction of ϕ with respect to Γ, then $\psi \leq_{\Gamma,\phi} \psi$.

Anti-Symmetry: Suppose $\psi \leq_{\Gamma,\phi} \psi'$ and $\psi' \leq_{\Gamma,\phi} \psi$. Then, $\psi \vdash \psi'$ and $\psi' \vdash \psi$ by definition. Therefore, $\psi \dashv\vdash \psi'$.

Transitivity: Suppose $\psi \leq_{\Gamma,\phi} \psi'$ and $\psi' \leq_{\Gamma,\phi} \psi''$. Then, $\psi' \vdash \psi$ and $\psi'' \vdash \psi'$. Therefore, $\psi'' \vdash \psi$. Hence, $\psi \leq_{\Gamma,\psi} \psi''$. ■

Observation 4.2.2 *If there is an expressible minimal abduction with respect to $\leq_{\Gamma,\phi}$ then it must be weaker than the new information ϕ.*

Proof:

Suppose there is an (expressible) minimal abduction ψ with respect to $\leq_{\Gamma,\phi}$ and that it is not weaker than ϕ. Consider the formula $\psi \vee \phi$. Now, since ψ and ϕ are abductions of ϕ with respect to Γ[1], then $\Gamma \cup \{\psi\} \vdash \phi$ and $\Gamma \cup \{\phi\} \vdash \phi$ so clearly then $\Gamma \cup \{\psi \vee \phi\} \vdash \phi$. Moreover, $\Gamma \cup \{\psi\} \not\vdash \bot$ and $\Gamma \cup \{\phi\} \not\vdash \bot$ so $\Gamma \cup \{\psi \vee \phi\} \not\vdash \bot$. So, $\psi \vee \phi$ is an abduction of ϕ with respect to Γ. Moreover, $\psi \vee \phi \leq_{\Gamma,\phi} \psi$. Contradiction. Hence, any minimal abduction under $\leq_{\Gamma,\phi}$ is weaker than ϕ. ■

Theorem 4.2.3 *For any abduction ψ of ϕ with respect to Γ weaker than ϕ and any $\delta \in \mathcal{L}$, $\Gamma \cup \{\psi\} \vdash \delta$ iff $\Gamma \cup \{\phi\} \vdash \delta$.*

Proof:

Let ψ be an abduction of ϕ with respect to Γ such that $\psi \leq_{\Gamma,\phi} \phi$.

[1]In the case where Γ and ϕ are inconsistent, $\Gamma \cup \{\phi\} \vdash \bot$, there can be no abductions whatsoever and the result is vacuously true.

(If)

Suppose $\Gamma \cup \{\phi\} \vdash \delta$. Therefore $\Gamma \vdash \phi \rightarrow \delta$ by (Deduction). $\Gamma \cup \{\psi\} \vdash \phi \rightarrow \delta$ (Monotonicity). $\Gamma \cup \{\psi\} \vdash \phi$ since ϕ is an abduction. $\Gamma \cup \{\psi\} \vdash \delta$ (Modus Ponens).

(Only If)

Suppose $\Gamma \cup \{\psi\} \vdash \delta$. Therefore $\Gamma \vdash \psi \rightarrow \delta$ by (Deduction). $\Gamma \cup \{\phi\} \vdash \psi \rightarrow \delta$ (Monotonicity). $\phi \vdash \psi$ since $\psi \leq_{\Gamma,\phi} \phi$. $\Gamma \cup \{\phi\} \vdash \psi$ (Monotonicity). $\Gamma \cup \{\phi\} \vdash \delta$ (Modus Ponens). ■

Observation 4.4.1 *The relative specificity relation $\preceq_{\Gamma,\phi}$ is a (partial) pre-order over the set of abductions of ϕ with respect to Γ.*

Proof:

Reflexivity: $\Gamma \cup \{\psi\} \vdash \psi$ by (Monotonicity) for any $\psi \in \mathcal{L}$. Therefore, if ψ is an abduction of ϕ with respect to Γ, then obviously $\psi \preceq_{\Gamma,\phi} \psi$.

Transitivity: Suppose $\psi \preceq_{\Gamma,\phi} \psi'$ and $\psi' \preceq_{\Gamma,\phi} \psi''$. Then $\Gamma \cup \{\psi'\} \vdash \psi$ and $\Gamma \cup \{\psi''\} \vdash \psi'$. By (Deduction) $\Gamma \vdash \psi' \rightarrow \psi$ and $\Gamma \vdash \psi'' \rightarrow \psi'$ so $\Gamma \vdash \psi'' \rightarrow \psi$. Therefore $\Gamma \cup \{\psi''\} \vdash \psi$ by (Deduction). Hence $\psi \preceq_{\Gamma,\phi} \psi''$. ■

Lemma 4.4.2 *If an abduction of ϕ with respect to Γ exists, then an abduction ψ of ϕ with respect to Γ is a relatively least specific abduction of ϕ with respect to Γ iff $\Gamma \vdash \phi \leftrightarrow \psi$.*

Proof:

Let an abduction of ϕ with respect to Γ exist. Then $\Gamma \nvdash \neg\phi$.

(If)

Let ψ be a relatively least specific abduction of ϕ with respect to Γ. That is, $\psi \preceq_{\Gamma,\phi} \psi'$ for any abduction ψ' of ϕ with respect to Γ. Now ϕ is an abduction of ϕ with respect to Γ. So $\psi \preceq_{\Gamma,\phi} \phi$. Therefore $\Gamma \cup \{\phi\}\psi$ and by (Deduction) $\Gamma \vdash \phi \rightarrow \psi$. Since ψ is an abduction of ϕ with respect to Γ we have by definition that $\Gamma \cup \{\psi\} \vdash \phi$ and by (Deduction) again $\Gamma \vdash \psi \rightarrow \phi$. Hence it follows that $\Gamma \vdash \phi \leftrightarrow \psi$.

(Only If)

Let $\Gamma \vdash \phi \leftrightarrow \psi$. Clearly then $\Gamma \vdash \psi \rightarrow \phi$ and by (Deduction) $\Gamma \cup \{\psi\} \vdash \phi$. Suppose for *reductio ad absurdum* that $\Gamma \cup \{\psi\} \vdash \bot$. Then $\Gamma \vdash \neg\psi$. However, $\Gamma \vdash \phi \rightarrow \psi$ so by (Contraposition) $\Gamma \vdash \neg\psi \rightarrow \neg\phi$ and by (Modus Ponens) $\Gamma \vdash \neg\phi$. Contradiction. Therefore, $\Gamma \cup \{\psi\} \nvdash \bot$ and so ψ is an abduction of ϕ with respect to Γ. Now consider any abduction ψ' of ϕ with respect to Γ. By definition $\Gamma \cup \{\psi'\} \vdash \phi$. Since $\Gamma \vdash \phi \leftrightarrow \psi$, by (Monotonicity) $\Gamma \cup \{\psi'\} \vdash \phi \leftrightarrow \psi$ and by (Modus Ponens) $\Gamma \cup \{\psi'\} \vdash \psi$. Therefore, $\psi \preceq_{\Gamma,\phi} \psi'$ for any abduction ψ' of ϕ with respect to Γ. Hence ψ is a least specific abduction. ■

Observation 4.6.1 *Any abduction, other than the new data, generated by the above procedure is a conjunction of literals.*

Proof:

The crucial part of the algorithm is step 4 which is the only point in the algorithm besides the initialisation where additions are made to the hypothesis set HS containing abductions. We see that a clause C which is not a tautology and not subsumed by any other clause in the working set WS is negated and added to HS. Since C is a clause and therefore a disjunction of literals, clearly its negation is a conjunction of literals. It is easy to see that the algorithm computes abductions through the soundness of the resolution rule and the contrapositive argument for computing abductions. ■

Theorem 4.6.2 *If Ψ is a conjunctive minimal abduction of Φ with respect to Γ then Ψ is in the hypothesis set HS.*

Proof:

(A similar result is given by Jackson [52] for his framework.) Suppose for *reductio ad absurdum* that there is some conjunctive minimal abduction Ψ of Φ with respect to Γ not in HS at the termination of the algorithm. So $\Gamma \cup \Psi \vdash \Phi$ and $\Gamma \cup \Psi' \not\vdash \Phi$ for any $\Psi' \subset \Psi$ however $\Psi \notin HS$. The result is given by the refutation completeness of propositional resolution [12] which ensures that every branch of $\neg\Psi$'s proof tree will eventually be generated (in step 4). This means that, at some point, Ψ will be determined (in fact, its negation initially) and added to HS. Contradiction. Hence any conjunctive minimal abduction will be added to HS in due course. ■

Observation 4.7.1 *Any abduction generated by the above procedure (i.e., in GS) is a disjunction of literals (i.e., a clause).*

Proof:

The elements of GS can be seen to be abductions through the analysis preceding the algorithm and due to the soundness of the resolution principle. Now consider any element of GS. Either it was the new data C or it is the result of negating an element of the construction set CS. In the former case, since the new data C is assumed to be a clause, the result follows automatically. In the latter case, due to the way clause D is selected at step 3, it is clear that after $m - 1$ resolution steps at step 5 the construction set CS will contain a set of singleton sets (i.e., a conjunction of literals). This can be seen as follows. Suppose $C = \{c_1, \ldots, c_m\}$. At step 3, without loss of generality, D is chosen to be $D = \{c_1, \ldots, c_i, l\}$ where $i < m$.

At step 4, C is negated giving $\neg C = \{\{\neg c_1\}, \ldots, \{\neg c_m\}\}$ and at step 5 repeated resolution with D gives $\{\{\neg l\}, \{\neg c_{i+1}\}, \ldots, \{\neg c_n\}\}$. When these are negated at step 6 we obtain (via de Morgan's laws) a clause. Hence, in either case we get a disjunction of literals. ■

Lemma 4.7.2 *Let the domain theory* Γ *consist of consistent and non-tautological clauses. If a clause* A *is in the generated set* GS*, then it satisfies the separability assumption with respect to the clause from the domain theory that was used to generate it.*

Proof:

Let clause A be in the generated set GS and assume it was generated from clause $D \in \Gamma$ and new data clause C. We need to show that $D \setminus \{l\} \cap A \setminus \{\neg l\} = \emptyset$ where l is the literal resolved upon. Suppose for *reductio ad absurdum* that A and D do not satisfy the separability assumption. So there is some literal k say, such that $k \in D$ and $k \in A$. Without loss of generality, suppose $D = \{k,\ d_1, \ldots, d_m\}$ and $A = \{k,\ a_1, \ldots, a_n\}$ for $n,\ m \geq 0$. Now A results from negating the result of step 5 which would have had to be $\{\{\neg k\},\ \{\neg a_1\}, \ldots, \{\neg a_n\}\}$. The literal $\neg k$ either comes from D in which case D is a tautology for $k \in D$ but D comes from Γ which does not contain tautologies. Otherwise $\{\neg k\} \in \neg C$. Therefore $k \in C$. However, if this were so, since $k \in D$ then $\neg k$ would be resolved away in step 5. Therefore no such k exists contradicting our original supposition. Hence $D \setminus \{l\} \cap A \setminus \{\neg l\} = \emptyset$ as required. ■

Theorem 4.7.3 *Let the domain theory* Γ *consist of consistent and non-tautological clauses. If a clause* A *is the result of an absorption or identification of the new data* C *together with a clause from the domain theory* Γ *then it will be in the generated set* GS*.*

Proof:

We verify that any clause resulting from the absorption and identification schema in Table 4.1 using a clause from the domain theory and the new data will be calculated by the algorithm.

Absorption:

> In this case the new data $C = \alpha \wedge \beta \rightarrow j$ and $D = \alpha \rightarrow k$ is in the domain theory. We need to show $\beta \wedge k \rightarrow j$ results. Without loss of generality, suppose $C = \{\neg a_1, \ldots, \neg a_m,\ \neg b_1, \ldots, \neg b_n,\ j\}$ and $D = \{\neg a_1, \ldots, \neg a_m,\ k\}$ (i.e., $\alpha = \neg a_1 \wedge \ldots \neg a_m$ and $\beta = \neg b_1 \wedge \ldots \neg b_n$). So we need to show $\{\neg b_1, \ldots, \neg b_m,\ \neg k,\ j\} \in GS$ eventually. Clearly $D \setminus \{k\} \subseteq C$ so would eventually be selected at step 3. Now $CS = \neg C \cup \{D\} =$ $\{\{a_1\}, \ldots \{a_m\},\ \{b_1\}, \ldots, \{b_n\},\ \{\neg j\},\ \{\neg a_1, \ldots, \neg a_m,\ k\}\}$ initially. After repeated resolution at step 5 we obtain $\{\{b_1\}, \ldots, \{b_n\},\ \{j\},\ \{k\}\}$ which negated gives $\{\neg b_1, \ldots, \neg b_n,\ j,\ \neg k\} \in GS.$ as desired.

Identification:

> The new data $C = \alpha \wedge \beta \rightarrow j$ and $D = \beta \wedge k \rightarrow j$ is in the domain theory. We need to show $\alpha \rightarrow k$ results. Without loss of generality, suppose $C = \{\neg a_1, \ldots, \neg a_m,\ \neg b_1, \ldots, \neg b_n,\ j\}$ and $D = \{\neg b_1, \ldots, \neg b_n,\ \neg k,\ j\}$ for $m,\ n \geq 0$ (i.e., $\alpha = \neg a_1 \wedge \ldots \neg a_m$ and $\beta = \neg b_1 \wedge \ldots \neg b_n$). So we need to show $\{\neg a_1, \ldots, \neg a_m,\ k\}$ results. Clearly $D \setminus \{\neg k\} \subseteq C$ so would eventually be selected at step 3. Now $CS = \neg C \cup \{D\} =$ $\{\{a_1\}, \ldots \{a_m\},\ \{b_1\}, \ldots, \{b_n\},\ \{\neg j\},\ \{\neg b_1, \ldots, \neg b_m,\ \neg k,\ j\}\}$ initially. After repeated resolution at step 5 we obtain $\{\{a_1\}, \ldots, \{a_m\},\ \{\neg k\}\}$ which negated gives $\{\neg a_1, \ldots, \neg a_m,\ k\}$ as required.

■

Theorem 4.7.4 *Let Γ_1 be a set of propositional Horn clauses over the language $\mathcal{L}(\Gamma_1)$, Γ_2 be the result of performing Inter-construction or Intra-construction on Γ_1, and l be the newly introduced literal (i.e., $l \in \mathcal{L}(\Gamma_2)$). If $l \notin \mathcal{L}(\Gamma_1)$, then, for any formula $\phi \in \mathcal{L}(\Gamma_1)$,*

$\Gamma_2 \vdash \phi$ *implies* $\Gamma_1 \vdash \phi$.

Proof:

Let Γ_1 be a set of Horn clauses over the language $\mathcal{L}(\Gamma_1)$, Γ_2 be the result of performing Inter-construction or Intra-construction on Γ_1 and l the newly introduced literal. We shall proceed as follows. We know that $\Gamma_1 \subseteq \Gamma_2$ by the way inverse resolution is defined here. So, supposing for *reductio ad absurdum*, that there is some formula $\phi \in \mathcal{L}(\Gamma_1)$ such that $\Gamma_2 \vdash \phi$ but $\Gamma_1 \not\vdash \phi$, then the only way this can occur is through interaction of the newly introduced clauses in Γ_2 (i.e., those formulae in $\Gamma_2 \setminus \Gamma_1$). Therefore, considering, in turn, inter-construction and intra-construction we show, using propositional resolution which is a sound inference procedure, that any clause resulting through the resolution of a newly introduced formula could also be obtained through resolution of the initial set of formulae. Due to the soundness of the resolution rule, this shows that our supposition above is not possible and hence $\Gamma_1 \vdash \phi$ as required.

We first consider inter-construction. In order for inter-construction to apply there must be Horn clauses $\alpha \wedge \beta \to j,\ \alpha \wedge \gamma \to k \in \Gamma_1$. This gives $\Gamma_2 = \Gamma_1 \cup \{\beta \wedge l \to j,\ \gamma \wedge l \to k,\ \alpha \to l\}$. In order for resolution to take place we only need consider three cases (remembering that $l \in \mathcal{L}(\Gamma_1))$.

Case (*i*) Either j or k appears in the antecedent of some Horn clause in Γ_1.

> That is, there is a clause $\delta \wedge j \to m \in \Gamma_1 \subseteq \Gamma_2$ or $\delta \wedge k \to m \in \Gamma_1 \subseteq \Gamma_2$ (note that l does not occur here since $l \notin \mathcal{L}(\Gamma_1)$). In the former case, the only resolution in $\Gamma_2 \setminus \Gamma_1$ is with $\beta \wedge l \to j \in \Gamma_2 \setminus \Gamma_1$ producing $\delta \wedge \beta \wedge l \to m$. However, this clause is not in $\mathcal{L}(\Gamma_1)$. The only possible way of obtaining a formula in $\mathcal{L}(\Gamma_1)$ is to resolve away l and this can only be done through resolution with $\alpha \to l$ which produces $\alpha \wedge \beta \wedge \delta \to m$. (It is very important to note here that the formula $\delta \wedge \beta \wedge l \to m$ may well resolve with formulae in Γ_1 but this will not result in a formula in $\mathcal{L}(\Gamma_1)$. Moreover, even though such resolutions could produce formulae in $\mathcal{L}(\Gamma_1)$ if resolved with $\alpha \to l$ such formulae could easily be reproduced by applying the same sequence of

resolutions to $\alpha \wedge \beta \wedge \delta \rightarrow m$. In short, such a strategy would make no difference so we shall not consider it further in the cases that follow.) Now $\delta \wedge j \rightarrow m \in \Gamma_1$ resolves with $\alpha \wedge \beta \rightarrow j \in \Gamma_1$ producing $\alpha \wedge \beta \wedge \delta \rightarrow m$ as desired. The latter case is similar.

Case (*ii*) Part of α occurs in the consequent of some Horn clause in Γ_1.

That is, there is a rule in $\delta \rightarrow a$ and $\alpha \equiv \alpha' \wedge a$. In $\Gamma_2 \setminus \Gamma_1$ this only resolves with $\alpha \rightarrow l$ (i.e., $\alpha' \wedge a \rightarrow l$) producing $\delta \wedge \alpha' \rightarrow l$. This formula is not in $\mathcal{L}(\Gamma_1)$ and the only possible way of obtaining a formula in $\mathcal{L}(\Gamma_1)$ is to resolve away l using one of two clauses in Γ_1: $\beta \wedge l \rightarrow j$ producing $\alpha' \wedge \beta \wedge \delta \rightarrow j$; or, $\gamma \wedge l \rightarrow k$ producing $\alpha' \wedge \gamma \wedge \delta \rightarrow k$. Now, in Γ_1, $\delta \rightarrow a$ resolves with $\alpha \wedge \beta \rightarrow j$ (i.e., $\alpha' \wedge a \wedge \beta \rightarrow j$) producing $\alpha' \wedge \beta \wedge \delta \rightarrow j$, and $\alpha \wedge \gamma \rightarrow k$ (i.e., $\alpha' \wedge a \wedge \gamma \rightarrow k$) producing $\alpha' \wedge \gamma \wedge \delta \rightarrow k$ as required.

Case (*iii*) Part of β or γ occurs in the consequent of some Horn clause in Γ_1.

Without loss of generality, assume part of β occurs in the consequent of some clause (the other case is similar). That is, there is some Horn clause $\delta \rightarrow b \in \Gamma_1 \subseteq \Gamma_2$, and $\beta \equiv \beta' \wedge b$. In $\Gamma_2 \setminus \Gamma_1$ this only resolves with $\beta \wedge l \rightarrow j$ (i.e., $\beta' \wedge b \wedge l \rightarrow j$) producing $\beta' \wedge \delta \wedge l \rightarrow j$ which is not $\mathcal{L}(\Gamma_1)$. The only way to resolve away l is through resolution with $\alpha \rightarrow l$ producing $\beta' \wedge \delta \wedge \alpha \rightarrow l$. Now in Γ_1, $\delta \rightarrow b$ resolves with $\alpha \wedge \beta \rightarrow l$ (i.e., $\alpha \wedge \beta' \wedge b \rightarrow j$) producing $\alpha \wedge \beta' \wedge \delta \rightarrow j$ as required.

Hence (due to the soundness of propositional resolution) anything provable from Γ_2 (inter-construction applied to Γ_1) and belonging to $\mathcal{L}(\Gamma_1)$ is provable from Γ_1.

We now consider intra-construction. In order for it to apply there must be Horn clauses $\alpha \wedge \beta \rightarrow j,\ \alpha \wedge \gamma \rightarrow j \in \Gamma_1$ which gives, after intra-construction, $\Gamma_2 = \Gamma_1 \cup \{\alpha \wedge l \rightarrow j,\ \beta \rightarrow l,\ \gamma \rightarrow l\}$. Again we consider three cases.

Case (*i*) Literal j appears in the antecedent of some Horn clause in Γ_1.

That is, there is a clause $\delta \wedge j \to m \in \Gamma_1 \subseteq \Gamma_2$. This resolves with $\alpha \wedge l \to j \in \Gamma_2 \setminus \Gamma_1$ producing $\alpha \wedge \delta \wedge l \to m$ which is not in $\mathcal{L}(\Gamma_1)$. To get a formula in $\mathcal{L}(\Gamma_1)$ this resolves only with $\beta \to l$ producing $\alpha \wedge \beta \wedge \delta \to m$ and $\gamma \to m$ producing $\alpha \wedge \gamma \wedge \delta \to m$. Now, in Γ_1, $\delta \wedge j \to m$ resolves with $\alpha \wedge \beta \to j$ producing $\alpha \wedge \beta \wedge \delta \to m$ and $\alpha \wedge \gamma \to j$ producing $\alpha \wedge \gamma \wedge \delta \to m$ as required.

Case (*ii*) Part of α appears in the consequent of some Horn clause in Γ_1.

That is, there is a Horn clause $\delta \to a \in \Gamma_1 \subseteq \Gamma_2$ and $\alpha \equiv \alpha' \wedge a$. In $\Gamma_2 \setminus \Gamma_1$, this resolves with $\alpha \wedge l \to j$ (i.e., $\alpha' \wedge a \wedge l \to j$) producing $\alpha' \wedge \delta \wedge l \to j$ which however is not in $\mathcal{L}(\Gamma_1)$. To get a formula in $\mathcal{L}(\Gamma_1)$ this resolves with only $\beta \to l$ producing $\alpha' \wedge \beta \wedge \delta \to l$ and $\gamma \to l$ producing $\alpha' \wedge \gamma \wedge \delta \to l$. Now, in Γ_1, $\delta \to a$ resolves with $\alpha \wedge \beta \to l$ (i.e., $\alpha' \wedge a \wedge \beta \to l$) producing $\alpha' \wedge \beta \wedge \delta \to l$ and $\alpha \wedge \gamma \to l$ (i.e., $\alpha' \wedge a \wedge \gamma \to l$) producing $\alpha' \wedge \gamma \wedge \delta \to l$ as required.

Case (*iii*) Part of β or γ appears in the consequent of some Horn clause Γ_1.

Without loss of generality assume part of β occurs in the consequent of some rule (the other case is similar). So there is some Horn clause $\delta \to b \in \Gamma_1$ and $\beta \equiv \beta' \wedge b$. In $\Gamma_2 \setminus \Gamma_1$ this resolves with $\beta \to l$ (i.e., $\beta' \wedge b \to l$) producing $\beta' \wedge \delta \to l$. This, however, is not in $\mathcal{L}(\Gamma_1)$. The only way to obtain a formula in $\mathcal{L}(\Gamma_1)$ is to resolve this with $\alpha \wedge l \to j$ producing $\alpha \wedge \beta' \wedge \delta \to j$. Now in Γ_1, $\delta \to b$ resolves with $\alpha \wedge \beta \to j$ (i.e., $\alpha \wedge \beta' \wedge b \to j$) producing $\alpha \wedge \beta' \wedge \delta \to j$ as required.

Hence again, anything provable from Γ_2 (intra-construction applied to Γ_1) and belonging to $\mathcal{L}(\Gamma_1)$ is provable from Γ_1. ∎

Observation 4.8.1 *If* $\Gamma \vdash \phi$ *then any* $\delta \in \mathcal{L}$ *consistent with* Γ *(i.e.,* $\Gamma \cup \{\delta\} \nvdash \perp$*) is an abduction of* ϕ *with respect to* Γ.

Proof:

Suppose $\Gamma \vdash \phi$. Now, $\vdash \phi \rightarrow (\delta \rightarrow \phi)$ (a paradox of material implication). $\Gamma \vdash \delta \rightarrow \phi$ (Modus Ponens). $\Gamma \cup \{\delta\} \vdash \phi$ (Deduction Theorem). ■

Appendix B

Proofs for Chapter 5

Note: This chapter contains the proofs for claims made in Chapter 5. Proofs of observations that will be of help in providing clarity and shortening the main proofs will also be supplied as needed and named Lemma B.x.

• The following postulates are equivalent given postulates ($K^{\oplus}1$)—($K^{\oplus}4$).

($K^{\oplus}5$)	If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$	(consistency)
(5.1)	If $\neg\alpha \notin K$, then $K^{\oplus}_{\alpha} \neq K_{\perp}$	(consistency$'$)
(5.2)	If $K \neq K_{\perp}$, then $K^{\oplus}_{\alpha} \neq K_{\perp}$	(consistency$''$)

Proof:

($K^{\oplus}5$)$\Rightarrow$(5.1)

Let $\neg\alpha \notin K$. $\neg\alpha \notin K^{\oplus}_{\alpha}$ by ($K^{\oplus}5$). Therefore, $K^{\oplus}_{\alpha} \neq K_{\perp}$ since $\neg\alpha \in K_{\perp}$.

(5.1)$\Rightarrow$ (5.2)

Let $K \neq K_{\perp}$. If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$ by ($K^{\oplus}4$) and so $K^{\oplus}_{\alpha} \neq K_{\perp}$. Otherwise, $\neg\alpha \notin K$ and $K^{\oplus}_{\alpha} \neq K_{\perp}$ by (5.1). Again, $K^{\oplus}_{\alpha} \neq K_{\perp}$.

(5.2)$\Rightarrow$($K^{\oplus}5$)

Let $\neg\alpha \notin K$. So $K \neq K_\perp$ and therefore $K^\oplus_\alpha \neq K_\perp$ by (5.2). Moreover, $\alpha \in K^\oplus_\alpha$ by (K$^\oplus$2). Putting these last two facts together, we obtain $\neg\alpha \notin K^\oplus_\alpha$ as desired.

It therefore follows that any of the postulates above can be derived from any of the other postulates above. ■

Lemma B.1 *Let K be a belief set and $\alpha \in \mathcal{L}$. If $\neg\alpha \notin K$ then there is a $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \perp$.*

Proof:

We claim that if $\neg\alpha \notin K$ then α is a suitable β. We need to verify that $K \cup \{\alpha\} \vdash \alpha$ and $K \cup \{\alpha\} \nvdash \perp$.

Firstly, $K \cup \{\alpha\} \vdash \alpha$ by (Reflexivity).

To show $K \cup \{\alpha\} \nvdash \perp$, suppose for *reductio ad absurdum* that $K \cup \{\alpha\} \vdash \perp$. Then, $K \cup \{\alpha\} \vdash \neg\alpha$ and so $K \vdash \alpha \to \neg\alpha$ by (Deduction). But, $\vdash (\alpha \to \neg\alpha) \leftrightarrow \neg\alpha$. Therefore, $K \vdash \neg\alpha$ by (Modus Ponens). Contradiction.

Hence, there is a $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \perp$ ■

Lemma B.2 *Let K be a belief set and $\alpha \in \mathcal{L}$. If $\neg\alpha \in K$ then there is no $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \perp$.*

Proof:

Let $\neg\alpha \in K$. For *reductio ad absurdum* suppose such a β exists. Now, since $\neg\alpha \in K$, then $K \vdash \neg\alpha$ and therefore $K \cup \{\beta\} \vdash \neg\alpha$ by (Monotonicity). But $K \cup \{\beta\} \vdash \alpha$ so $K \cup \{\beta\} \vdash \bot$. Contradiction. ■

Theorem 5.2.1 *The function* $\oplus$ *satisfies postulates* (K$^{\oplus}$1)—(K$^{\oplus}$5) *iff*

$$K^{\oplus}_{\alpha} = \begin{cases} Cn(K \cup \{\beta\}) & \textit{for some } \beta \in \mathcal{L} \textit{ such that:} \\ & \quad \textit{(i) } K \cup \{\beta\} \vdash \alpha \\ & \quad \textit{(ii) } K \cup \{\beta\} \nvdash \bot \\ & \\ K & \textit{if no such } \beta \textit{ exists} \end{cases}$$

Proof:

(If)

We suppose $K^{\oplus}_{\alpha}$ is defined as above and verify that $\oplus$ satisfies each of $(K^{\oplus}1)$ — $(K^{\oplus}5)$.

(K$^{\oplus}$1) $K^{\oplus}_{\alpha}$ is a belief set.

> Assume there is a β such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \bot$. So, $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$ by definition. Now, $Cn(K^{\oplus}_{\alpha}) = Cn(Cn(K \cup \{\beta\}))$ by (Monotonicity). But, $Cn(Cn(K \cup \{\beta\})) = Cn(K \cup \{\beta\})$ by (Iteration). So, $Cn(K^{\oplus}_{\alpha}) = Cn(K \cup \{\beta\})$ and therefore $Cn(K^{\oplus}_{\alpha}) = K^{\oplus}_{\alpha}$. Thus $K^{\oplus}_{\alpha}$ is a belief set.
>
> Otherwise, no such β exists and $K^{\oplus}_{\alpha} = K$. Therefore $K^{\oplus}_{\alpha}$ is a belief set since K is a belief set.

(K$^{\oplus}$2) If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$

> Let $\neg\alpha \notin K$. By Lemma B.1 we have that there is a $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \bot$. Therefore, $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$ for some

such β by definition. But $K \cup \{\beta\} \vdash \alpha$ and so $\alpha \in Cn(K \cup \{\beta\})$. Hence $\alpha \in K^{\oplus}_{\alpha}$ as desired.

(K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha}$

Suppose there is a $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \perp$. So $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$ for some such β by definition. Now $Cn(K) \subseteq Cn(K \cup \{\beta\})$ by (Monotonicity). But $K = Cn(K)$ since K is a belief set and so $K \subseteq Cn(K \cup \{\beta\})$. Therefore $K \subseteq K^{\oplus}_{\alpha}$.

Otherwise no such β exists and so $K^{\oplus}_{\alpha} = K$ by definition. Therefore $K \subseteq K^{\oplus}_{\alpha}$ trivially.

(K$^{\oplus}$4) If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$

Let $\neg\alpha \in K$. Then, by Lemma B.2, there is no $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \perp$. Therefore $K^{\oplus}_{\alpha} = K$ for some such β by definition as desired.

(K$^{\oplus}$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then, by Lemma B.1, there exists a $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \perp$. Therefore $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$ by definition. For *reductio ad absurdum* suppose $\neg\alpha \in K^{\oplus}_{\alpha}$. That is, $\neg\alpha \in Cn(K \cup \{\beta\})$. So $K \cup \{\beta\} \vdash \neg\alpha$. But $K \cup \{\beta\} \vdash \alpha$ and so $K \cup \{\beta\} \vdash \perp$. Contradiction. Hence $\neg\alpha \notin K^{\oplus}_{\alpha}$.

(Only If)

We suppose $\oplus$ satisfies (K$^{\oplus}$1)—(K$^{\oplus}$5) and show that, for any $\alpha \in \mathcal{L}$, it satisfies the definition above.

We consider two cases:

(I) $K \cup \{\alpha\} \nvdash \perp$

In this case, $\neg\alpha \notin K$ since K is a belief set and therefore, applying Lemma B.1, there is a $\beta \in \mathcal{L}$ such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \bot$. We exhibit such a β and show that $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$.

Now $K \subseteq K^{\oplus}_{\alpha}$ by (K$^{\oplus}$3) so consider the set $K' = K^{\oplus}_{\alpha} \setminus K$. Since $\mathcal{L}$ is a finite language, $Cn(K')$ is finitely axiomatisable. (In particular, if $K' = \emptyset$, then $Cn(K')$ is axiomatisable by any tautology.) Suppose $K'' = \{k_1, \ldots, k_m\}$ is such a finite axiomatisation and consider $\beta = k_1 \wedge \ldots \wedge k_m$ (which is also a finite axiomatisation of $Cn(K')$). We need to show (i) $K \cup \{\beta\} \vdash \alpha$, (ii) $K \cup \{\beta\} \nvdash \bot$ and (iii) $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$.

(i) $K \cup \{\beta\} \vdash \alpha$

> If $\alpha \in K$, then $K \cup \{\beta\} \vdash \alpha$ by (Reflexivity). Otherwise, $\alpha \notin K$. But $\alpha \in K^{\oplus}_{\alpha}$ by (K$^{\oplus}$2). Therefore $\alpha \in K'$ and so $K' \vdash \alpha$. So $\{\beta\} \vdash \alpha$ since β is a finite axiomatisation of K'.

(ii) $K \cup \{\beta\} \nvdash \bot$

> Since $\neg\alpha \notin K$, then by (K$^{\oplus}$5) $\neg\alpha \notin K^{\oplus}_{\alpha}$. Suppose, for *reductio ad absurdum*, that $K \cup \{\beta\} \vdash \bot$. Then $K \vdash \neg\beta$ and so $K \cup K' \vdash \neg\beta$ by (Monotonicity). But $K' \vdash \beta$ (since β is a finite axiomatisation of K' and $\{\beta\} \vdash \beta$). So $K \cup K' \vdash \beta$ by (Monotonicity). Therefore, $K \cup K' \vdash \beta$ and $K \cup K' \vdash \neg\beta$ and, since $K^{\oplus}_{\alpha} = K \cup K'$. Then $K^{\oplus}_{\alpha} \vdash \beta$ and $K^{\oplus}_{\alpha} \vdash \neg\beta$. So $K^{\oplus}_{\alpha} \vdash \neg\alpha$ and since, by (K$^{\oplus}$1), $K^{\oplus}_{\alpha}$ is closed under Cn, then $\neg\alpha \in K^{\oplus}_{\alpha}$. Contradiction. Therefore $K \cup \{\beta\} \nvdash \bot$ as desired.

(iii) $K^{\oplus}_{\alpha} = Cn(K \cup \{\beta\})$

> We need to show $K^{\oplus}_{\alpha} \subseteq Cn(K \cup \{\beta\})$ and $Cn(K \cup \{\beta\} \subseteq K^{\oplus}_{\alpha}$.
>
> $K^{\oplus}_{\alpha} \subseteq Cn(K \cup \{\beta\})$ (i.e., need to show that if $\gamma \in K^{\oplus}_{\alpha}$, then $\gamma \in Cn(K \cup \{\beta\})$).
>
> Let $\gamma \in K^{\oplus}_{\alpha}$. Now $K^{\oplus}_{\alpha} = K \cup K'$ so $\gamma \in K$ or $\gamma \in K'$. If $\gamma \in K$, then $K \vdash \gamma$ and so $K \cup \{\beta\} \vdash \gamma$ by (Monotonicity). Otherwise

$\gamma \in K'$. Then $\{\beta\} \vdash \gamma$ since β is a finite axiomatisation of K' and so $K \cup \{\beta\} \vdash \gamma$ by (Monotonicity). Therefore $K \cup \{\beta\} \vdash \gamma$.

$K^{\oplus}_{\alpha} \subseteq Cn(K \cup \{\beta\}$ (i.e., need too show that if $\gamma \in Cn(K \cup \{\beta\})$ then $\gamma \in K^{\oplus}_{\alpha}$).

Suppose $\gamma \in Cn(K \cup \{\beta\})$. Then $\beta \rightarrow \gamma \in Cn(K) = K$ by (Deduction). Therefore $\beta \rightarrow \gamma \in K^{\oplus}_{\alpha}$ since by (K$^{\oplus}$3), $K \subseteq K^{\oplus}_{\alpha}$. Now $K' \vdash \beta$ since $\{\beta\} \vdash \beta$ (and β is a finite axiomatisation of K) and so, since $K^{\oplus}_{\alpha} = K \cup K'$, we have $K' \subseteq K^{\oplus}_{\alpha} \vdash \beta$. But, by (K$^{\oplus}$1), $K^{\oplus}_{\alpha}$ is closed under Cn. So $\beta \in K^{\oplus}_{\alpha}$ and furthermore $\gamma \in K^{\oplus}_{\alpha}$ by (Deduction).

(II) $K \cup \alpha \vdash \bot$

Therefore $K \vdash \alpha \rightarrow \bot$ by (Deduction) and since $\vdash \neg\alpha \leftrightarrow (\alpha \rightarrow \bot)$, then $K \vdash \neg\alpha$. That is, $\neg\alpha \in K$ since K is a belief set and therefore, by Lemma B.2, there is no β such that $K \cup \{\beta\} \vdash \alpha$ and $K \cup \{\beta\} \nvdash \bot$. Hence, we need to show that $K^{\oplus}_{\alpha} = K$. But this follows directly by (K$^{\oplus}$4). Thus satisfying the definition.

■

Lemma B.3 *Let K be a belief set and $\alpha \in \mathcal{L}$. $K \top \alpha = \emptyset$ iff $\neg\alpha \in K$.*

Proof:

(If)

Let $\neg\alpha \in K$. Suppose for *reductio ad absurdum* that $K \top \alpha \neq \emptyset$. Then for any $K' \in K \top \alpha$, $\neg\alpha \in K'$ by Definition 5.3.1(i) and since $\neg\alpha \in K$. So $\neg\alpha \in Cn(K')$ by (Inclusion). But $\alpha \in Cn(K')$ by Definition 5.3.1(ii). Therefore $Cn(K') = K_{\bot}$ contradicting Definition 5.3.1(iii). Hence $K \top \alpha = \emptyset$.

(Only If)

Let $K \top \alpha = \emptyset$. Suppose for *reductio ad abdurdum* that $\neg\alpha \notin K$. Then $K \cup \{\alpha\} \nvdash \bot$ since if $K \cup \{\alpha\} \vdash \bot$ then $K \vdash \alpha \rightarrow \bot$ by (Deduction) and since $\vdash \neg\alpha \leftrightarrow (\alpha \rightarrow \bot)$ and K is a belief set then $\neg\alpha \in K$ which contradicts our assumption. Since $K \cup \{\alpha\}$ is consistent then there is some $K' \supseteq K$ such that $\alpha \in K'$. Moreover, there is such a K' that is maximally consistent. Therefore $K' \in K \top \alpha$ and so $K \top \alpha \neq \emptyset$. Contradiction. Therefore $\neg\alpha \in K$ as desired. ■

Observation 5.3.1 *Let K be a belief set and $\alpha \in \mathcal{L}$. Any $K' \in K \top \alpha$ is a belief set.*

Proof:

If $\neg\alpha \in K$ then $K \top \alpha = \emptyset$ and the observation becomes vacuously true.

Otherwise, $\neg\alpha \notin K$ and so $K \top \alpha \neq \emptyset$. Suppose for *reductio ad absurdum* that some $K' \in K \top \alpha$ is not a belief set (i.e, $K' \neq Cn(K')$). We know that $K' \subseteq Cn(K')$ by (Inclusion) but, since $K' \neq Cn(K')$, then $Cn(K') \supset K'$. But, this violates Definition 5.3.1 (iv) for K' being an element of $K \top \alpha$ (since $Cn(K')$ satisfies at least Definition 5.3.1 (i)—(iii) and it is a proper superset of K').

Hence any $K' \in K \top \alpha$ is a belief set ■

Lemma B.4 *Let K be a belief set and $\alpha \in \mathcal{L}$. $K \top \alpha = K \top \beta$ iff $K \vdash \alpha \leftrightarrow \beta$.*

Proof:

(If)

We need to show that if $K \vdash \alpha \leftrightarrow \beta$, then $K \top \alpha = K \top \beta$.

If $\neg\alpha \in K$ then $\neg\beta \in K$ since $K \vdash \alpha \leftrightarrow \beta$ and so $K\top\alpha = K\top\beta = \emptyset$ by Definition 5.3.2.

Otherwise, we assume $\neg\alpha \notin K$ and show that $K\top\alpha \subseteq K\top\beta$ and $K\top\beta \subseteq K\top\alpha$.

$K\top\alpha \subseteq K\top\beta$

(That is, we need to show that if $K' \in K\top\alpha$, then $K' \in K\top\beta$)

Suppose $K' \in K\top\alpha$. Then $\alpha \in Cn(K')$ by Definition 5.3.1 (ii). So, $\beta \in Cn(K')$ since $K \vdash \alpha \leftrightarrow \beta$ and $K \subseteq K'$ by Definition 5.3.1 (i). Moreover, $K \subseteq K'$, $K' \neq K_\perp$ and K' maximal by Definition 5.3.1. Therefore, K' is a maximally consistent superset of K implying β. That is, $K' \in K\top\beta$.

$K\top\beta \subseteq K\top\alpha$

Proved in the same manner.

(Only If)

We need to show that if $K\top\alpha = K\top\beta$, then $K \vdash \alpha \leftrightarrow \beta$.

Let $K\top\alpha = K\top\beta$. If $K\top\alpha = K\top\beta = \emptyset$, then $\neg\alpha, \neg\beta \in K$ by Lemma B.3. Therefore $K \vdash \alpha \leftrightarrow \beta$.

Otherwise, $K\top\alpha = K\top\beta \neq \emptyset$. Now, we claim that $K \cup \{\alpha\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \nvdash \perp$ iff $K \cup \{\beta\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \nvdash \perp$. For suppose this is not the case. Then there is some maximal K' with $\neg(\alpha \leftrightarrow \beta) \in K'$ such that either both $K' \in K\top\alpha$ and $K' \notin K\top\beta$ or both $K' \notin K\top\alpha$ and $K' \in K\top\beta$. But $K\top\alpha = K\top\beta(\neq \emptyset)$ so our supposition is not possible. Now suppose, for *reductio ad absurdum*, that $K \nvdash \alpha \leftrightarrow \beta$. We now consider two cases.

(i) $K \cup \{\alpha\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \vdash \perp$ and $K \cup \{\beta\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \vdash \perp$

Now, since $K \cup \{\alpha\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \vdash \perp$, surely $K \vdash \alpha \rightarrow \beta$ (two applications of (Deduction) and note that $\vdash [\alpha \rightarrow \beta] \leftrightarrow [\alpha \rightarrow (\neg(\alpha \leftrightarrow \beta) \rightarrow \perp)]$). Similarly, since $K \cup \{\beta\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \vdash \perp$, surely $K \vdash \beta \rightarrow \alpha$. Therefore $K \vdash \alpha \leftrightarrow \beta$. Contradiction.

(ii) $K \cup \{\alpha\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \not\vdash \bot$ and $K \cup \{\beta\} \cup \{\neg(\alpha \leftrightarrow \beta)\} \not\vdash \bot$

(That is, $\neg(\alpha \leftrightarrow \beta)$ is consistent with both $K \cup \{\alpha\}$ and $K \cup \{\beta\}$.) So there is some $K' \in K\top\alpha$ $(= K\top\beta)$ such that $\neg(\alpha \leftrightarrow \beta) \in K'$. But $\alpha,\ \beta \in K'$ since $K' \in K\top\alpha = K\top\beta$ (and by Definition 5.3.1 (ii) and Observation 5.3.1). Contradiction. ■

Lemma 5.3.2 *Any maxichoice abductive expansion function satisfies postulates* (K$^{\oplus}$1) — (K$^{\oplus}$6).

Proof:

We shall verify that $\oplus$, defined by (Def Max) (i.e., Definition 5.3.3), satisfies each of (K$^{\oplus}$1)—(K$^{\oplus}$6).

Recall that γ selects a single $K' \in K\top\alpha$ and

$$K^{\oplus}_{\alpha} = \begin{cases} \gamma(K\top\alpha) & \text{whenever } K\top\alpha \neq \emptyset \\ \text{K} & \text{otherwise} \end{cases}$$

(K$^{\oplus}$1) $K^{\oplus}_{\alpha}$ is a belief set.

If $\neg\alpha \notin K$ then $K^{\oplus}_{\alpha} = \gamma(K\top\alpha) = K'$ by Definition 5.3.3. By Observation B.3 $K\top\alpha \neq \emptyset$ and by Observation 5.3.1 K' is a belief set. Therefore, $K^{\oplus}_{\alpha}$ is a belief set.

Otherwise, $\neg\alpha \in K$ and $K^{\oplus}_{\alpha} = K$ by Definition 5.3.3. Since K is a belief set, $K^{\oplus}_{\alpha}$ is too.

(K$^{\oplus}$2) If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Now $K^{\oplus}_{\alpha} = \gamma(K\top\alpha) = K'$ by Definition 5.3.3. Now $K\top\alpha \neq \emptyset$ by Observation B.3. By Definition 5.3.1, any $K' \in K\top\alpha$ has $\alpha \in Cn(K')$ $(= K'$ by Observation 5.3.1). Therefore, $\alpha \in K^{\oplus}_{\alpha}$

(K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha}$

If $\neg\alpha \notin K$ then $K^{\oplus}_{\alpha} = \gamma(K\top\alpha) = K'$ by Definition 5.3.3 and $K\top\alpha \neq \emptyset$ by Observation B.3. By Definition 5.3.1(i), $K \subseteq K'$ for any $K' \in K\top\alpha$. Therefore, $K \subseteq K^{\oplus}_{\alpha}$

Otherwise, $\neg\alpha \in K$ and $K^{\oplus}_{\alpha} = K$ by Definition 5.3.3. Therefore $K \subseteq K^{\oplus}_{\alpha}$ trivially.

(K$^{\oplus}$4) If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$

Let $\neg\alpha \in K$. Then by Definition 5.3.3 $K^{\oplus}_{\alpha} = K$ as desired.

(K$^{\oplus}$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then $K^{\oplus}_{\alpha} = \gamma(K\top\alpha)$ by Definition 5.3.3 and $K\top\alpha \neq \emptyset$ by Observation B.3. Suppose for *reductio ad absurdum* that $\neg\alpha \in K'$ for any $K' \in K\top\alpha$. So $\neg\alpha \in Cn(K')$ by (Inclusion). But by Definition 5.3.1(ii), $\alpha \in Cn(K')$. Therefore, $Cn(K') = K_{\perp}$. This contradicts Definition 5.3.1 (iii) and so $\neg\alpha \notin K'$ for any $K' \in K\top\alpha$. Hence, $\neg\alpha \notin K^{\oplus}_{\alpha}$

(K$^{\oplus}$6) If $K \vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$

Let $K \vdash \alpha \leftrightarrow \beta$.

Suppose $\neg\alpha \notin K$. Then $\neg\beta \notin K$ since $K \vdash \alpha \leftrightarrow \beta$ and so $K^{\oplus}_{\alpha} = \gamma(K\top\alpha)$ and $K^{\oplus}_{\beta} = \gamma(K\top\beta)$ by Definition 5.3.3. Now $K\top\alpha \neq \emptyset$ and $K\top\beta \neq \emptyset$ by Observation B.3. Also $K\top\alpha = K\top\beta$ by Lemma B.4. So $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$ since γ is a function.

Otherwise $\neg\alpha \in K$. Then $\neg\beta \in K$ since $K \vdash \alpha \leftrightarrow \beta$ and so $K^{\oplus}_{\alpha} = K$ and $K^{\oplus}_{\beta} = K$ by Definition 5.3.3. Hence $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta} = K$. ■

Observation 5.3.3 *Let K be a belief set and $\alpha \in \mathcal{L}$. $\bigcap(K \top \alpha)$ is a belief set whenever $K \top \alpha$ is nonempty.*

Proof:

Let $K \top \alpha \neq \emptyset$. $K \top \alpha = \{K_1,\ K_2,\ \ldots\}$ where K_i is closed under Cn (i.e., $K_i = Cn(K_i)$) by Observation 5.3.1. Therefore, $\bigcap(K \top \alpha) = \bigcap\{K_1,\ K_2,\ \ldots\} = K_1 \cap K_2 \cap \ldots$. So $Cn(\bigcap(K \top \alpha)) = Cn(K_1 \cap K_2 \cap \ldots) = Cn(K_1) \cap Cn(K_2) \cap \ldots$ (as noted in the preliminaries) $= K_1 \cap K_2 \cap \ldots = \bigcap(K \top \alpha)$ as desired. ■

Lemma 5.3.4 *Any full meet abductive expansion function satisfies postulates* (K$^{\oplus}$1)—(K$^{\oplus}$6).

Proof:

We shall verify that $\oplus$, defined by (Def Meet) (i.e., Definition 5.3.4), satisfies each of $(K^{\oplus}1)$ — $(K^{\oplus}6)$.

Recall that γ selects all $K' \in K \top \alpha$ and therefore

$$K^{\oplus}_{\alpha} = \begin{cases} \bigcap(K \top \alpha) & \text{whenever } K \top \alpha \neq \emptyset \\ \text{K} & \text{otherwise} \end{cases}$$

(K$^{\oplus}$1) $K^{\oplus}_{\alpha}$ is a belief set.

> If $\neg\alpha \notin K$, then $K^{\oplus}_{\alpha} = \bigcap(K \top \alpha)$ by Definition 5.3.4 and by Lemma B.3 $K \top \alpha \neq \emptyset$. Now $\bigcap(K \top \alpha)$ is a belief set by Observation 5.3.3. Hence $K^{\oplus}_{\alpha}$ is a belief set.
>
> Otherwise, $\neg\alpha \in K$ and $K^{\oplus}_{\alpha} = K$ by Definition 5.3.4. Since K is a belief set, then $K^{\oplus}_{\alpha}$ is too.

$(K^{\oplus}2)$ If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then $K^{\oplus}_{\alpha} = \bigcap(K\top\alpha)$ by Definition 5.3.4. $K\top\alpha \neq \emptyset$ by Lemma B.3 and, moreover, $\alpha \in K'$ for any $K' \in K\top\alpha$ by Definition 5.3.1 and Observation 5.3.1. So $\alpha \in \bigcap(K\top\alpha)$ and therefore $\alpha \in K^{\oplus}_{\alpha}$.

(K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha}$

If $\neg\alpha \notin K$ then, $K^{\oplus}_{\alpha} = \bigcap(K\top\alpha)$ by Definition 5.3.4. Also, $K\top\alpha \neq \emptyset$ by Lemma B.3 and, for any $K' \in K\top\alpha$, $K \subseteq K'$ by Definition 5.3.1 (i). Therefore $K \subseteq \bigcap(K\top\alpha)$ and so $K \subseteq K^{\oplus}_{\alpha}$.

Otherwise, $\neg\alpha \in K$ and so $K^{\oplus}_{\alpha} = K$ by Definition 5.3.4. Therefore $K \subseteq K^{\oplus}_{\alpha}$ trivially.

(K$^{\oplus}$4) If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$

Directly from Definition 5.3.4 (Def Meet).

(K$^{\oplus}$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then $K^{\oplus}_{\alpha} = \bigcap(K\top\alpha)$ by Definition 5.3.4 and $K\top\alpha \neq \emptyset$ by Lemma B.3. Now, for any $K' \in K\top\alpha$, $\neg\alpha \notin K'$ since, otherwise, $\neg\alpha \in Cn(K')$ by (Inclusion) and $\alpha \in Cn(K')$ by Definition 5.3.1 (ii) contradicting Definition 5.3.1 (iii) that K' is consistent. Therefore $\neg\alpha \notin \bigcap(K\top\alpha)$ and so $\neg\alpha \notin K^{\oplus}_{\alpha}$ as desired.

(K$^{\oplus}$6) If $K \vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$

Let $K \vdash \alpha \leftrightarrow \beta$.

If $\neg\alpha \notin K$, then $\neg\beta \notin K$ since $K \vdash \alpha \leftrightarrow \beta$. So $K^{\oplus}_{\alpha} = \bigcap(K\top\alpha)$ and $K^{\oplus}_{\beta} = \bigcap(K\top\beta)$ by Definition 5.3.4. By Lemma B.4 $K\top\alpha = K\top\beta$ and so $\bigcap(K\top\alpha) = \bigcap(K\top\beta)$. Therefore $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$.

Otherwise $\neg\alpha \in K$, then $\neg\beta \in K$ since $K \vdash \alpha \leftrightarrow \beta$ and so $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta} = K$ by Definition 5.3.4.

■

Theorem 5.3.5 *Let $\oplus$ be an abductive expansion function. For any formula $\alpha \in \mathcal{L}$ and belief sets K and H such that $\neg\alpha \notin K$ and $\neg\alpha \notin H$, the operation $\oplus$ is a full meet abductive expansion for K with respect to α* iff *$\oplus$ satisfies postulates* (K$^+$1)—(K$^+$6) *for AGM expansion over K.*

Proof:

Now $\oplus$ satisfies (K$^+$1)—(K$^+$6) iff $K^{\oplus}_{\alpha} = Cn(K \cup \{\alpha\})$ by Theorem 2.2.1 and, since $\neg\alpha \notin K$, $K^{\oplus}_{\alpha}$ is a full meet abductive expansion function iff $K^{\oplus}_{\alpha} = \bigcap(K \top \alpha)$ by Definition 5.3.4. So it will suffice to show that $\bigcap(K \top \alpha) = Cn(K \cup \{\alpha\})$. Note that since $\neg\alpha \notin K$, then $K \neq K_{\perp}$ and $K \top \alpha \neq \emptyset$ by Lemma B.3.

$\bigcap(K \top \alpha) \subseteq Cn(K \cup \{\alpha\})$

> (That is, we need to show that, if $\beta \in \bigcap(K \top \alpha)$, then $\beta \in Cn(K \cup \{\alpha\})$.)
>
> Suppose $\beta \in \bigcap(K \top \alpha)$. Then $\beta \in K'$ for every $K' \in K \top \alpha$. Now $\alpha \in K'$ for every $K' \in K \top \alpha$ by Definition 5.3.1 (ii) and using Observation 5.3.1. So $\alpha \rightarrow \beta \in K'$ for every $K' \in K \top \alpha$. Now, we claim that $K \vdash \alpha \rightarrow \beta$. To show this we consider two cases. Firstly, if $K \cup \{\alpha\} \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \perp$, then $K \vdash \alpha \rightarrow \beta$ straightforwardly (two applications of (Deduction) and since $\vdash (\alpha \rightarrow \beta) \leftrightarrow (\alpha \rightarrow (\alpha \rightarrow \beta) \rightarrow \perp)$ using (Monotonicity) and (Modus Ponens)). Otherwise $K \cup \{\alpha\} \cup \{\neg(\alpha \rightarrow \beta)\} \nvdash \perp$ and so there is some $K' \in K \top \alpha$ such that $\neg(\alpha \rightarrow \beta) \in K'$ (that is, $\alpha \wedge \neg\beta \in K'$ using Observation 5.3.1) meaning that $\neg\beta \in K'$ contradicting the above. Therefore, this latter case is not possible. So it follows that $\alpha \rightarrow \beta \in K$ since K is a belief set and therefore $\beta \in Cn(K \cup \{\alpha\})$ by (Deduction) as required.

$Cn(K \cup \{\alpha\}) \subseteq \bigcap(K \top \alpha)$

Now $K \subseteq K'$ for every $K' \in K\top\alpha$ by Definition 5.3.1 (i). Also $\alpha \in Cn(K')$ for every $K' \in K\top\alpha$ by Definition 5.3.1 (ii) and, since $K' = Cn(K')$ by Observation 5.3.1, then $\alpha \in K'$ for every $K' \in K\top\alpha$. Therefore $K \cup \{\alpha\} \subseteq K'$ and so $Cn(K \cup \{\alpha\}) \subseteq Cn(K')$ by (Monotonicity). So $Cn(K \cup \{\alpha\}) \subseteq K'$ by Observation 5.3.1 again. This holds for every $K' \in K\top\alpha$ so $Cn(K \cup \{\alpha\}) \subseteq \bigcap(K\top\alpha)$ as desired.

Note: Although we have not explicitly used the fact that $\neg\alpha \notin H$ this is required otherwise it is possible to find a $H \supset K$ such that $K^{\oplus}_{\alpha} \not\subseteq H^{\oplus}_{\alpha}$ contradicting postulate (K^{+}5). ■

Observation 5.3.6 *Let K be a belief set and $\alpha \in \mathcal{L}$. Then $\bigcap\gamma(K\top\alpha)$ is a belief set whenever $K\top\alpha$ is nonempty.*

Proof:

Let $K\top\alpha \neq \emptyset$. $\gamma(K\top\alpha) = \{K_1,\ K_2,\ \ldots\}$ where K_i is closed (i.e., $K_i = Cn(K_i)$) by Observation 5.3.1. We nee to show that $Cn(\bigcap\gamma(K\top\alpha)) = \bigcap\gamma(K\top\alpha)$. Therefore, $\bigcap\gamma(K\top\alpha) = \bigcap\{K_1,\ K_2,\ \ldots\} = K_1 \cap K_2 \cap \ldots$. So $Cn(\bigcap\gamma(K\top\alpha)) = Cn(K_1 \cap K_2 \cap \ldots) = Cn(K_1) \cap Cn(K_2) \cap \ldots$ (as noted in the preliminaries) $= K_1 \cap K_2 \cap \ldots = \bigcap\gamma(K\top\alpha)$ as desired. ■

Theorem 5.3.7 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a partial meet abductive expansion function* if and only if $\oplus$ *satisfies postulates* (K$^{\oplus}$1)—(K$^{\oplus}$6) *for abductive expansion over K.*

Proof:

(If)

We suppose $\oplus$ satisfies postulates (K$^{\oplus}$1)—(K$^{\oplus}$6) for abductive expansion over K and show that $\oplus$ is a partial meet abductive expansion function (i.e., satisfies Definition 5.3.5). The case where $\neg\alpha \in K$ holds trivially by (K$^{\oplus}$4). So we need to find a selection function γ such that $K^{\oplus}_{\alpha} = \bigcap \gamma(K\top\alpha)$ if $\neg\alpha \notin K$. In this case we know, by Observation B.3, that $K\top\alpha \neq \emptyset$. Let γ be defined as follows:

$$\gamma(K\top\alpha) = \{K' \mid K' \in K\top\alpha \text{ and } K^{\oplus}_{\alpha} \subseteq K'\} \text{ when } K\top\alpha \neq \emptyset$$
$$\gamma(K\top\alpha) = \{K\} \text{ when } K\top\alpha = \emptyset$$

(In fact, it is not necessary to consider this latter case as $K\top\alpha = \emptyset$ iff $\neg\alpha \in K$ by Observation B.3 and this is taken care of by Definition 5.3.5 (Def Meet)).

We need to show that: (i) γ is a well-defined function; (ii) γ is a selection function; and, (iii) $\bigcap \gamma(K\top\alpha) = K^{\oplus}_{\alpha}$ for all α such that $\neg\alpha \notin K$.

(i) γ is a well-defined function

We need to show that $\gamma(K\top\alpha) = \gamma(K\top\beta)$ if $K\top\alpha = K\top\beta$.

Suppose $K\top\alpha = K\top\beta$ ($\neq \emptyset$ since $\neg\alpha \notin K$ as noted previously). By Observation B.4 $K \vdash \alpha \leftrightarrow \beta$ and so $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$ by (K$^{\oplus}$6). Therefore $\gamma(K\top\alpha) = \gamma(K\top\beta)$ by the definition of γ.

(ii) γ is a selection function

(i.e., that $\gamma(K\top\alpha)$ is a nonempty subset of $K\top\alpha$ when $K\top\alpha$ is nonempty.) We need to show that if $K\top\alpha \neq \emptyset$, then $\gamma(K\top\alpha) \neq \emptyset$.

We have noted above that $K\top\alpha \neq \emptyset$ and so $\neg\alpha \notin K$. By (K$^{\oplus}$2) and (K$^{\oplus}$3) we have then, that $K^{\oplus}_{\alpha} \subseteq K' \in K\top\alpha$ for some $K' \in K\top\alpha$. By the Definition of γ it follows that $K' \in \gamma(K\top\alpha)$ and therefore $\gamma(K\top\alpha) \neq \emptyset$ as required.

(iii) $\bigcap \gamma(K\top\alpha) = K^{\oplus}_{\alpha}$ for all α such that $\neg\alpha \notin K$

Recall that $K\top\alpha \neq \emptyset$ and $\neg\alpha \notin K$.

$\bigcap \gamma(K\top\alpha) \subseteq K^{\oplus}_{\alpha}$

We need to show that if $\delta \in \bigcap \gamma(K \top \alpha)$, then $\delta \in K^{\oplus}_{\alpha}$.

Suppose $\delta \in \bigcap \gamma(K \top \alpha)$ and, for *reductio ad absurdum*, that $\delta \notin K^{\oplus}_{\alpha}$. Since $\delta \in \bigcap \gamma(K \top \alpha)$, then $\delta \in K'$ for every $K' \in \gamma(K \top \alpha)$. Also, since $\delta \notin K^{\oplus}_{\alpha}$, then $K^{\oplus}_{\alpha} \nvdash \delta$ since $K^{\oplus}_{\alpha}$ is closed by (K$^{\oplus}$1) and so $K^{\oplus}_{\alpha} \cup \{\neg\delta\} \nvdash \bot$ (i.e., $K^{\oplus}_{\alpha}$ and $\neg\delta$ are consistent). Therefore, there is a maximally consistent superset of $K^{\oplus}_{\alpha}$, say $K^{\#}$, such that $K^{\oplus}_{\alpha} \subseteq K^{\#}$ and $\neg\delta \in K^{\#}$. (Note, we know that $K^{\oplus}_{\alpha}$ is consistent since $\neg\alpha \notin K^{\oplus}_{\alpha}$ by (K$^{\oplus}$5).) But $\alpha \in K^{\oplus}_{\alpha}$ by (K$^{\oplus}$2) and so $\alpha \in K^{\#}$ and by (K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha} \subseteq K^{\#}$. Therefore $K^{\#} \in K \top \alpha$ and, moreover, $K^{\#} \in \gamma(K \top \alpha)$ by definition of γ. This contradicts the fact that $\delta \in K'$ for every $K' \in K \top \alpha$ since by Definition 5.3.1 (iii) every such K' is consistent. Hence $\delta \in K^{\oplus}_{\alpha}$ and so $\bigcap \gamma(K \top \alpha) \subseteq K^{\oplus}_{\alpha}$ as required.

$K^{\oplus}_{\alpha} \subseteq \bigcap \gamma(K \top \alpha)$

Holds by the definition of γ. This definition tells us that $K^{\oplus}_{\alpha} \subseteq K'$ for any $K' \in \gamma(K \top \alpha)$. So it follows directly that $K^{\oplus}_{\alpha} \subseteq \bigcap \gamma(K \top \alpha)$.

(Only if)

We shall verify that $\oplus$, defined by (Def Part) (i.e., Definition 5.3.5), satisfies (K$^{\oplus}$1)—(K$^{\oplus}$6). Recall that γ selects some nonempty subset of $K \top \alpha$ and

$$K^{\oplus}_{\alpha} = \begin{cases} \bigcap \gamma(K \top \alpha) & \text{whenever } K \top \alpha \neq \emptyset \\ \text{K} & \text{otherwise} \end{cases}$$

(K$^{\oplus}$1) $K^{\oplus}_{\alpha}$ is a belief set

If $\neg\alpha \notin K$ then $K^{\oplus}_{\alpha} = \bigcap \gamma(K \top \alpha)$ by Definition 5.3.5 and $K \top \alpha \neq \emptyset$ by Lemma B.3. Therefore $\bigcap \gamma(K \top \alpha)$ is a belief set by Observation 5.3.6 and so $K^{\oplus}_{\alpha}$ is a belief set.

Otherwise, $\neg\alpha \in K$ and $K^{\oplus}_{\alpha} = K$ by Definition 5.3.5. Since K is a belief set, so is $K^{\oplus}_{\alpha}$.

(K$^{\oplus}$2) If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then $K^{\oplus}_{\alpha} = \bigcap\gamma(K\top\alpha)$ by Definition 5.3.5 and $K\top\alpha \neq \emptyset$ by Lemma B.3. Now $\alpha \in K'$ for any $K' \in K\top\alpha$ by Definition 5.3.1 (ii) and Observation 5.3.1. So $\alpha \in \bigcap\gamma(K\top\alpha)$. Hence $\alpha \in K^{\oplus}_{\alpha}$.

(K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha}$

If $\neg\alpha \notin K$ then $K^{\oplus}_{\alpha} = \bigcap\gamma(K\top\alpha)$ by Definition 5.3.5 and $K\top\alpha \neq \emptyset$ by Lemma B.3. Now for any $K' = K\top\alpha$, $K \subseteq K'$ by Definition 5.3.1 (ii). Therefore $K \subseteq \bigcap\gamma(K\top\alpha)$ and so $K \subseteq K^{\oplus}_{\alpha}$.

Otherwise $\neg\alpha \in K$ and $K^{\oplus}_{\alpha} = K$ by Definition 5.3.5. Therefore $K \subseteq K^{\oplus}_{\alpha}$ trivially.

(K$^{\oplus}$4) If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$

Directly from Definition 5.3.5 (Def Part).

(K$^{\oplus}$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then $K^{\oplus}_{\alpha} = \bigcap\gamma(K\top\alpha)$ by Definition 5.3.5 and $K\top\alpha \neq \emptyset$ by Lemma B.3. Now, for any $K' \in K\top\alpha$, $\neg\alpha \notin K'$ otherwise $\neg\alpha \in Cn(K')$ by (Inclusion) and $\alpha \in Cn(K')$ by Definition 5.3.1 (ii) contradicting Definition 5.3.1 (iii). Therefore $\neg\alpha \notin \bigcap\gamma(K\top\alpha)$ and so $\neg\alpha \notin K^{\oplus}_{\alpha}$.

(K$^{\oplus}$6) If $K \vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$

Let $K \vdash \alpha \leftrightarrow \beta$.

If $\neg\alpha \notin K$, then $\neg\beta \notin K$ since $K \vdash \alpha \leftrightarrow \beta$. Therefore $K^{\oplus}_{\alpha} = \bigcap\gamma(K\top\alpha)$ and $K^{\oplus}_{\beta} = \bigcap\gamma(K\top\beta)$ by Definition 5.3.5. Also $K\top\alpha \neq \emptyset$ and $K\top\beta \neq \emptyset$ by Lemma B.3. But, by Lemma B.4, $K\top\alpha = K\top\beta$ and so $\gamma(K\top\alpha) = \gamma(K\top\beta)$ since γ is a function. Therefore $\bigcap\gamma(K\top\alpha) = \bigcap\gamma(K\top\beta)$ and so $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$.

Otherwise, $\neg\alpha \in K$ and so $\neg\beta \in K$ since $K \vdash \alpha \leftrightarrow \beta$. Therefore $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta} = K$ by Definition 5.3.5.

■

(5.6) $K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta} \subseteq K^{\oplus}_{\alpha\vee\beta}$

Proof:

We need to show that if $\delta \in K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$, then $\delta \in K^{\oplus}_{\alpha\vee\beta}$.

Suppose $\delta \in K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$. So $\delta \in K^{\oplus}_{\alpha}$ and $\delta \in K^{\oplus}_{\beta}$. We consider two cases.

$\neg(\alpha \vee \beta) \in K$

Then $\neg\alpha \wedge \neg\beta \in K$ since K is a belief set. That is, $\neg\alpha \in K$ and $\neg\beta \in K$. But, then $K^{\oplus}_{\alpha} = K$ and $K^{\oplus}_{\beta} = K$ by (K$^{\oplus}$4). Therefore $\delta \in K$ and so $\delta \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$3).

$\neg(\alpha \vee \beta) \notin K$

Since $\delta \in K^{\oplus}_{\alpha}$, then $\delta \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ by (K$^{\oplus}$7) and therefore $\alpha \rightarrow \delta \in Cn(K^{\oplus}_{\alpha\vee\beta})$ by (Deduction). That is, $\alpha \rightarrow \delta \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$1). Similarly, since $\delta \in K^{\oplus}_{\beta}$, then $\delta \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\beta\})$ by (K$^{\oplus}$7) and so $\beta \rightarrow \delta \in K^{\oplus}_{\alpha\vee\beta}$ by (Deduction) and by (K$^{\oplus}$1). Therefore $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$1) and, due to the fact that $\vdash [(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)] \leftrightarrow [(\alpha \vee \beta) \rightarrow \delta]$, we have by (K$^{\oplus}$1) again that $(\alpha \vee \beta) \rightarrow \delta \in K^{\oplus}_{\alpha\vee\beta}$. Now $\alpha \vee \beta \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$2) since $\neg(\alpha \vee \beta) \notin K$. Therefore, putting these last together, we obtain $\delta \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$1) as desired.

■

(5.7) If $\alpha \in K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha} \subseteq K^{\oplus}_{\alpha\vee\beta}$

Proof:

Let $\alpha \in K^{\oplus}_{\alpha\vee\beta}$. So $K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\} = K^{\oplus}_{\alpha\vee\beta}$ and therefore $Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\}) = Cn(K^{\oplus}_{\alpha\vee\beta}) = K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$1). Now $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ by (K$^{\oplus}$7). Therefore $K^{\oplus}_{\alpha} \subseteq K^{\oplus}_{\alpha\vee\beta}$. ■

Lemma 5.3.8 *Any relational partial meet abductive expansion function satisfies* (K$^{\oplus}$7).

Proof:

Let $\oplus$ be a relational partial meet abductive expansion function. We need to show that $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ holds.

If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$ by Definition 5.3.5 (Def Part) and since $K \subseteq K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$3) which has previously been shown to be satisfied (see Theorem 5.3.7 (Only If)), then $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ is satisfied trivially.

Otherwise $\neg\alpha \notin K$. It follows that $\neg(\alpha \vee \beta) \notin K$. So $K^{\oplus}_{\alpha} = \bigcap\gamma(K\top\alpha)$ and $K^{\oplus}_{\alpha\vee\beta} = \bigcap\gamma(K\top\alpha \vee \beta)$ by Definition 5.3.5 (Def Part). We need to show $\bigcap\gamma(K\top\alpha) \subseteq Cn(\bigcap\gamma(K\top\alpha\vee\beta)\cup\{\alpha\})$. Suppose $\delta \in \bigcap\gamma(K\top\alpha)$. We need to show $\delta \in Cn(\bigcap\gamma(K\top\alpha\vee\beta) \cup \{\alpha\})$.

Now consider any $K' \in \gamma(K\top\alpha \vee \beta)$. Either $\alpha \notin K'$ or $\alpha \in K'$. In the first case, $\neg\alpha \in K'$ since K' is maximal. So $\alpha \to \delta \in K'$ by Observation 5.3.1 (since obviously $\neg\alpha \vee \delta \in K'$). In the latter case, $\alpha \in K'$. Clearly $K' \in K\top\alpha$. Moreover, this K' is such that $K^{*} \preceq K'$ for all $K^{*} \in K\top\alpha\vee\beta$ by relationality (i.e., (Def γ)). Now $K\top\alpha \subseteq K\top\alpha\vee\beta$ (see Lemma B.5). Let K^{+} be an arbitrary element of $K\top\alpha$. Then $K^{+} \in K\top\alpha \vee \beta$ and so $K^{+} \preceq K'$ by relationality (i.e., (Def γ)). Therefore $K' \in \gamma(K\top\alpha)$. But, since $\delta \in \bigcap\gamma(K\top\alpha)$, then $\delta \in K'$. By Observation 5.3.1, it follows that $\alpha \to \delta \in K'$. So, in either case $\alpha \to \delta \in K'$ and consequently $\alpha \to \delta \in K'$ for any $K' \in \gamma(K\top\alpha \vee \beta)$. Therefore $\alpha \to \delta \in \bigcap\gamma(K\top\alpha \vee \beta)$. So by (Inclusion) it follows that $\alpha \to \delta \in Cn(\bigcap\gamma(K\top\alpha \vee \beta))$.

Therefore $\delta \in Cn(\bigcap \gamma(K\top\alpha \vee \beta) \cup \{\alpha\})$ by (Deduction) as desired. ■

(5.8) Either $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$

Proof:

Suppose $\neg(\alpha \vee \beta) \in K$. Then $\neg\alpha \wedge \neg\beta \in K$ since K is a belief set. That is, $\neg\alpha \in K$ and $\neg\beta \in K$, again since K is a belief set. By (K$^{\oplus}$4) we have $K^{\oplus}_{\alpha\vee\beta} = K$, $K^{\oplus}_{\alpha} = K$ and $K^{\oplus}_{\beta} = K$. So $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ and $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ trivially.

Otherwise $\neg(\alpha \wedge \beta) \notin K$. By (K$^{\oplus}$5) we have $\neg(\alpha \vee \beta) \notin K^{\oplus}_{\alpha\vee\beta}$. So $\neg\alpha \wedge \neg\beta \notin K^{\oplus}_{\alpha\vee\beta}$ since (K$^{\oplus}$1) satisfied. That is, either $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ or $\neg\beta \notin K^{\oplus}_{\alpha\vee\beta}$. So by (K$^{\oplus}$8) $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ as desired. ■

(5.9) If $\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$

Proof:

Let $\alpha \notin K^{\oplus}_{\alpha\vee\beta}$.

If $\neg(\alpha \vee \beta) \in K$, then $\alpha \notin K^{\oplus}_{\alpha\vee\beta} = K$ by (K$^{\oplus}$4) and since, by (K$^{\oplus}$3), $K \subseteq K^{\oplus}_{\beta}$ then $\alpha \notin K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ trivially.

Otherwise, $\neg(\alpha \vee \beta) \notin K$. So $\alpha \vee \beta \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$2). Now, $\neg\beta \notin K^{\oplus}_{\alpha\vee\beta}$ since otherwise $\alpha \in K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$1) contradicting our original hypothesis. Therefore, by (K$^{\oplus}$8), $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ as desired. ■

Lemma 5.3.9 *Any transitively relational partial meet abductive expansion function satisfies* ($K^{\oplus}8$).

Proof:

Let $\oplus$ be a transitively relational partial meet abductive expansion function. We need to show that

If $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ then $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$

holds.

Let $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$.

If $\neg(\alpha \vee \beta) \in K$, then $K^{\oplus}_{\alpha\vee\beta} = K$ since ($K^{\oplus}4$) is satisfied. Also, since ($K^{\oplus}3$) is satisfied, $K \subseteq K^{\oplus}_{\alpha}$. So $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ trivially.

Otherwise $\neg(\alpha \vee \beta) \notin K$. Now, since $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then by ($K^{\oplus}3$), $\neg\alpha \notin K$ and therefore $K\top\alpha \neq \emptyset$ by Lemma B.3. Also, since $\neg(\alpha \vee \beta) \notin K$, $K\top\alpha \vee \beta \neq \emptyset$ by Lemma B.3. Using Definition 5.3.5 we have that $K^{\oplus}_{\alpha\vee\beta} = \bigcap\gamma(K\top\alpha \vee \beta)$ and $K^{\oplus}_{\alpha} = \bigcap\gamma(K\top\alpha)$. So, since $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, $\neg\alpha \notin \bigcap\gamma(K\top\alpha \vee \beta)$. Therefore there is some $K' \in \gamma(K\top\alpha \vee \beta)$ such that $\neg\alpha \notin K'$. That is, $\alpha \in K'$ since K' is maximal. It follows then that $K' \in K\top\alpha$ and, consequently, $K' \in K\top\alpha \ \cap \ \gamma(K\top\alpha \vee \beta)$. So $K' \in K\top\alpha \ \cap \ \gamma(K\top\alpha \vee \beta) \neq \emptyset$ and by Lemma B.6 we have that $\gamma(K\top\alpha) \subseteq \gamma(K\top\alpha\vee\beta)$. Therefore $\bigcap\gamma(K\top\alpha\vee\beta) \subseteq \bigcap\gamma(K\top\alpha)$ by Lemma B.7. Hence $K^{\oplus}_{\alpha\vee\beta} = \bigcap\gamma(K\top\alpha \vee \beta) \subseteq \bigcap\gamma(K\top\alpha) = K^{\oplus}_{\alpha}$ as desired. ∎

(5.10) If $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\}) \subseteq K^{\oplus}_{\alpha}$

Proof:

We need to show that (5.10) is equivalent to ($K^{\oplus}8$) in the presence of the other postulates.

($K^{\oplus}8$)$\Rightarrow$(5.10)

Let $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$. Therefore $\neg\alpha \notin K$ by (K$^{\oplus}$3) and $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ by (K$^{\oplus}$8). Now $Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\}) \subseteq Cn(K^{\oplus}_{\alpha} \cup \{\alpha\})$ by (Monotonicity). But, since $\neg\alpha \notin K$, $\alpha \in K^{\oplus}_{\alpha}$ by (K$^{\oplus}$2). It follows then that $Cn(K^{\oplus}_{\alpha} \cup \{\alpha\}) = Cn(K^{\oplus}_{\alpha}) = K^{\oplus}_{\alpha}$ by (K$^{\oplus}$1). Hence $Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\}) \subseteq K^{\oplus}_{\alpha}$.

(5.10)⇒(K$^{\oplus}$8)

Let $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$. Now $K^{\oplus}_{\alpha\vee\beta} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ by (Inclusion). Therefore $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ directly by (5.10).

■

(5.11) $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\beta}$ or $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$

Proof:

In the case that $\neg(\alpha \vee \beta) \in K$ (i.e., $\neg\alpha \wedge \neg\beta \in K$ since K is a belief set) we have that $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta} = K^{\oplus}_{\alpha\vee\beta} = K$ by (K$^{\oplus}$4) and so the above is trivially satisfied. We therefore need only consider when $\neg(\alpha \vee \beta) \notin K$.

We consider four cases:

(i) $\alpha \in K^{\oplus}_{\alpha\vee\beta}$ and $\beta \in K^{\oplus}_{\alpha\vee\beta}$

In this case we have $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ and $\neg\beta \notin K^{\oplus}_{\alpha\vee\beta}$ due to (K$^{\oplus}$5). So by (K$^{\oplus}$8) it follows that $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ and $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$. Also, since $\alpha \in K^{\oplus}_{\alpha\vee\beta}$ and $\beta \in K^{\oplus}_{\alpha\vee\beta}$, by (5.7) $K^{\oplus}_{\alpha} \subseteq K^{\oplus}_{\alpha\vee\beta}$ and $K^{\oplus}_{\beta} \subseteq K^{\oplus}_{\alpha\vee\beta}$. Therefore $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$.

(ii) $\alpha \in K^{\oplus}_{\alpha\vee\beta}$ and $\beta \notin K^{\oplus}_{\alpha\vee\beta}$

Using (5.7) and the fact that $\alpha \in K^{\oplus}_{\alpha\vee\beta}$ we get $K^{\oplus}_{\alpha} \subseteq K^{\oplus}_{\alpha\vee\beta}$. Also, since $\alpha \in K^{\oplus}_{\alpha\vee\beta}$, (K$^{\oplus}$5) gives $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$. So $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ by (K$^{\oplus}$8). Therefore $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha}$.

(iii) $\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ and $\beta \in K^{\oplus}_{\alpha\vee\beta}$

A similar argument to case (ii) gives $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\beta}$.

(iv) $\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ and $\beta \notin K^{\oplus}_{\alpha\vee\beta}$

So $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ and $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ by (5.9). Therefore $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$. Now $K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta} \subseteq K^{\oplus}_{\alpha\vee\beta}$ by (5.6). Hence $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$.

Therefore $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\beta}$ or $K^{\oplus}_{\alpha\vee\beta} = K^{\oplus}_{\alpha} \cap K^{\oplus}_{\beta}$ as desired. ■

Lemma 5.3.10 *Postulates* ($K^{\oplus}7$) *and* ($K^{\oplus}8$) *together with (5.3) imply postulate* ($K^{\oplus}6$) *in the presence of the other postulates (($K^{\oplus}1$) — ($K^{\oplus}5$)) for abductive expansion over K.*

Proof:

Let $K \vdash \alpha \leftrightarrow \beta$. We need to show that $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$. Consider the following two cases:

(i) $\neg\alpha \in K$

It follows that $\neg\beta \in K$ since $K \vdash \alpha \leftrightarrow \beta$. So, using ($K^{\oplus}4$), $K^{\oplus}_{\alpha} = K$ and $K^{\oplus}_{\beta} = K$. Therefore $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$.

(ii) $\neg\alpha \notin K$

So $\neg\beta \notin K$ since $K \vdash \alpha \leftrightarrow \beta$ and consequently $\neg(\alpha \vee \beta) \notin K$. Now, since $K \vdash \alpha \leftrightarrow \beta$, then $K \vdash \alpha \leftrightarrow (\alpha \vee \beta)$ and $K \vdash \beta \leftrightarrow (\beta \vee \alpha)$. By ($K^{\oplus}3$) we get $K \subseteq K^{\oplus}_{\alpha\vee\beta}$ and $K \subseteq K^{\oplus}_{\beta\vee\alpha}$. So, with the help of ($K^{\oplus}1$), $\alpha \leftrightarrow (\alpha \vee \beta) \in K^{\oplus}_{\alpha\vee\beta}$ and $\beta \leftrightarrow (\beta \vee \alpha) \in K^{\oplus}_{\beta\vee\alpha}$. ($K^{\oplus}2$) together with the fact that $\neg(\alpha \vee \beta) \notin K^{\oplus}_{\alpha\vee\beta}$ gives $\alpha \vee \beta \in K^{\oplus}_{\alpha\vee\beta}$. Putting these last two together with the help of ($K^{\oplus}1$) yields $\alpha \in K^{\oplus}_{\alpha\vee\beta}$ and $\beta \in K^{\oplus}_{\beta\vee\alpha}$. So, by (5.1) (which is equivalent to (K$^\oplus$5) in the presence of postulates (K$^\oplus$1) — (K$^\oplus$4)), $\neg\alpha \notin K^\oplus_{\alpha\vee\beta}$ and $\neg\beta \notin K^\oplus_{\beta\vee\alpha}$ and therefore (K$^\oplus$8) gives $K^\oplus_{\alpha\vee\beta} \subseteq K^\oplus_\alpha$ and $K^\oplus_{\beta\vee\alpha} \subseteq K^\oplus_\beta$. Now, since $\alpha \in K^\oplus_{\alpha\vee\beta}$, then $Cn(K^\oplus_{\alpha\vee\beta} \cup \{\alpha\}) = K^\oplus_{\alpha\vee\beta}$ using (K$^\oplus$1). By (K$^\oplus$7) $K^\oplus_\alpha \subseteq Cn(K^\oplus_{\alpha\vee\beta} \cup \{\alpha\})$ and therefore $K^\oplus_\alpha \subseteq K^\oplus_{\alpha\vee\beta}$. Since $\beta \in K^\oplus_{\beta\vee\alpha}$, a similar argument gives $K^\oplus_\beta \subseteq K^\oplus_{\beta\vee\alpha}$. So we have $K^\oplus_\alpha = K^\oplus_{\alpha\vee\beta}$ and $K^\oplus_\beta = K^\oplus_{\beta\vee\alpha}$. However, $K^\oplus_{\alpha\vee\beta} = K^\oplus_{\beta\vee\alpha}$ by (5.3). Therefore $K^\oplus_\alpha = K^\oplus_\beta$ as desired.

■

Lemma B.5 *Let K be a belief set and $\alpha,\ \beta \in \mathcal{L}$. Then $K\top\alpha \vee \beta = K\top\alpha\ \cup\ K\top\beta$.*

Proof:

$K\top\alpha \vee \beta \subseteq K\top\alpha\ \cup\ K\top\beta$

(Need to show that if $K' \in K\top\alpha \vee \beta$, then $K' \in K\top\alpha\ \cup\ K\top\beta$.)

Suppose $K' \in K\top\alpha \vee \beta$. Now $\alpha \vee \beta \in K'$ by Definition 5.3.1 (ii) and Observation 5.3.1. So, since K' is maximal, either $\alpha \in K'$ or $\beta \in K'$. Therefore $K' \in K\top\alpha$ or $K' \in K\top\beta$ and so $K' \in K\top\alpha\ \cup\ K\top\beta$.

$K\top\alpha\ \cup\ K\top\beta \subseteq K\top\alpha \vee \beta$

(Need to show that if $K' \in K\top\alpha\ \cup\ K\top\beta$, then $K' \in K\top\alpha \vee \beta$.)

Suppose $K' \in K\top\alpha \cup K\top\beta$. Then $K' \in K\top\alpha$ or $K' \in K\top\beta$. If $K' \in K\top\alpha$, then $\alpha \in K'$ by Definition 5.3.1 (ii) and Observation 5.3.1. Also, $\alpha \vee \beta \in K'$ by Observation 5.3.1. Therefore $K' \in K\top\alpha \vee \beta$. A similar argument shows that $K' \in K\top\alpha \vee \beta$ if $K' \in K\top\beta$. Therefore $K\top\alpha\ \cup\ K\top\beta \subseteq K\top\alpha \vee \beta$ as desired.

■

Lemma B.6 *Let K be a belief set and $\alpha,\ \beta \in \mathcal{L}$. Let γ be a selection function defined via* (Def γ) *using a transitively relational marking-off identity $\preceq$. If $K\top\alpha \cap \gamma(K\top\alpha \vee \beta) \neq \emptyset$, then $\gamma(K\top\alpha) \subseteq \gamma(K\top\alpha \vee \beta)$.*

Proof:

Let $K\top\alpha \cap \gamma(K\top\alpha \vee \beta) \neq \emptyset$. Need to show that if $K' \in \gamma(K\top\alpha)$, then $K' \in \gamma(K\top\alpha \vee \beta)$.

Suppose $K' \in \gamma(K\top\alpha)$ and, for *reductio ad absurdum*, $K' \notin \gamma(K\top\alpha \vee \beta)$. Now $K' \in \gamma(K\top\alpha) \subseteq K\top\alpha \subseteq K\top\alpha \vee \beta$ (the latter part by Lemma B.5) but $K' \notin \gamma(K\top\alpha \vee \beta)$. Therefore there is some $K'' \in K\top\alpha \vee \beta$ with $K'' \not\preceq K'$ by relationality of $\preceq$. Also, $K\top\alpha \cap \gamma(K\top\alpha \vee \beta) \neq \emptyset$ by hypothesis so there is a $K^{\#} \in K\top\alpha$ with $K^{\#} \in \gamma(K\top\alpha \vee \beta)$ such that $K'' \preceq K^{\#}$. Also $K^{\#} \preceq K'$, since $K' \in \gamma(K\top\alpha)$, by relationality. So $K'' \preceq K'$ since $\preceq$ is transitive. Contradiction. Therefore $K' \in \gamma(K\top\alpha \vee \beta)$ as desired. ■

Lemma B.7 *Let K be a belief set and $\alpha,\ \beta \in \mathcal{L}$. Whenever $K\top\alpha \neq \emptyset$, if $\gamma(K\top\alpha) \subseteq \gamma(K\top\beta)$, then $\bigcap\gamma(K\top\beta) \subseteq \bigcap\gamma(K\top\alpha)$.*

Proof:

Let $\gamma(K\top\alpha) \subseteq \gamma(K\top\beta)$. We need to show that $\bigcap\gamma(K\top\beta) \subseteq \bigcap\gamma(K\top\alpha)$ (i.e., if $\delta \in \bigcap\gamma(K\top\beta)$, then $\delta \in \bigcap\gamma(K\top\alpha)$).

Suppose $\delta \in \bigcap\gamma(K\top\beta)$. So $\delta \in K'$ for every $K' \in \gamma(K\top\beta)$ and, since $\gamma(K\top\alpha) \subseteq \gamma(K\top\beta)$, then $\delta \in K'$ for every $K' \in \gamma(K\top\alpha)$. Therefore $\delta \in \bigcap\gamma(K\top\alpha)$ as desired. ■

Note: Since (K$^{\oplus}$1)—(K$^{\oplus}$6) are satisfied by any partial meet abductive expansion function, due to Theorem 5.3.7, they are also satisfied by any relational or transitively relational partial meet abductive expansion function. Therefore we shall use them freely in the following proofs.

Theorem 5.3.11 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a transitively relational partial meet abductive expansion function* if and only if $\oplus$ *satisfies postulates* (K$^{\oplus}$1)—(K$^{\oplus}$8) *for abductive expansion over K.*

Proof:

(If)

We suppose $\oplus$ satisfies postulates (K$^{\oplus}$1)—(K$^{\oplus}$8) for abductive expansion over K and show that $\oplus$ is a transitively relational partial meet abductive expansion function. As in Theorem 5.3.7 the case where $\neg\alpha \in K$ holds trivially by (K$^{\oplus}$4). So, we need to find a transitively relational selection function γ such that $K^{\oplus}_{\alpha} = \bigcap \gamma(K \top \alpha)$ if $\neg\alpha \notin K$. By Theorem 5.3.7 (If) we have that γ is a selection function and that $K^{\oplus}_{\alpha} = \bigcap \gamma(K \top \alpha)$ if $\neg\alpha \notin K$. We need to show that γ is transitively relational.

We define $\preceq$ over all maximal consistent supersets of K as follows:

For all K' and $K'' \in \mathcal{K}$, $K'' \preceq K'$ iff the following three conditions hold.

(i) $K', K'' \in K \top \alpha$ for some α such that $\neg\alpha \notin K$.

(ii) $K' \in K \top \alpha$ and $K^{\oplus}_{\alpha} \subseteq K'$ for some α such that $\neg\alpha \notin K$.

(iii) For all $\alpha \in \mathcal{L}$, if $K', K'' \in K \top \alpha$ and $K \subseteq K^{\oplus}_{\alpha} \subseteq K''$, then $K^{\oplus}_{\alpha} \subseteq K'$

We define the completion γ^* of a selection function in the following manner.

$$\gamma^*(K \top \alpha) = \{K' \in K \top \alpha : \bigcap \gamma(K \top \alpha) \subseteq K'\} \text{ for all } \alpha \text{ such that } \neg\alpha \notin K$$
$$\gamma^*(K \top \alpha) = \{K\} \text{ when } \neg\alpha \in K$$

Clearly γ^* is also a selection function for K and determines the same partial meet abductive expansion function as γ.

We need to show that

(a) the relation $\preceq$ satisfies the marking-off identity as defined by (Def γ) (i.e., is relational with respect to γ^*); and,

(b) the relation $\preceq$ is transitive with respect to γ^* for all α such that $\neg\alpha \notin K$.

(a) the relation $\preceq$ satisfies the marking-off identity as defined by (Def γ).

Recall that the marking-off identity is defined by (Def γ) in the following way.

$$\gamma^*(K\top\alpha) = \{K' \in K\top\alpha : K'' \preceq K' \text{ for all } K'' \in K\top\alpha\}$$

To show that $\preceq$ satisfies the marking-off identity, we first show that if $K' \in \gamma^*(K\top\alpha)$ and $K'' \in K\top\alpha$, then $K'' \preceq K'$. We then show the converse.

To show the former, suppose $K' \in \gamma^*(K\top\alpha)$. Let $K'' \in K\top\alpha$. We need to show that $K'' \preceq K'$. (We note that if $K' = K$ or $K'' = K$ then K is maximal and so $K\top\alpha = \{K\}$ for any $\alpha \in \mathcal{L}$. That is, $K' = K'' = K$. In this case, the results holds trivially by the definition of $\preceq$.) Immediately, we have

(i) $K'' \in K\top\alpha$

(ii) $K' \in K\top\alpha$ and $K^{\oplus}_{\alpha} \subseteq K'$

Now, let $\beta \in \mathcal{L}$ and suppose $K',\ K'' \in K\top\beta$ and $K^{\oplus}_{\beta} \subseteq K''$. To show that $K'' \preceq K'$, all we need show is that $K^{\oplus}_{\beta} \subseteq K'$ (that is, we need to show that if $\delta \in K^{\oplus}_{\beta}$, then $\delta \in K'$) and, putting all these together, use our definition of $\preceq$ above.

Now either $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ or $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ by (5.8). Considering the latter case gives $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta} \subseteq K'' \in K\top\alpha$. So $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ (otherwise $\neg\alpha \in K''$ contradicting Definition 5.3.1 (iii)). Therefore $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ by (K$^{\oplus}$8). Hence, in either case, $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha} \subseteq K'$.

Suppose, for *reductio ad absurdum*, that there is a $\delta \in K^{\oplus}_{\beta}$ with $\delta \notin K'$ (so $\neg\delta \in K'$ since K' is maximal). Now, by (K$^{\oplus}$7), $K^{\oplus}_{\beta} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\beta\})$. So $\delta \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\beta\})$ and consequently $\beta \rightarrow \delta \in Cn(K^{\oplus}_{\alpha\vee\beta})$. That is, by (K$^{\oplus}$1), $\beta \rightarrow \delta \in K^{\oplus}_{\alpha\vee\beta}$. So, $\beta \rightarrow \delta \in K'$ since $K^{\oplus}_{\alpha\vee\beta} \subseteq K'$ by above. However, $K' \in K\top\beta$ so $\beta \in K'$. Therefore, by Observation 5.3.1, $\delta \in K'$ contradicting our supposition as desired.

To show the converse we suppose that $K' \notin \gamma^*(K\top\alpha)$ and $K' \in K\top\alpha$. We need to find a $K'' \in K\top\alpha$ with $K'' \not\preceq K'$. Now we have that $\neg\alpha \notin K$ (this is also implied by our hypothesis that $K' \in K\top\alpha$). So it follows that $K\top\alpha \neq \emptyset$ and therefore $\gamma^*(K\top\alpha) \neq \emptyset$ since γ^* is a selection function. Let $K'' \in \gamma^*(K\top\alpha)$. Noting that

(i) $K', K'' \in K\top\alpha$

(ii) $K'' \in \gamma^*(K\top\alpha)$, and

(iii) $K' \notin \gamma^*(K\top\alpha)$

we see that condition (iii) in our definition of $\preceq$ above fails in the case that β is replaced by α. Therefore $K'' \not\preceq K'$ as desired.

(b) the relation $\preceq$ is transitive with respect to γ^* for all α such that $\neg\alpha \notin K$.

Suppose $K^{\#} \preceq K''$ and $K'' \preceq K'$. We need to show that $K^{\#} \preceq K'$. (We note that if any of $K' = K$ or $K'' = K$ or $K^{\#} = K$, then K is maximal and so $K' = K'' = K^{\#} = K$ and the result holds trivially by the definition of $\preceq$.)

Since $K'' \preceq K'$ we have that $K'' \in K\top\delta$ and $K^{\oplus}_{\delta} \subseteq K'$ for some $\delta \in \mathcal{L}$ so condition (ii) of our definition of $\preceq$ is satisfied. Since $K^{\#} \preceq K''$ we have that $K^{\#} \in K\top\delta$ for some $\delta \in \mathcal{L}$ so condition (i) of our definition of $\preceq$ is satisfied. We require condition (iii) of our definition of $\preceq$ be satisfied.

Suppose $K^{\#}, K' \in K\top\beta$ and $K^{\oplus}_{\beta} \subseteq K^{\#}$. We need to show that $K^{\oplus}_{\beta} \subseteq K'$ in order to satisfy condition (iii) of our definition of $\preceq$. Since $K' \in K\top\beta$, then $\neg\beta \notin K$ (otherwise Definition 5.3.1 (iii) is contradicted). Now, due to

$K^{\#} \preceq K''$ there is an $\alpha \in \mathcal{L}$ with $K'' \in K\top\alpha$ and $K^{\oplus}_{\alpha} \subseteq K''$. Also, since $K\top\alpha \vee \beta = K\top\alpha \cup K\top\beta$ by Lemma B.5 we have that $K^{\#}$, K'', $K' \in K\top\alpha \vee \beta$.

Now either $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\beta}$ or $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ by (5.8). The former case gives $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\#}$ by the above and so, because $K^{\#} \preceq K''$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K''$. Again, since $K'' \preceq K'$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K'$. Similarly, the latter case gives $K^{\oplus}_{\alpha\vee\beta} \subseteq K''$ and, since $K'' \preceq K'$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K'$. So, in either case, $K^{\oplus}_{\alpha\vee\beta} \subseteq K'$.

Let $\delta \in K^{\oplus}_{\beta}$. We need to show that $\delta \in K'$. Now $K^{\oplus}_{\beta} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\beta\})$ by ($K^{\oplus}$7). So $\delta \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\beta\})$ and consequently $\beta \rightarrow \delta \in Cn(K^{\oplus}_{\alpha\vee\beta})$ by (Deduction). That is, $\beta \rightarrow \delta \in K^{\oplus}_{\alpha\vee\beta}$ by ($K^{\oplus}$1). So $\beta \rightarrow \delta \in K'$ by above and therefore $\delta \in K'$ by Observation 5.3.1 and since $\beta \in K'$.

(Only If)

Directly from Theorem 5.3.7, Lemma 5.3.8 and Lemma 5.3.9. ■

Lemma B.8 *Let K be a belief set and $\alpha \in \mathcal{L}$. $[K] \cap [\alpha] = \emptyset$ if and only if $\neg\alpha \in K$.*

Proof:

(If)

Let $\neg\alpha \in K$. We need to show $[K] \cap [\alpha] = \emptyset$.

If $K = K_{\perp}$, then $[K] = \emptyset$ by definition and so $[K] \cap [\alpha] = \emptyset$ trivially.

Otherwise $K \neq K_{\perp}$. Now, for any $m \in [K]$, we have $\neg\alpha \in m$. Therefore $\alpha \notin m$ for $m \in [K]$ since they are maximal consistent extensions of $\mathcal{L}$. On the other hand, $\alpha \in m$ for all $m \in [\alpha]$. Hence $[K] \cap [\alpha] = \emptyset$.

(Only If)

Suppose $\neg\alpha \notin K$ (i.e., $K \nvdash \neg\alpha$ and also $K \neq K_{\perp}$). We need to show $[K] \cap [\alpha] \neq \emptyset$. Now

$K \cup \{\alpha\} \nvdash \bot$. So, there is some $m \in \mathcal{M}_{\mathcal{L}}$ such that $K \subseteq m$ and moreover $\alpha \in m$. But, such an m is also an element of $[\alpha]$. Therefore $m \in [K] \cap [\alpha]$. Hence $[K] \cap [\alpha] \neq \emptyset$. ■

Lemma B.9 *Let K be a belief set and $\alpha,\ \beta \in \mathcal{L}$. $[K] \cap [\alpha] = [K] \cap [\beta]$ if and only if $K \vdash \alpha \leftrightarrow \beta$.*

Proof:

(If)

Let $K \vdash \alpha \leftrightarrow \beta$. We need to show $[K] \cap [\alpha] = [K] \cap [\beta]$ and do so by showing two cases:

(i) $[K] \cap [\alpha] \subseteq [K] \cap [\beta]$

> That is, we need to show that if $m \in [K] \cap [\alpha]$, then $m \in [K] \cap [\beta]$.
>
> Suppose $m \in [K] \cap [\alpha]$ for some $m \in \mathcal{M}_{\mathcal{L}}$. So $m \in [K]$ and $m \in [\alpha]$. That is, $\alpha \in m$. But since $m \in [K]$ and $K \vdash \alpha \leftrightarrow \beta$, then surely $\beta \in m$. So $m \in [\beta]$. Therefore $m \in [K] \cap [\beta]$ as desired.

(ii) $[K] \cap [\beta] \subseteq [K] \cap [\alpha]$

> Proved in a similar manner to Case(i).

(Only If)

Let $[K] \cap [\alpha] = [K] \cap [\beta]$. So $th([K] \cap [\alpha]) = th([K] \cap [\beta])$ by Lemma 2.2.14 (iv). Now $th([K] \cap [\alpha]) = Cn(th([K]) \cup \{\alpha\}) = Cn(K \cup \{\alpha\})$ by Lemma 2.2.14 (iii) and (i) respectively. Similarly $th([K] \cup [\beta]) = Cn(K \cup \{\beta\})$. It follows that $Cn(K \cup \{\alpha\}) = Cn(K \cup \{\beta\})$. Now $\alpha \in Cn(K \cup \{\alpha\})$ by (Inclusion) and so $\alpha \in Cn(K \cup \{\beta\})$. Then $\alpha \to \beta \in Cn(K)$ by (Deduction) and, since K is a belief set, $\alpha \to \beta \in K$. A similar argument, since $\beta \in Cn(K \cup \{\beta\})$, gives $\beta \to \alpha \in K$. Therefore $\alpha \leftrightarrow \beta \in K$ (i.e., $K \vdash \alpha \leftrightarrow \beta$) as desired. ■

Theorem 5.3.12 *Let $K \in \mathcal{K}$ be some belief set and $\mathcal{IS}$ any internal system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred within $[K]$. If for any $\alpha \in \mathcal{L}$ we define $K^{\oplus}_{\alpha}$ to be $th(f_{\mathcal{IS}}(\alpha))$, then postulates* (K$^{\oplus}$1)—(K$^{\oplus}$8) *are satisfied.*

Proof: Let $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha))$. We show that each of (K$^{\oplus}$1)—(K$^{\oplus}$8) are satisfied.

(K$^{\oplus}$1) $K^{\oplus}_{\alpha}$ is a belief set

Now, $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha))$.

The result follows directly from the definition of th (since $th : 2^{\mathcal{M}_{\mathcal{L}}} \to \mathcal{L}$). However, we can also argue as follows.

If $[K] \cap [\alpha] = \emptyset$ (i.e., $\neg\alpha \in K$ by Lemma B.8), then $f_{\mathcal{IS}}(\alpha) = [K]$ by definition (Def $f_{\mathcal{IS}}$). Therefore $th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (i). So $K^{\oplus}_{\alpha} = K$ and hence $K^{\oplus}_{\alpha}$ is a belief set since K is.

Otherwise $[K] \cap [\alpha] \neq \emptyset$ (i.e., $\neg\alpha \notin K$ by Lemma B.8), and so $f_{\mathcal{IS}}(\alpha) = [\alpha] \cap c_{\mathcal{IS}}(\alpha)$ by definition (Def $f_{\mathcal{IS}}$). Therefore $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([\alpha] \cap c_{\mathcal{IS}}(\alpha)) = Cn(th(c_{\mathcal{IS}}(\alpha) \cup \{\alpha\})$ by Lemma 2.2.14 (iii). So $K^{\oplus}_{\alpha}$ is closed under Cn and hence a belief set.

(K$^{\oplus}$2) If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. So $[K] \cap [\alpha] \neq \emptyset$ by Lemma B.8 and therefore $f_{\mathcal{IS}}(\alpha) = [\alpha] \cap c_{\mathcal{IS}}(\alpha)$ by definition (Def $f_{\mathcal{IS}}$). Now $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([\alpha] \cap c_{\mathcal{IS}}(\alpha)) = Cn(th(c_{\mathcal{IS}}(\alpha)) \cup \{\alpha\})$ (the latter by Lemma 2.2.14 (iii)). But $\alpha \in Cn(th(c_{\mathcal{IS}}(\alpha)) \cup \{\alpha\})$ by (Inclusion). Therefore $\alpha \in K^{\oplus}_{\alpha}$ as desired.

(K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha}$

If $[K] \cap [\alpha] = \emptyset$ (i.e., $\neg\alpha \in K$ by Lemma B.8), then $f_{\mathcal{IS}}(\alpha) = [K]$ by definition (Def $f_{\mathcal{IS}}$). Therefore $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ (the latter part by Lemma 2.2.14 (i)). So $K \subseteq K^{\oplus}_{\alpha}(= K)$ trivially.

Otherwise $[K] \cap [\alpha] \neq \emptyset$ (i.e., $\neg\alpha \notin K$ by Lemma B.8), and so, by definition (Def $f_{\mathcal{IS}}$), $f_{\mathcal{IS}}(\alpha) = [\alpha] \cap c_{\mathcal{IS}}(\alpha)$. Now $c_{\mathcal{IS}}(\alpha) \subseteq [K]$ by definition (it is guaranteed by (IS2) and (IS3)) and so surely $[\alpha] \cap c_{\mathcal{IS}}(\alpha) \subseteq [K]$. Therefore $th([K]) \subseteq th([\alpha] \cap c_{\mathcal{IS}}(\alpha))$ by Lemma 2.2.14 (iv) and $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([\alpha] \cap c_{\mathcal{IS}}(\alpha))$. But $th([K]) = K$ by Lemma 2.2.14 (i). Hence $K \subseteq K^{\oplus}_{\alpha}$ as desired.

($K^{\oplus}4$) If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$

Let $\neg\alpha \in K$. Then $[K] \cap [\alpha] = \emptyset$ by Lemma B.8 and so $f_{\mathcal{IS}}(\alpha) = [K]$ by definition (Def $f_{\mathcal{IS}}$). Now $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ (the latter part by Lemma 2.2.14 (i)) as desired.

($K^{\oplus}5$) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Then $[K] \cap [\alpha] \neq \emptyset$ by Lemma B.8 and so $f_{\mathcal{IS}}(\alpha) = [\alpha] \cap c_{\mathcal{IS}}(\alpha)$ by definition (Def $f_{\mathcal{IS}}$). Now, for any $m \in [\alpha]$, $\neg\alpha \notin m$ (since m is maximal consistent and $\alpha \in m$) and so $\neg\alpha \notin m$ for any $m \in [\alpha] \cap c_{\mathcal{IS}}(\alpha)$. Moreover, since $[K] \cap [\alpha] \neq \emptyset$, then by (IS3) there is a smallest sphere, $c_{\mathcal{IS}}(\alpha)$, intersecting $[\alpha]$ (i.e, $[\alpha] \cap c_{\mathcal{IS}}(\alpha) \neq \emptyset$). Therefore $\neg\alpha \notin th([\alpha] \cap c_{\mathcal{IS}}(\alpha))$ by definition of th (i.e., $\neg\alpha \notin \bigcap\{m \in [\alpha] \cap c_{\mathcal{IS}}(\alpha)\}$). Hence, since $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([\alpha] \cap c_{\mathcal{IS}}(\alpha))$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$.

($K^{\oplus}6$) If $K \vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$

Let $K \vdash \alpha \leftrightarrow \beta$. So, by Lemma B.9, $[K] \cap [\alpha] = [K] \cap [\beta]$.

If $[K] \cap [\alpha] = \emptyset$ then surely $[K] \cap [\beta] = \emptyset$ (i.e., $\neg\alpha \in K$ and $\neg\beta \in K$ by Lemma B.8). So, by definition (Def $f_{\mathcal{IS}}$), $f_{\mathcal{IS}}(\alpha) = [K]$ and $f_{\mathcal{IS}}(\beta) = [K]$. Therefore $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ (the latter part by

Lemma 2.2.14 (i)). Similarly $K^{\oplus}_{\beta} = th(f_{\mathcal{IS}}(\beta)) = th([K]) = K$. Hence $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$.

Otherwise $[K] \cap [\alpha] \neq \emptyset$ and $[K] \cap [\beta] \neq \emptyset$ (i.e., $\neg\alpha \notin K$ and $\neg\beta \notin K$ by Lemma B.8) and there is some smallest sphere $c_{\mathcal{IS}}(\alpha)$ intersecting $[\alpha]$ and some smallest sphere $c_{\mathcal{IS}}(\beta)$ intersecting $[\beta]$ (since $c_{\mathcal{IS}}(\alpha), c_{\mathcal{IS}}(\beta) \subseteq [K]$ by (IS2) and (IS3)). Now since $[K] \cap [\alpha] = [K] \cap [\beta]$, then, for any $m \in \mathcal{M}_{\mathcal{L}}$, $m \in [K] \cap [\alpha]$ if and only if $m \in [K] \cap [\beta]$. It follows that $c_{\mathcal{IS}}(\alpha) = c_{\mathcal{IS}}(\beta)$ and consequently $[\alpha] \cap c_{\mathcal{IS}}(\alpha) = [\beta] \cap c_{\mathcal{IS}}(\beta)$. So $f_{\mathcal{IS}}(\alpha) = f_{\mathcal{IS}}(\beta)$ and $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th(f_{\mathcal{IS}}(\beta)) = K^{\oplus}_{\beta}$ by Lemma 2.2.14 (iv). Hence $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$.

(K$^{\oplus}$7) $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$

If $[K] \cap [\alpha] = \emptyset$ (i.e., $\neg\alpha \in K$ by Lemma B.8) then $f_{\mathcal{IS}}(\alpha) = [K]$ by definition (Def $f_{\mathcal{IS}}$). Now $K^{\oplus}_{\alpha} = th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ (the latter part by Lemma 2.2.14 (i)). Also $K \subseteq K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$3) which is satisfied (see above). Therefore $K^{\oplus}_{\alpha} \subseteq K^{\oplus}_{\alpha\vee\beta}$ and so certainly $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$.

Otherwise $[K] \cap [\alpha] \neq \emptyset$ (i.e., $\neg\alpha \notin K$ by Lemma B.8). Now, in the logics that we are considering here, $\vdash \alpha \rightarrow \alpha \vee \beta$. Therefore, any maximal consistent extension containing α also contains $\alpha \vee \beta$ (i.e., $[\alpha] \subseteq [\alpha \vee \beta]$ — in fact, $[\alpha \vee \beta] = [\alpha] \cup [\beta]$) meaning that any sphere that intersects $[\alpha]$ also intersects $[\alpha \vee \beta]$. It follows then, that $[K] \cap [\alpha \vee \beta] \neq \emptyset$ (i.e., $\neg(\alpha \vee \beta) \notin K$ by Lemma B.8) and so, according to (IS3), there is some smallest sphere $c_{\mathcal{IS}}(\alpha)$ intersecting $[\alpha]$ and some smallest sphere $c_{\mathcal{IS}}(\alpha \vee \beta)$ intersecting $[\alpha \vee \beta]$. Moreover, by the same reasoning, it follows that $c_{\mathcal{IS}}(\alpha \vee \beta) \subseteq c_{\mathcal{IS}}(\alpha)$. Surely then, $[\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta) \cap [\alpha] \subseteq [\alpha] \cap c_{\mathcal{IS}}(\alpha)$. Therefore $th([\alpha] \cap c_{\mathcal{IS}}(\alpha)) \subseteq th([\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta) \cap [\alpha])$ by Lemma 2.2.14 (iv). Now $th([\alpha] \cap c_{\mathcal{IS}}(\alpha)) = th(f_{\mathcal{IS}}(\alpha)) = K^{\oplus}_{\alpha}$ by definition. Also, $th([\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta) \cap [\alpha]) = Cn(th([\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta)) \cup \{\alpha\})$ by Lemma 2.2.14 (iii) and $th([\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta)) = th(f_{\mathcal{IS}}(\alpha \vee \beta)) = K^{\oplus}_{\alpha\vee\beta}$. So $Cn(th([\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta)) \cup \{\alpha\}) = Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$. Therefore $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ as desired.

(K$^{\oplus}$8) If $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$. We need to show $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$.

Since (K$^{\oplus}$3) is satisfied (see above), then $\neg\alpha \notin K$ and so certainly $\neg\alpha\wedge\neg\beta \notin K$ (i.e., $\neg(\alpha \vee \beta) \notin K$). Therefore $[\alpha \vee \beta] \cap [K] \neq \emptyset$ by Lemma B.8 and so $f_{\mathcal{IS}}(\alpha \vee \beta) = [\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta)$ by definition (Def $f_{\mathcal{IS}}$). Similarly, $[\alpha] \cap [K] \neq \emptyset$ by Lemma B.8 and so $f_{\mathcal{IS}}(\alpha) = [\alpha] \cap c_{\mathcal{IS}}(\alpha)$ by definition (Def $f_{\mathcal{IS}}$).

Now, since $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, there is some $m \in [K^{\oplus}_{\alpha\vee\beta}]$ such that $\neg\alpha \notin m$ (i.e., since m is maximal consistent, $\alpha \in m$). Moreover, since $K^{\oplus}_{\alpha\vee\beta} = th(f_{\mathcal{IS}}(\alpha\vee\beta)) = th([\alpha\vee\beta]\cap c_{\mathcal{IS}}(\alpha\vee\beta))$ by definition, then $m \in c_{\mathcal{IS}}(\alpha\vee\beta)$. It follows that $c_{\mathcal{IS}}(\alpha) \subseteq c_{\mathcal{IS}}(\alpha \vee \beta)$ since $\alpha \in m$. So, keeping in mind that $[\alpha] \subseteq [\alpha \vee \beta]$, $[\alpha] \cap c_{\mathcal{IS}}(\alpha) \subseteq [\alpha] \cap c_{\mathcal{IS}}(\alpha \vee \beta) \subseteq [\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta)$. Therefore $th([\alpha\vee\beta]\cap c_{\mathcal{IS}}(\alpha\vee\beta)) \subseteq th([\alpha]\cap c_{\mathcal{IS}}(\alpha))$ by Lemma 2.2.14 (iv). Now $th([\alpha \vee \beta] \cap c_{\mathcal{IS}}(\alpha \vee \beta)) = th(f_{\mathcal{IS}}(\alpha \vee \beta)) = K^{\oplus}_{\alpha\vee\beta}$ by definition and $th([\alpha] \cap c_{\mathcal{IS}}(\alpha)) = th(f_{\mathcal{IS}}(\alpha)) = K^{\oplus}_{\alpha}$ by definition. Hence $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$ as desired.

■

Theorem 5.3.13 *Let $\oplus : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$ be any function satisfying postulates* (K$^{\oplus}$1)—(K$^{\oplus}$8). *Then for any belief set $K \in \mathcal{K}$ there is an internal system of spheres $\mathcal{IS}$ on $\mathcal{M}_{\mathcal{L}}$ centred within $[K]$ which, for all $\alpha \in \mathcal{L}$, satisfies $K^{\oplus}_{\alpha} = th(f_{\mathcal{S}}(\alpha))$.*

Proof:

Let $\oplus$ satisfy postulates (K$^{\oplus}$1) — (K$^{\oplus}$8). We show how to construct such an internal system of spheres $\mathcal{IS}$ centred within $[K]$.

We begin by letting $\mathcal{IS}'$ be the class of all nonempty subsets U of $\mathcal{M}_{\mathcal{L}}$ such that the following two conditions hold (this is based on a similar construction by Grove [39] Theorem 2):

1. $\forall u \in U, \exists\alpha \in \mathcal{L}$ such that $u \in [K^{\oplus}_{\alpha}]$

2. If $[\alpha] \cap U \neq \emptyset$ for any $\alpha \in \mathcal{L}$, then $[K^{\oplus}_{\alpha}] \subseteq u$

Now, we let

$$\mathcal{IS} = \begin{cases} \mathcal{IS}' \cup \{[K]\} & \text{if } K \neq K_{\perp} \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

It is straightforward to show that $\mathcal{IS}$, thus defined is an internal system of spheres centred within $[K]$. We need to verify that $K^{\oplus}_{\alpha}$ can be obtained from $\mathcal{IS}$. That is, we need to show that, for all $\alpha \in \mathcal{L}$, $K^{\oplus}_{\alpha} = th(f_{\mathcal{S}}(\alpha))$. We consider two cases.

Case (i): $\neg\alpha \in K$

> Then by (K$^{\oplus}$4), $K^{\oplus}_{\alpha} = K$. Moreover, $[K] \cap [\alpha] = \emptyset$ by Lemma B.8 and so $f_{\mathcal{IS}}(\alpha) = [K]$ by definition (Def $f_{\mathcal{IS}}$). Now $th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ (the latter part by Lemma 2.2.14 (i)) as desired.

Case (ii): $\neg\alpha \notin K$ (and therefore $K \neq K_{\perp}$)

> We need to show $K^{\oplus}_{\alpha} = th(f_{\mathcal{S}}(\alpha)) = th([\alpha] \cap c_{\mathcal{IS}}(\alpha))$ and do so by showing how to find such a $c_{\mathcal{IS}}(\alpha)$.
>
> Now $[K^{\oplus}_{\alpha}] \subseteq [K] \cap (\bigcap\{U \in \mathcal{IS} : U \cap [\alpha] \neq \emptyset\})$ from the definition above. To show equality it is sufficient to find a sphere $U \in \mathcal{IS}$ such that $[K^{\oplus}_{\alpha}] = [K] \cap U$ (this U will correspond to the sphere $c_{\mathcal{IS}}(\alpha)$ that we are attempting to find). We let U be the following
>
> $$U = \bigcap\{[K^{\oplus}_{\beta}] : [\alpha] \subseteq [\beta] \text{ and } \beta \in \mathcal{L}\}$$
>
> We note that $U \subseteq [K]$ (i.e., $K \subseteq th(U)$ by Lemma 2.2.14 (i) and (iv)) since $\oplus$ satisfies (K$^{\oplus}$3) and so $K \subseteq K^{\oplus}_{\beta}$ for every $\beta \in \mathcal{L}$ (i.e., $[K^{\oplus}_{\beta}] \subseteq [K]$ by Lemma 2.2.14 (v)). We first show that U is a sphere and then proceed to show $[K^{\oplus}_{\alpha}] \subseteq [\alpha] \cap U$.
>
> Condition 1 above holds directly from the definition of U so it remains to show Condition 2. Suppose $[\delta] \cap U \neq \emptyset$ for some $\delta \in \mathcal{L}$ (i.e., $[\delta] \cap [K^{\oplus}_{\beta}] \neq \emptyset$ for some $\beta \in \mathcal{L}$ such that $[\alpha] \subseteq [\beta]$) we need to show $[K^{\oplus}_{\delta}] \subseteq U$. Consider $\beta \vee \delta$. Now

$[\alpha] \subseteq [\beta] \subseteq [\beta \vee \delta]$, so $[K^{\oplus}_{\beta\vee\delta}] \subseteq U$ by definition of U (certainly $[\alpha] \subseteq [\beta \vee \delta]$ and so, if $m \in [K^{\oplus}_{\beta\vee\delta}]$, then $m \in U$). It follows, by the ensuing argument, that $[\delta] \cap [K^{\oplus}_{\beta\vee\delta}] \neq \emptyset$.

> Suppose for *reductio ad absurdum* that $[\delta] \cap [K^{\oplus}_{\beta\vee\delta}] = \emptyset$. Then $\neg\delta \in K^{\oplus}_{\beta\vee\delta}$ by Lemma B.8. Now, since $[\delta] \cap U \neq \emptyset$ then surely $[\beta \vee \delta] \cap U \neq \emptyset$ and, since $U \subseteq [K]$, then $[\beta \vee \delta] \cap [K] \neq \emptyset$. That is, $\neg(\beta \vee \delta) \notin K$ by Lemma B.8 and, by (K$^{\oplus}$2), $\beta \vee \delta \in K^{\oplus}_{\beta\vee\delta}$. Putting this together with the fact that $\neg\delta \in K^{\oplus}_{\beta\vee\delta}$ and using (K$^{\oplus}$1) we obtain $\beta \in K^{\oplus}_{\beta\vee\delta}$ and so by (K$^{\oplus}$5) $\neg\beta \notin K^{\oplus}_{\beta\vee\delta}$. Now by (K$^{\oplus}$7), $K^{\oplus}_{\beta} \subseteq Cn(K^{\oplus}_{\beta\vee\delta} \cup \{\beta\})$. But $\beta \in K^{\oplus}_{\beta\vee\delta}$ and so using (K$^{\oplus}$1) we get $K^{\oplus}_{\beta} \subseteq K^{\oplus}_{\beta\vee\delta}$. Using (K$^{\oplus}$8) gives the converse, $K^{\oplus}_{\beta\vee\delta} \subseteq K^{\oplus}_{\beta}$ and so $K^{\oplus}_{\beta} = K^{\oplus}_{\beta\vee\delta}$. It follows then, that $\neg\delta \in K^{\oplus}_{\beta}$. That is, by Lemma B.3 $[\delta] \cap [K^{\oplus}_{\beta}] = \emptyset$. This contradicts our initial supposition. Hence $[\delta] \cap [K^{\oplus}_{\beta\vee\delta}] \neq \emptyset$.

By Lemma B.3 $\neg\delta \notin K^{\oplus}_{\beta\vee\delta}$. So by (K$^{\oplus}$8) $K^{\oplus}_{\beta\vee\delta} \subseteq K^{\oplus}_{\delta}$ and by Lemma 2.2.14 (v) it follows that $[K^{\oplus}_{\delta}] \subseteq [K^{\oplus}_{\beta\vee\delta}]$. But we noted above that $[K^{\oplus}_{\beta\vee\delta}] \subseteq U$. Therefore $[K^{\oplus}_{\delta}] \subseteq U$ as desired. Hence U is a sphere.

We now show that $[K^{\oplus}_{\alpha}] = [\alpha] \cap U$. If $[\alpha] \cap [K^{\oplus}_{\beta}] \neq \emptyset$ (by the definition of U there is not much point in considering the case where $[\alpha] \cap [K^{\oplus}_{\beta}] = \emptyset$), then by Lemma B.8 $\neg\alpha \notin K^{\oplus}_{\beta}$. Moreover, by (K$^{\oplus}$3), $K \subseteq K^{\oplus}_{\beta}$ and so $\neg\alpha \notin K$. So, again by Lemma B.8, $[\alpha] \cap [K] \neq \emptyset$. Also, since $[\alpha] \subseteq [\beta]$ then $\vdash \alpha \rightarrow \beta$ and so $\vdash \beta \leftrightarrow \alpha \vee \beta$. So $K^{\oplus}_{\beta} = K^{\oplus}_{\alpha\vee\beta}$ by (K$^{\oplus}$6) (which is also a consequence of (K$^{\oplus}$7) and (K$^{\oplus}$8)). Therefore $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ and by (K$^{\oplus}$8) $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$. Also then $K^{\oplus}_{\beta} \subseteq K^{\oplus}_{\alpha}$ and so $Cn(K^{\oplus}_{\beta} \cup \{\alpha\}) \subseteq Cn(K^{\oplus}_{\alpha} \cup \{\alpha\})$ by (Monotonicity). But, since $\neg\alpha \notin K$ and by (K$^{\oplus}$2), then $\alpha \in K^{\oplus}_{\alpha}$. Therefore (with the help of (K$^{\oplus}$1)) $Cn(K^{\oplus}_{\beta} \cup \{\alpha\}) \subseteq K^{\oplus}_{\alpha}$. Now (K$^{\oplus}$7) gives $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ which, by the above, gives $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\beta} \cup \{\alpha\})$. So (K$^{\oplus}$7) and (K$^{\oplus}$8) give $K^{\oplus}_{\alpha} = Cn(K^{\oplus}_{\beta} \cup \{\alpha\})$ (i.e., $[K^{\oplus}_{\alpha}] = [K^{\oplus}_{\beta}] \cap [\alpha]$ by Lemma 2.2.14 (v)). Therefore $[\alpha] \cap U = \bigcup\{[\alpha] \cap [K^{\oplus}_{\beta}] : [\alpha] \subseteq [\beta]\}$. That is, $[\alpha] \cap U$ is the union of a number of terms where each term is either $\emptyset$ or $[K^{\oplus}_{\alpha}]$ and at least one of these terms is $[K^{\oplus}_{\alpha}]$ (viz., $[\alpha] \cap [K^{\oplus}_{\alpha}]$) — note that we have already dealt with the situation in which $\neg\alpha \in K$ in case (i) and so $\neg\alpha \notin K$ and by (K$^{\oplus}$5) $\neg\alpha \notin K^{\oplus}_{\alpha}$ guaranteeing $[\alpha] \cap [K^{\oplus}_{\alpha}] \neq \emptyset$ by Lemma B.8). Therefore $[\alpha] \cap U = [K^{\oplus}_{\alpha}]$ as desired. (Applying Lemma 2.2.14 (iv) one obtains $K^{\oplus}_{\alpha} = th([\alpha] \cap U)$ and U can be equated with $c_{\mathcal{IS}}(\alpha)$ as noted above.) ■

Lemma 5.3.14 *If $\leq$ satisfies postulates* (SEE1)—(SEE3) *and* (AE4)*. then it also has the following properties:*

(*i***)** *If $\alpha \in K$ and $\beta \notin K$ then, $\beta < \alpha$*

(*ii***)** *When $K \neq K_{\perp}$, if $\neg\alpha \in K$ then $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$*

(*iii***)** *When $K \neq K_{\perp}$, if $K \cup \{\alpha\} \vdash \beta$, then $\alpha \leq \beta$.*

Proof:

(*i*) If $\alpha \in K$ and $\beta \notin K$ then, $\beta < \alpha$

Suppose $\alpha \in K$ and $\beta \notin K$. Clearly $K \neq K_{\perp}$. Since $\alpha \in K$, then $\gamma \leq \alpha$ for all $\gamma \in \mathcal{L}$ by (AE4) and so $\beta \leq \alpha$. Suppose, for *reductio ad absurdum*, that $\alpha \leq \beta$. Therefore, by (SEE1) and the conclusion above that $\gamma \leq \alpha$ for all $\gamma \in \mathcal{L}$, then $\alpha \leq \beta$ for all $\gamma \in \mathcal{L}$. So $\beta \in K$ by (AE4) contradicting our initial supposition. Therefore $\alpha \not\leq \beta$ and consequently $\beta < \alpha$.

(*ii*) When $K \neq K_{\perp}$, if $\neg\alpha \in K$ then $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$

Let $K \neq K_{\perp}$. Suppose $\neg\alpha \in K$. Now $\vdash (\alpha \wedge \neg\alpha) \leftrightarrow \perp$ so $\alpha \wedge \neg\alpha = \perp$. Also, $\alpha \wedge \neg\alpha = min\{\alpha, \neg\alpha\}$ by Lemma 2.2.18 (xii). Since $\neg\alpha \in K$ and $K \neq K_{\perp}$, then $\alpha \notin K$. So, by part (i), $\alpha < \neg\alpha$ and so $\alpha \wedge \neg\alpha = \alpha = \perp$. However, $\perp \vdash \beta$ for all $\beta \in \mathcal{L}$. So, by (SEE2), $\perp \leq \beta$ for all $\beta \in \mathcal{L}$. Therefore $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$.

(*iii*) When $K \neq K_\perp$, if $K \cup \{\alpha\} \vdash \beta$, then $\alpha \leq \beta$.

Let $K \neq K_\perp$. Suppose $K \cup \{\alpha\} \vdash \beta$. So $K \vdash \alpha \rightarrow \beta$ by (Deduction) and consequently $\alpha \rightarrow \beta \in K$. By (AE4), $\gamma \leq \alpha \rightarrow \beta$ for all $\gamma \in \mathcal{L}$. In particular, $\alpha \leq \alpha \rightarrow \beta$. So $\alpha \wedge \alpha \leq (\alpha \rightarrow \beta) \wedge \alpha$ by 2.2.18 (x). Now $\vdash (\alpha \wedge \alpha) \leftrightarrow \alpha$ so $\alpha \wedge \alpha = \alpha$. Also $(\alpha \wedge \alpha) \leftrightarrow \alpha \vdash \beta$ and consequently $(\alpha \wedge \alpha) \leftrightarrow \alpha \leq \beta$ by (SEE2). Putting these together we get $\alpha \leq \alpha \wedge \alpha \leq (\alpha \wedge \alpha) \leftrightarrow \alpha \leq \beta$. Therefore, (SEE1) gives, $\alpha \leq \beta$ as desired.

■

Lemma 5.3.15 *Let $\leq$ be an SEE relation satisfying* (SEE1)—(SEE3). *For any $\alpha \in \mathcal{L}$, either $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$ or $\neg\alpha \leq \beta$ for all $\beta \in \mathcal{L}$.*

Proof: Now $\vdash (\alpha \wedge \neg\alpha) \leftrightarrow \perp$. So $\alpha \wedge \neg\alpha = \perp$. Also, $\alpha \wedge \neg\alpha = min\{\alpha, \neg\alpha\}$ by Lemma 2.2.18 (xii). Suppose $\alpha \leq \neg\alpha$ so $\alpha \wedge \neg\alpha = \alpha = \perp$. However, $\perp \vdash \beta$ for all $\beta \in \mathcal{L}$. So, by (SEE2), $\perp \leq \beta$ for all $\beta \in \mathcal{L}$. Therefore, $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$. Otherwise $\neg\alpha \leq \alpha$ and a similar argument gives $\neg\alpha \leq \beta$ for all $\beta \in \mathcal{L}$. ■

Lemma B.10 *Let $\leq$ be an abductive entrenchment relation. Suppose $\alpha \leq \beta$ and $\beta \wedge \delta < \alpha \wedge \gamma$ for any $\alpha,\ \beta,\ \gamma,\ \delta \in \mathcal{L}$. Then the following properties hold:*

(i) $\delta < \gamma$

(ii) $\delta < \alpha$

(iii) $\delta < \beta$

Proof:

Now since $\beta \wedge \delta < \alpha \wedge \gamma$ then either $\beta < \alpha \wedge \gamma$ or $\delta < \alpha \wedge \gamma$ by Lemma 2.2.18 (ix). The first case is clearly not possible since $\alpha \leq \beta$ and $\alpha \wedge \gamma = min\{\alpha, \gamma\}$ by Lemma 2.2.18 (xii). So $\delta < \alpha \wedge \gamma$ and, again since $\alpha \wedge \gamma = min\{\alpha, \gamma\}$ by Lemma 2.2.18 (xii), clearly (i) $\delta < \gamma$ and (ii) $\delta < \alpha$. It remains to show part (iii). Now since $\delta < \alpha$ and $\alpha \leq \beta$, then $\delta < \beta$ by Lemma 2.2.18 (ii). ■

Lemma B.11 *Let $\leq$ be an abductive entrenchment relation. If $\alpha < \beta$ and $\gamma < \delta$, then $\alpha \wedge \gamma < \beta \wedge \delta$ for any $\alpha,\ \beta,\ \gamma \in \mathcal{L}$.*

Proof:

Let $\alpha < \beta$ and $\gamma < \delta$. Now $\{\alpha \wedge \gamma\} \vdash \alpha$ so $\alpha \wedge \gamma \leq \alpha$ by (SEE2). Also $\alpha < \beta$ therefore it follows that $\alpha \wedge \gamma < \beta$ by Lemma 2.2.18 (ii). Similarly $\{\alpha \wedge \gamma\} \vdash \gamma$ so $\alpha \wedge \gamma \leq \gamma$ by (SEE2) and since $\gamma < \delta$, then $\alpha \wedge \gamma < \gamma$ by Lemma 2.2.18 (ii). Therefore $\alpha \wedge \gamma < \gamma \wedge \delta$ by Lemma 2.2.18 (iii). ■

Theorem 5.3.16 *Let $K \in \mathcal{K}$ be some belief set and $\leq$ an abductive entrenchment for K. If for any $\alpha \in \mathcal{L}$, we define $K^{\oplus}_{\alpha}$ using $(C\oplus)$, then the operation $\oplus$ so defined satisfies postulates* (K$^{\oplus}$1)—(K$^{\oplus}$8) *as well as the condition* (C$\leq$).

Proof:

Let $\alpha \in \mathcal{L}$ and $K^{\oplus}_{\alpha}$ be defined using condition (C$\oplus$). We first show that each of the (K$^{\oplus}$1)—(K$^{\oplus}$8) are satisfied and then show that condition (C$\leq$) is satisfied.

Recall condition (C$\oplus$): $\beta \in K^{\oplus}_{\alpha}$ iff $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$.

($\mathrm{K}^{\oplus}1$) $K^{\oplus}_{\alpha}$ is a belief set

Suppose $K^{\oplus}_{\alpha} \vdash \gamma$. So there are $\beta_1, \ldots, \beta_n \in K^{\oplus}_{\alpha}$ with $\{\beta_1, \ldots, \beta_n\} \vdash \gamma$ by (Compactness). We need to show $\gamma \in K^{\oplus}_{\alpha}$. We can do this if we can show, by (C$\oplus$), $\gamma \in K$ or both $\neg\alpha \notin K$ and $\alpha \to \neg\gamma < \alpha \to \gamma$.

If $\gamma \in K$ then $\gamma \in K^{\oplus}_{\alpha}$ directly so we suppose $\gamma \notin K$ and we need to show $\neg\alpha \notin K$ and $\alpha \to \neg\gamma < \alpha \to \gamma$. If $\neg\alpha \in K$ then, by (C$\oplus$), $\beta_1, \ldots, \beta_n \in K^{\oplus}_{\alpha}$ iff $\beta_1, \ldots, \beta_n \in K$. Therefore, since K is a belief set, $\gamma \in K$. This contradicts the supposition above and we may conclude $\neg\alpha \notin K$.

It remains to show $\alpha \to \neg\gamma < \alpha \to \gamma$. Now $\{\beta_1, \ldots, \beta_n\} \vdash \gamma$ for $\beta_1, \ldots, \beta_n \in K^{\oplus}_{\alpha}$. In the limiting case where $n = 0$ we get $\gamma \in K$ since K is a belief set. But we are supposing $\gamma \notin K$ so this case is not possible. Without loss of generality assume $\beta_1, \ldots, \beta_k \in K$. If $k = n$ then $\gamma \in K$ since K is a a belief set. Again, this contradicts our supposition so we may assume $k < n$, $\beta_{k+1}, \ldots, \beta_n \notin K$ and, for each $i = k+1 \leq i \leq n$, $\alpha \to \neg\beta_i < \alpha \to \beta_i$ using (C$\oplus$).

Note that since $\beta_1, \ldots, \beta_k \in K$, then $\delta < \beta_i$ for all $\delta \notin K$ and $1 \leq i \leq k$ by Lemma 5.3.14 (ii). Now $\beta_i \vdash \alpha \to \beta_i$ so $\alpha \to \beta_i \in K$ for $1 \leq i \leq k$ and therefore $\delta < \alpha \to \beta_i$ for all $\delta \notin K$. Consider $\alpha \to \neg\beta_i$ for all $1 \leq i \leq k$. If $\alpha \to \neg\beta_i \in K$ then since K is a belief set and $\alpha \to \beta_i \in K$ it follows that $\neg\alpha \in K$. But this contradicts our supposition above so $\alpha \to \neg\beta_i \notin K$ and consequently $\alpha \to \neg\beta_i < \alpha \to \beta_i$. Combining this with the fact above, that $\alpha \to \neg\beta_i < \alpha \to \beta_i$ for $k+1 \leq i \leq n$ and Lemma 2.2.18 (xii), we see that $(\alpha \to \neg\beta_i) \wedge \ldots \wedge (\alpha \to \neg\beta_n) < (\alpha \to \beta_i) \wedge \ldots \wedge (\alpha \to \beta_n)$ However $\vdash [(\alpha \to \neg\beta_i) \wedge \ldots \wedge (\alpha \to \neg\beta_n)] \leftrightarrow [\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n)]$ and $\vdash [(\alpha \to \beta_i) \wedge \ldots \wedge (\alpha \to \beta_n)] \leftrightarrow [\alpha \to (\beta_1 \wedge \ldots \wedge \beta_n)]$. Therefore, through the use of (SEE2), $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) < \alpha \to (\beta_1 \wedge \ldots \wedge \beta_n)$. (1)

Now since $\beta_1 \wedge \ldots \wedge \beta_n \vdash \gamma$ then $\alpha \to (\beta_1 \wedge \ldots \wedge \beta_n) \vdash \alpha \to \gamma$. So $\alpha \to (\beta_1 \wedge \ldots \wedge \beta_n) \leq \alpha \to \gamma$ by (SEE2). (2)

Suppose $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) < \alpha \to \neg\gamma$. Now (1) and (2), by Lemma 2.2.18 (ii), produce $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) < \alpha \to \gamma$. Also

$\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) \wedge (\alpha \to \gamma) \vdash \alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n)$ so $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) \wedge (\alpha \to \gamma) \leq \alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n)$ by (SEE2). By Lemma 2.2.18 (iii), $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) < (\alpha \to \neg\gamma) \wedge (\alpha \to \gamma)$. Then $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) \wedge (\alpha \to \gamma) < (\alpha \to \neg\gamma) \wedge (\alpha \to \gamma)$ by Lemma 2.2.18 (ii). But $\vdash [\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) \wedge (\alpha \to \gamma)] \leftrightarrow [\alpha \to (\neg\beta_1 \wedge \ldots \neg\beta_n \wedge \gamma)]$ and $\vdash [(\alpha \to \neg\gamma) \wedge (\alpha \to \gamma)] \leftrightarrow \neg\alpha$. Therefore $\alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \gamma) < \neg\alpha$. However $\neg\alpha \vdash \alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \gamma)$ so $\neg\alpha \leq \alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \gamma)$ by (SEE2) which is a contradiction. So our initial supposition was incorrect and it follows that $\alpha \to \neg\gamma \leq \alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n)$. (3)

Putting together (1), (2) and (3) we get $\alpha \to \neg\gamma \leq \alpha \to (\neg\beta_1 \wedge \ldots \wedge \neg\beta_n) < \alpha \to (\beta_1 \wedge \ldots \beta_n) \leq \alpha \to \gamma$. Therefore $\alpha \to \neg\gamma < \alpha \to \gamma$ by (SEE1) as desired.

(K$^\oplus$2) If $\neg\alpha \notin K$, then $\alpha \in K^\oplus_\alpha$

If $\alpha \in K$ then $\alpha \in K^\oplus_\alpha$ by (C$\oplus$) directly.

Otherwise $\alpha \notin K$ (and so $K \neq K_\perp$). We need to show by (C$\oplus$) that $\alpha \to \neg\alpha < \alpha \to \alpha$. Now $\vdash \alpha \to \alpha$ so $\alpha \to \alpha \in K$. Also $\vdash (\alpha \to \neg\alpha) \to \neg\alpha$ and so $\alpha \to \neg\alpha \notin K$ since $\neg\alpha \notin K$ and K is a belief set. Therefore $\alpha \to \neg\alpha < \alpha \to \alpha$ by Lemma 5.3.14 (ii) as required.

(K$^\oplus$3) $K \subseteq K^\oplus_\alpha$

Suppose $\beta \in K$. Then, directly by (C$\oplus$), $\beta \in K^\oplus_\alpha$. Therefore $K \subseteq K^\oplus_\alpha$.

(K$^\oplus$4) If $\neg\alpha \in K$, then $K^\oplus_\alpha = K$

Let $\neg\alpha \in K$. (K$^\oplus$3) gives one direction (viz., $K \subseteq K^\oplus_\alpha$). We need to show $K^\oplus \subseteq K$. By (C$\oplus$), if $\beta \in K^\oplus_\alpha$ then either $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \to \neg\beta < \alpha \to \beta$. But $\neg\alpha \in K$ so the latter half is not possible and so if $\beta \in K^\oplus_\alpha$ then $\beta \in K$. So $K^\oplus_\alpha \subseteq K$. Therefore $K^\oplus_\alpha = K$.

(K$^{\oplus}$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K$. Suppose for *reductio ad absurdum* that $\neg\alpha \in K^{\oplus}_{\alpha}$. So $\alpha \to \neg(\neg\alpha) < \alpha \to \neg\alpha$ by condition (C$\oplus$). But $\vdash (\alpha \to \neg(\neg\alpha)) \leftrightarrow \top$ and $\vdash (\alpha \to \neg\alpha) \leftrightarrow \neg\alpha$ and therefore $\top < \neg\alpha$. However $\top \in K$ since it is a belief set and since $\neg\alpha \notin K$ it follows that $\alpha < \top$ by Lemma 5.3.14 (ii), contradicting the above. Hence $\neg\alpha \notin K^{\oplus}_{\alpha}$.

(K$^{\oplus}$6) If $K \vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$

Let $K \vdash \alpha \leftrightarrow \beta$. Suppose $\gamma \in K^{\oplus}_{\alpha}$. We need to show $\gamma \in K^{\oplus}_{\beta}$. Now, by Condition (C$\oplus$), either $\gamma \in K$ or both $\neg\alpha \notin K$ and $\alpha \to \neg\gamma < \alpha \to \gamma$. We need to show, according to Condition (C$\oplus$), that either $\gamma \in K$ or both $\neg\beta \notin K$ and $\beta \to \neg\gamma < \beta \to \gamma$. In the former case ($\gamma \in K$) we get by (C$\oplus$) that $\gamma \in K^{\oplus}_{\beta}$. So, assume $\gamma \notin K$, $\neg\alpha \notin K$ and $\alpha \to \neg\gamma < \alpha \to \gamma$. Clearly $K \neq K_{\perp}$. Now, since $K \vdash \alpha \leftrightarrow \beta$, we also have $K \vdash (\alpha \to \gamma) \leftrightarrow (\beta \to \gamma)$. and $K \vdash (\alpha \to \neg\gamma) \leftrightarrow (\beta \to \neg\gamma)$ In the first case we have, since K is a belief set, $(\alpha \to \gamma) \leftrightarrow (\beta \to \gamma) \in K$ and consequently $(\alpha \to \gamma) \leq (\beta \to \gamma)$ by Lemma 5.3.14 (iii). Similarly, in the latter case we have $(\beta \to \neg\gamma) \to (\alpha \to \neg\gamma) \in K$ and so Lemma 5.3.14 (iii) gives $(\beta \to \neg\gamma) \leq (\alpha \to \neg\gamma)$. Putting these together we get $(\beta \to \neg\gamma) \leq (\alpha \to \neg\gamma) < (\alpha \to \gamma) \leq (\beta \to \gamma)$. Therefore, $(\beta \to \neg\gamma) < (\beta \to \gamma)$ by transitivity.

(K$^{\oplus}$7) $K^{\oplus}_{\alpha} \subseteq Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$

Suppose $\gamma \in K^{\oplus}_{\alpha}$. We need to show $\gamma \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$.

If $\gamma \in K$, then by condition (C$\oplus$), $\gamma \in K^{\oplus}_{\alpha\vee\beta}$. Therefore $\gamma \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$ by (Monotonicity).

Otherwise $\gamma \notin K$. Now $\gamma \in K^{\oplus}_{\alpha}$ so $\neg\alpha \notin K$ and $\alpha \to \neg\gamma < \alpha \to \gamma$ by condition (C$\oplus$). We need to show $\gamma \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$. We can do this by showing that $\alpha \to \gamma \in Cn(K^{\oplus}_{\alpha\vee\beta})$ by (Deduction) and, since (K$^{\oplus}$1) is satisfied (see above), that would be the same as showing that $\alpha \to \gamma \in K^{\oplus}_{\alpha\vee\beta}$.

We note that since $\neg\alpha \notin K$, then $\neg\alpha \wedge \neg\beta \notin K$ (i.e., $\neg(\alpha \vee \beta) \notin K$) so, by condition (C$\oplus$), we need to show $(\alpha\vee\beta) \rightarrow \neg(\alpha \rightarrow \gamma) < (\alpha\vee\beta) \rightarrow (\alpha \rightarrow \gamma)$. Now $\{(\alpha \vee \beta) \rightarrow \neg(\alpha \rightarrow \gamma)\} \vdash \alpha \rightarrow \neg\gamma$ so $(\alpha \vee \beta) \rightarrow \neg(\alpha \rightarrow \gamma) \leq \alpha \rightarrow \neg\gamma$ by (SEE2). Also $\{\alpha \rightarrow \gamma\} \vdash (\alpha \vee \beta) \rightarrow (\alpha \rightarrow \gamma)$ therefore $\alpha \rightarrow \gamma \leq (\alpha \vee \beta) \rightarrow (\alpha \rightarrow \gamma)$. So $(\alpha \vee \beta) \rightarrow \neg(\alpha \rightarrow \gamma) \leq \alpha \rightarrow \neg\gamma < \alpha \rightarrow \gamma \leq (\alpha \vee \beta) \rightarrow (\alpha \rightarrow \gamma)$. which, by (SEE1) and Lemma 2.2.18 (ii) gives $(\alpha \vee \beta) \rightarrow \neg(\alpha \rightarrow \gamma) < \alpha \vee \beta) \rightarrow (\alpha \rightarrow \gamma)$. Therefore $\alpha \rightarrow \gamma \in K^{\oplus}_{\alpha\vee\beta}$ and consequently $\gamma \in Cn(K^{\oplus}_{\alpha\vee\beta} \cup \{\alpha\})$.

(K$^{\oplus}$8) If $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, then $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$

Let $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$. Then, by condition (C$\oplus$), $\neg\alpha \notin K$ and either $\neg(\alpha \vee \beta) \in K$ or $(\alpha \vee \beta) \rightarrow \alpha \not< (\alpha \vee \beta) \rightarrow \neg\alpha$. We need to show $K^{\oplus}_{\alpha\vee\beta} \subseteq K^{\oplus}_{\alpha}$. That is, if $\gamma \in K^{\oplus}_{\alpha\vee\beta}$, then $\gamma \in K^{\oplus}_{\alpha}$. By condition (C$\oplus$), $\gamma \in K^{\oplus}_{\alpha\vee\beta}$ if and only if $\gamma \in K$ or both $\neg(\alpha \vee \beta) \notin K$ and $(\alpha \vee \beta) \rightarrow \neg\gamma < (\alpha \vee \beta) \rightarrow \gamma$.

If $\gamma \in K$, then $\gamma \in K^{\oplus}_{\alpha}$ by condition (C$\oplus$). Otherwise $\gamma \notin K$ and so $\neg(\alpha \vee \beta) \notin K$ and $(\alpha \vee \beta) \rightarrow \neg\gamma < (\alpha \vee \beta) \rightarrow \gamma$. Now since $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$ and (K$^{\oplus}$3) satisfied (see above) then $\neg\alpha \notin K$.

Suppose for *reductio ad absurdum* that $\gamma \notin K^{\oplus}_{\alpha}$. So, by condition (C$\oplus$), $\gamma \notin K$ and either $\neg\alpha \in K$ or $\alpha \rightarrow \neg\gamma \not< \alpha \rightarrow \gamma$. However, we have seen that $\neg\alpha \notin K$ so $\gamma \notin K$ and $\alpha \rightarrow \neg\gamma \not< \alpha \rightarrow \gamma$ (i.e., $\alpha \rightarrow \gamma \leq \alpha \rightarrow \neg\gamma$ by Lemma 2.2.17 (i)).(1)

We also know, as noted above, that $(\alpha \vee \beta) \rightarrow \neg\gamma < (\alpha \vee \beta) \rightarrow \gamma$. Now $\vdash [(\alpha \vee \beta) \rightarrow \neg\gamma] \leftrightarrow [(\alpha \rightarrow \neg\gamma) \wedge (\beta \rightarrow \neg\gamma)]$ and $\vdash [(\alpha \vee \beta) \rightarrow \gamma] \leftrightarrow [(\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)]$. Consequently $(\alpha \rightarrow \neg\gamma) \wedge (\beta \rightarrow \neg\gamma) < (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$.(2)

Moreover, since $\neg\alpha \notin K^{\oplus}_{\alpha\vee\beta}$, the latter part of condition (C$\oplus$) gives $(\alpha \vee \beta) \rightarrow \neg(\neg\alpha) \not< (\alpha \vee \beta) \rightarrow \neg\alpha$. But $\vdash [(\alpha \vee \beta) \rightarrow \neg(\neg\alpha)] \leftrightarrow (\beta \rightarrow \alpha)$ and $\vdash [(\alpha \vee \beta) \rightarrow \neg\alpha] \leftrightarrow \neg\alpha$. So $\beta \rightarrow \alpha \not< \neg\alpha$ or, by Lemma 2.2.17 (i), $\neg\alpha \leq \beta \rightarrow \alpha$.(3)

Therefore, using (1) and (2) together with Lemma B.10, we get (a) $\beta \rightarrow \neg\gamma < \beta \rightarrow \gamma$, (b) $\beta \rightarrow \neg\gamma < \alpha \rightarrow \gamma$ and (c) $\beta \rightarrow \neg\gamma < \alpha \rightarrow \neg\gamma$. Now (b) and (c) give $(\beta \to \neg\gamma) \wedge (\beta \to \neg\gamma) < (\alpha \to \gamma) \wedge (\alpha \to \neg\gamma)$ by Lemma B.11. But $\vdash [(\beta \to \neg\gamma) \wedge (\beta \to \neg\gamma)] \leftrightarrow (\beta \to \neg\gamma)$ and $\vdash [(\alpha \to \gamma) \wedge (\alpha \to \neg\gamma)] \leftrightarrow \neg\alpha$. Therefore $\beta \to \neg\gamma < \neg\alpha$. Now $\neg\beta \vdash \beta \to \neg\gamma$ so $\neg\beta \leq \beta \to \neg\gamma$ by (SEE2) and it follows that $\neg\beta < \neg\alpha$ by Lemma 2.2.18 (ii). Now $\vdash \neg\beta \leftrightarrow [(\beta \to \alpha) \wedge (\beta \to \neg\alpha)]$ so $\neg\beta = min\{\beta \to \alpha,\ \beta \to \neg\alpha\}$ using this logical equivalence and Lemma 2.2.18 (xii). If $\neg\beta = \beta \to \alpha$ then $\beta \to \alpha < \neg\alpha$ since $\neg\beta < \neg\alpha$. But this contradicts (3) $\neg\alpha \leq \beta \to \alpha$. So $\neg\beta = \beta \to \neg\alpha$ and so $\beta \to \neg\alpha < \neg\alpha$ since $\neg\beta < \neg\alpha$. However, $\{\neg\alpha\} \vdash \beta \to \neg\alpha$ and therefore $\neg\alpha \leq \beta \to \neg\alpha$ by (SEE2). Therefore we have a contradiction in either case. Hence $\gamma \in K^{\oplus}_{\alpha}$.

Condition (C$\leq$) is satisfied.

Recall condition (C$\leq$): $\alpha \leq \beta$ iff $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$.

(If)

Let either $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$. We need to show $\alpha \leq \beta$.

If $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ then by condition (C$\oplus$) $\alpha \notin K$ and either $\alpha \wedge \beta \in K$ or $(\neg\alpha \vee \neg\beta) \to \neg\alpha \not< (\neg\alpha \vee \neg\beta) \to \alpha$. The case where $\alpha \notin K$ and $\alpha \wedge \beta \in K$ is obviously not possible since K is a belief set. Therefore suppose $\alpha \notin K$ and $(\neg\alpha \vee \neg\beta) \to \neg\alpha \not< (\neg\alpha \vee \neg\beta) \to \alpha$. Now, as noted above, $\vdash [(\neg\alpha \vee \neg\beta) \to \neg\alpha] \leftrightarrow (\alpha \to \beta)$ and $\vdash [(\neg\alpha \vee \neg\beta) \to \alpha] \leftrightarrow \alpha$. So $\alpha \to \beta \not< \alpha$ using (SEE2). Therefore using Lemma 2.2.17 (i) (connectivity of $\leq$) $\alpha \leq \alpha \to \beta$ and it follows that $\alpha \leq \alpha \wedge (\alpha \to \beta)$ by Lemma 2.2.17 (v). However $\vdash [\alpha \wedge (\alpha \to \beta)] \leftrightarrow (\alpha \wedge \beta)$. So $\alpha \leq \alpha \wedge \beta$ by (SEE2). Therefore $\alpha \leq \beta$ using Lemma 2.2.18 (xii) as desired.

Otherwise $K \vdash \alpha \wedge \beta$. Certainly then $\beta \in K$ and consequently $\gamma \leq \beta$ for all $\gamma \in \mathcal{L}$ by (AE4). In particular $\alpha \leq \beta$ as required.

(Only If)

Let $\alpha \leq \beta$ and suppose $\alpha \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$. We need to show $K \vdash \alpha \wedge \beta$. We note that it is possible to easily dispense with the case where $K = K_{\perp}$ since $K \vdash \alpha \wedge \beta$ trivially. Therefore, in addition to the above, we suppose $K \neq K_{\perp}$. Since $\alpha \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$, condition

(C$\oplus$) gives $\alpha \in K$ or both $\alpha \wedge \beta \notin K$ and $(\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha < (\neg\alpha \vee \neg\beta) \rightarrow \alpha$. Considering the former, if $\alpha \in K$, then $\gamma \leq \alpha$ for all $\gamma \in \mathcal{L}$ by (AE4). Now since $\alpha \leq \beta$ then $\gamma \leq \beta$ for all $\gamma \in \mathcal{L}$ by (SEE1). So by (AE4) again, $\beta \in K$. Therefore $K \vdash \alpha \wedge \beta$ as required.

Otherwise $\alpha \notin K$. Consequently $\alpha \wedge \beta \notin K$ and $(\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha < (\neg\alpha \vee \neg\beta) \rightarrow \alpha$. Now $\vdash [(\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha] \leftrightarrow (\alpha \rightarrow \beta)$ and $\vdash [(\neg\alpha \vee \neg\beta) \rightarrow \alpha] \leftrightarrow \alpha$. Therefore it follows by (SEE1) and (SEE2) that $\alpha \rightarrow \beta < \alpha$. Also $\beta \vdash \alpha \rightarrow \beta$ so by (SEE2) $\beta \leq \alpha \rightarrow \beta$. Therefore, since $\alpha \leq \beta$, $\alpha \leq \alpha \rightarrow \beta$ by (SEE1). This contradicts our conclusion above and so this case is not possible. That is, $\alpha \in K$ as desired. ■

Theorem 5.3.17 *Let* $\oplus : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$ *be any function satisfying* (K$^{\oplus}$1)—(K$^{\oplus}$8). *Then, for any belief set* $K \in \mathcal{K}$*, if we define* $\leq$ *using* (C$\leq$)*, then the relation* $\leq$ *so defined is an abductive entrenchment relation (i.e., it satisfies* (SEE1)—(SEE3) and (AE4) *and also satisfies condition* (C$\oplus$)*.*

Proof: Let $\oplus$ be a function satisfying the postulates for abductive expansion (K$^{\oplus}$1)—(K$^{\oplus}$8) and define $\leq$ using condition (C$\leq$). We first show that each of (SEE1)—(SEE3) and (AE4) are satisfied and then show that condition (C$\oplus$) is satisfied.

Recall condition (C$\leq$): $\alpha \leq \beta$ iff $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$.

(SEE1) If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ for any $\alpha,\ \beta,\ \gamma \in \mathcal{L}$.

> Suppose for *reductio ad absurdum* that $\alpha \leq \beta$ and $\beta \leq \gamma$ but $\alpha \not\leq \gamma$ for $\alpha,\ \beta,\ \gamma \in \mathcal{L}$. Using condition (C$\leq$) we get the following from our three hypotheses respectively
>
> (1) either $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$
> (2) either $\beta \notin K^{\oplus}_{\neg\beta\vee\neg\gamma}$ or $K \vdash \beta \wedge \gamma$
> (3) $\alpha \in K^{\oplus}_{\neg\alpha\vee\neg\gamma}$ and $K \nvdash \alpha \wedge \gamma$.

Consider $K \vdash \alpha \wedge \beta$ from hypothesis (1). If $K \vdash \alpha \wedge \beta$, then $K \vdash \alpha$ and $K \vdash \beta$. Therefore $\beta \in K$ since K is a belief set. So, by (K$^\oplus$3), $\beta \in K^\oplus_{\neg\beta\vee\neg\gamma}$. It follows from (2) that $K \vdash \beta \wedge \gamma$. Therefore $K \vdash \gamma$ and, since $K \vdash \alpha$, then $K \vdash \alpha \wedge \gamma$. However, this contradicts (3) so $K \nvdash \alpha \wedge \beta$.

Consider $K \vdash \beta \wedge \gamma$ from hypothesis (2). If $K \vdash \beta \wedge \gamma$, then $K \vdash \beta$ and $K \vdash \gamma$. Therefore $\gamma \in K$ since K is a belief set and consequently $\gamma \in K^\oplus_{\neg\alpha\vee\neg\gamma}$ by (K$^\oplus$3). Now $\alpha \in K^\oplus_{\neg\alpha\vee\neg\gamma}$ by (3) and so $\alpha \wedge \gamma \in K^\oplus_{\neg\alpha\vee\neg\gamma}$ by (K$^\oplus$1). So $\alpha \wedge \gamma \in K$ by (K$^\oplus$5) (apply its contrapositive) and use the fact that $\vdash (\alpha \wedge \gamma) \leftrightarrow \neg(\neg\alpha \vee \neg\gamma)$. Therefore $K \vdash \alpha \wedge \gamma$ which contradicts (3) and so $K \nvdash \beta \wedge \gamma$.

Now since $K \nvdash \alpha \wedge \beta$ and $K \nvdash \beta \wedge \gamma$ we have from (1), (2) and (3) above that the following hold $\alpha \notin K^\oplus_{\neg\alpha\vee\neg\beta}$, $\beta \notin K^\oplus_{\neg\beta\vee\neg\gamma}$, $\alpha \in K^\oplus_{\neg\alpha\vee\neg\gamma}$ and $K \nvdash \alpha \wedge \gamma$. We can obtain a contradiction by first using (K$^\oplus$7) to show $\alpha \in K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ and then using (K$^\oplus$8) twice to show the opposite.

Now $\alpha \in K^\oplus_{\neg\alpha\vee\neg\gamma}$ by (3). Therefore $\alpha \in Cn(K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma} \cup \{\neg\alpha \vee \neg\gamma\})$ by (K$^\oplus$7). Using (Deduction) we get $(\neg\alpha \vee \neg\gamma) \rightarrow \alpha \in Cn(K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma})$ and so, by (K$^\oplus$1), $(\neg\alpha \vee \neg\gamma) \rightarrow \alpha \in K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$. However $\vdash [(\neg\alpha \vee \neg\gamma) \rightarrow \alpha] \leftrightarrow \alpha$. Hence $\alpha \in K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$.

We now show that the opposite also holds thus obtaining a contradiction. That is, we need to show $\alpha \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$. Now $K \nvdash \alpha \wedge \beta$ by our argument above. Consequently $K \nvdash \alpha \wedge \beta \wedge \gamma$ and since K is a belief set $\alpha \wedge \beta \wedge \gamma \notin K$ (i.e., $\neg(\neg\alpha \vee \neg\beta \vee \neg\gamma) \notin K$ since K is a belief set and $\vdash (\alpha \wedge \beta \wedge \gamma) \leftrightarrow \neg(\neg\alpha \vee \neg\beta \vee \neg\gamma)$. Therefore certainly $\alpha \wedge \beta \wedge \gamma \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ by (K$^\oplus$5) and (K$^\oplus$1). So either $\alpha \wedge \beta \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ or $\gamma \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$. In the first case $\neg(\neg\alpha \vee \neg\beta) \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ by (K$^\oplus$1) and it follows by (K$^\oplus$8) that $K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma} \subseteq K^\oplus_{\neg\alpha\vee\neg\beta}$. In the second case ($\gamma \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$) we get $\beta \wedge \gamma \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ using (K$^\oplus$1). That is $\neg(\neg\beta \vee \neg\gamma) \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ again by (K$^\oplus$1). Therefore $K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma} \subseteq K^\oplus_{\neg\beta\vee\neg\gamma}$ by (K$^\oplus$8). Now since $\beta \notin K^\oplus_{\neg\beta\vee\neg\gamma}$ by (2), it follows that $\beta \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ and also $\alpha \wedge \beta \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ using (K$^\oplus$1). That is $\neg(\neg\alpha \wedge \neg\beta) \notin K^\oplus_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ again with the help of (K$^\oplus$1). So

$K^{\oplus}_{\neg\alpha\vee\neg\beta\vee\neg\gamma} \subseteq K^{\oplus}_{\neg\alpha\vee\neg\beta}$ by (K$^{\oplus}$8). In either case, then, $K^{\oplus}_{\neg\alpha\vee\neg\beta\vee\neg\gamma} \subseteq K^{\oplus}_{\neg\alpha\vee\neg\beta}$. But $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ by (1). Therefore $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta\vee\neg\gamma}$ as desired.

Hence our initial supposition was false and it follows that, if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.

(SEE2) If $\{\alpha\} \vdash \beta$, then $\alpha \leq \beta$ for any $\alpha,\ \beta \in \mathcal{L}$.

Let $\{\alpha\} \vdash \beta$. We need to show $\alpha \leq \beta$. By condition (C$\leq$) we can do this by showing either $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$. But, if $\alpha \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$ then by (K$^{\oplus}$1) and since $\{\alpha\} \vdash \beta$, $\alpha \wedge \beta \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$ (i.e., $\neg(\neg\alpha \vee \neg\beta) \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$. Therefore by (K$^{\oplus}$5) (apply its contrapositive) $\alpha \wedge \beta \in K$ and so $K \vdash \alpha \wedge \beta$ as required.

(SEE3) $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$ for any $\alpha,\ \beta \in \mathcal{L}$.

We need to show $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$. We can do this by showing, according to condition (C$\leq$), that $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg(\alpha\wedge\beta)}$ or $K \vdash \alpha \wedge (\alpha \wedge \beta)$ or $\beta \notin K^{\oplus}_{\neg\alpha\vee\neg(\alpha\wedge\beta)}$ or $K \vdash \alpha \wedge (\alpha \wedge \beta)$.

Now since $\vdash [\neg\alpha \vee \neg(\alpha \wedge \beta)] \leftrightarrow (\neg\alpha \vee \neg\beta)$ and $\vdash [\alpha \wedge (\alpha \wedge \beta)] \leftrightarrow (\alpha \wedge \beta)$ we have, by classical logic and (K$^{\oplus}$6), to show $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $\beta \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha\wedge\beta$. However, if the first two do not hold then certainly $\alpha\wedge\beta \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$ (i.e., $\neg(\neg\alpha \vee \neg\beta) \in K^{\oplus}_{\neg\alpha\vee\neg\beta}$ by (K$^{\oplus}$1)). So $\alpha \wedge \beta \in K$ by (K$^{\oplus}$5) (apply its contrapositive) and therefore $K \vdash \alpha \wedge \beta$ as required.

(SEE4) When $K \neq K_{\perp}$, $\alpha \in K$ if and only if $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$.

Let $K \neq K_{\perp}$.

(If)

Let $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$. We need to show $\alpha \in K$.

Now choose β to be such that $\vdash \beta$. Certainly $\beta \leq \alpha$. So, by condition (C$\leq$), either $\beta \notin K^{\oplus}_{\neg\beta\vee\neg\alpha}$ or $K \vdash \beta \wedge \alpha$. However, $\beta \in K$ since K is a belief set and so $\beta \in K^{\oplus}_{\neg\beta\vee\neg\alpha}$ by (K$^{\oplus}$3). So the first case is not possible and consequently $K \vdash \beta \wedge \alpha$. Therefore $K \vdash \alpha$ and, since K is a belief set, $\alpha \in K$ as desired.

(Only If)

Let $\alpha \in K$. We need to show $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$ We can do this, using condition (C$\leq$), by showing that $\beta \notin K^{\oplus}_{\neg\beta\vee\neg\alpha}$ or $K \vdash \alpha \wedge \beta$ for all $\beta \in \mathcal{L}$.

Suppose $K \nvdash \alpha \wedge \beta$ (i.e., $\alpha \wedge \beta \notin K$ since K is a belief set). So, since K is a belief set, $\neg(\neg\alpha \vee \neg\beta) \notin K$ and therefore, by (K$^{\oplus}$5), $\neg(\neg\alpha \vee \neg\beta) \notin K^{\oplus}_{\neg\beta\vee\neg\alpha}$. That is, $\alpha \wedge \beta \notin K^{\oplus}_{\neg\beta\vee\neg\alpha}$ by (K$^{\oplus}$1). However $\alpha \in K$ so $\alpha \in K^{\oplus}_{\neg\beta\vee\neg\alpha}$ by (K$^{\oplus}$3). Therefore $\beta \notin K^{\oplus}_{\neg\beta\vee\neg\alpha}$ as required.

Condition (C$\oplus$) is satisfied.

Recall condition (C$\oplus$): $\beta \in K^{\oplus}_{\alpha}$ iff $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$.

(If)

Let $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$. We need to show $\beta \in K^{\oplus}_{\alpha}$.

If $\beta \in K$, then $\beta \in K^{\oplus}_{\alpha}$ by (K$^{\oplus}$3). Otherwise $\beta \notin K$ and so $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$. The last part gives $\alpha \rightarrow \neg\beta \leq \alpha \rightarrow \beta$ and $\alpha \rightarrow \beta \not\leq \alpha \rightarrow \neg\beta$ by definition. The former inequality, by condition (C$\leq$), gives $\alpha \rightarrow \neg\beta \notin K^{\oplus}_{\neg(\alpha\rightarrow\neg\beta)\vee\neg(\alpha\rightarrow\beta)}$ or $K \vdash (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)$. However, $\vdash [\neg(\alpha \rightarrow \neg\beta) \vee \neg(\alpha \rightarrow \beta)] \leftrightarrow \alpha$ and $\vdash [(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)] \leftrightarrow \neg\alpha$. So we have $\alpha \rightarrow \neg\beta \notin K^{\oplus}_{\alpha}$ (this is justified by (K$^{\oplus}$6). The latter inequality gives $\alpha \rightarrow \beta \in K^{\oplus}_{\neg(\alpha\rightarrow\beta)\vee\neg(\alpha\rightarrow\neg\beta)}$ and $K \nvdash (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)$ by condition (C$\leq$). Using the logical equivalences above we get $\alpha \rightarrow \beta \in K^{\oplus}_{\alpha}$ and $K \nvdash \neg\alpha$. Putting these together we have the following $\alpha \rightarrow \neg\beta \notin K^{\oplus}_{\alpha}$, $\alpha \rightarrow \beta \in K^{\oplus}_{\alpha}$ and $\neg\alpha \notin K$. Now, since $\neg\alpha \notin K$ we have by (K$^{\oplus}$2) that $\alpha \in K^{\oplus}_{\alpha}$. Therefore, since $\alpha \rightarrow \beta \in K^{\oplus}_{\alpha}$, then by (K$^{\oplus}$1) $\beta \in K^{\oplus}_{\alpha}$ as required.

(Only If)

Let $\beta \in K^{\oplus}_{\alpha}$. We need to show $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$.

Suppose $\beta \notin K$. We need to show $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$. That is, for the latter part, we need to show $\alpha \rightarrow \neg\beta \leq \alpha \rightarrow \beta$ and $\alpha \rightarrow \beta \not\leq \alpha \rightarrow \neg\beta$ by definition. Now since $\beta \in K^{\oplus}_{\alpha}$ and $\beta \notin K$ surely $K^{\oplus}_{\alpha} \neq K$. It follows by (K$^{\oplus}$4) (apply its contrapositive) that $\neg\alpha \notin K$. It remains to show the two inequalities. Now we have that $\neg\alpha \notin K$ so, by (K$^{\oplus}$5), $\neg\alpha \notin K^{\oplus}_{\alpha}$ and therefore $K^{\oplus}_{\alpha} \neq K_{\perp}$. Also, since $\neg\alpha \notin K$ then $\alpha \in K^{\oplus}_{\alpha}$ by (K$^{\oplus}$2).

Certainly $\alpha \wedge \beta \in K^{\oplus}_{\alpha}$ using (K$^{\oplus}$1) and, since $K^{\oplus}_{\alpha} \neq K_{\perp}$, then $\neg(\alpha \wedge \beta) \notin K^{\oplus}_{\alpha}$. But $\vdash \neg(\alpha \wedge \beta) \leftrightarrow (\alpha \rightarrow \neg\beta)$ so $\alpha \rightarrow \neg\beta \notin K^{\oplus}_{\alpha}$. Now $\vdash \alpha \leftrightarrow [\neg(\alpha \rightarrow \neg\beta) \vee \neg(\alpha \rightarrow \beta)]$ so $\alpha \rightarrow \neg\beta \notin K^{\oplus}_{\neg(\alpha\rightarrow\neg\beta)\vee\neg(\alpha\rightarrow\beta)}$ by (K$^{\oplus}$6). Therefore $\alpha \rightarrow \neg\beta \leq \alpha \rightarrow \beta$ by condition (C$\leq$). It remains to show the second inequality.

Since $\neg\alpha \notin K$ and K is a belief set then certainly $K \vdash \neg\alpha$. Now $\vdash \neg\alpha \leftrightarrow [(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)]$ so $K \nvdash (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)$.(1)

The facts that $\beta \in K^{\oplus}_{\alpha}$ and $\beta \vdash \alpha \rightarrow \beta$ give, by (K$^{\oplus}$1), that $\alpha \rightarrow \beta \in K^{\oplus}_{\alpha}$. However $\vdash \alpha \leftrightarrow [\neg(\alpha \rightarrow \neg\beta) \vee \neg(\alpha \rightarrow \beta)]$ so, by (K$^{\oplus}$6), $\alpha \rightarrow \beta \in K^{\oplus}_{\neg(\alpha\rightarrow\neg\beta)\vee\neg(\alpha\rightarrow\beta)}$.(2)

With (1) and (2) condition (C$\leq$) gives us $\alpha \rightarrow \beta \not\leq \alpha \rightarrow \neg\beta$ as desired. ■

Theorem 5.3.18 *Let $K \in \mathcal{K}$ be any belief set. For any $\alpha,\ \beta \in \mathcal{L}$, $K^{\oplus}_{\alpha} = K^{\oplus_{\leq_{\oplus}}}_{\alpha}$.*

Proof:

$K^{\oplus}_{\alpha} \subseteq K^{\oplus_{\leq_{\oplus}}}_{\alpha}$

> Suppose $\beta \in K^{\oplus}_{\alpha}$. We need to show $\beta \in K^{\oplus_{\leq_{\oplus}}}_{\alpha}$.
>
> If $\beta \in K$, then $\beta \in K^{\oplus_{\leq_{\oplus}}}_{\alpha}$ directly by Condition (C$\oplus$).
>
> So, assume $\beta \notin K$. Therefore $K^{\oplus}_{\alpha} \neq K$ and, by (K$^{\oplus}$4), $\neg\alpha \notin K$. We get by (K$^{\oplus}$2) that $\alpha \in K^{\oplus}_{\alpha}$ and also, through (K$^{\oplus}$1), that $\alpha \rightarrow \beta \in K^{\oplus}_{\alpha}$. Now, $\vdash \alpha \leftrightarrow [\neg(\alpha \rightarrow \beta) \vee \neg(\alpha \rightarrow \neg\beta)]$ and $\vdash \neg\alpha \leftrightarrow [(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)]$. So, using the results obtained above, $\alpha \rightarrow \beta \in K^{\oplus}_{\neg(\alpha\rightarrow\beta)\vee\neg(\alpha\rightarrow\neg\beta)}$ and $K \nvdash (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)$. Therefore, by Condition (C$\leq$), we get $\alpha \rightarrow \beta \not\leq_{\oplus} \alpha \rightarrow \neg\beta$. By Lemma 2.2.17 (i) (properties of entrenchment) we get $\alpha \rightarrow \neg\beta <_{\oplus} \alpha \rightarrow \beta$. Together with the fact that $\neg\alpha \notin K$, Condition (C$\oplus$) gives us $\beta \in K^{\oplus_{\leq_{\oplus}}}_{\alpha}$ as desired.

$K^{\oplus_{\leq_{\oplus}}}_{\alpha} \subseteq K^{\oplus}_{\alpha} \subseteq$

Suppose $\beta \in K_\alpha^{\oplus_{\leq_\oplus}}$. We need to show $\beta \in K_\alpha^\oplus$.

It follows that either $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta <_\oplus \alpha \rightarrow \beta$ by Condition (C$\oplus$). In the former case, $\beta \in K_\alpha^\oplus$ by (K$^\oplus$3). In the latter case, $\alpha \rightarrow \neg\beta \leq_\oplus \alpha \rightarrow \beta$ and $\alpha \rightarrow \beta \not\leq_\oplus \alpha \rightarrow \neg\beta$. So $\alpha \rightarrow \beta \in K^\oplus_{\neg(\alpha\rightarrow\beta)\vee\neg(\alpha\rightarrow\neg\beta)}$ and $K \not\vdash (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta)$ by Condition (C$\leq$). Using the logical equivalences in the proof above, $\alpha \rightarrow \beta \in K_\alpha^\oplus$. Also, $\alpha \in K_\alpha^\oplus$ by (K$^\oplus$2) and due to $\neg\alpha \notin K$. Therefore $\beta \in K_\alpha^\oplus$ by (K$^\oplus$1).

■

Theorem 5.3.19 *Let $K \in \mathcal{K}$ be any belief set. For any $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \beta$ iff $\alpha \leq_{\oplus_\leq} \beta$.*

Proof:

(If)

Suppose $\alpha \leq_{\oplus_\leq} \beta$. We need to show $\alpha \leq \beta$.

By Condition (C$\leq$), either $\alpha \notin K^{\oplus_\leq}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$. Considering first the latter case, we have $\beta \in K$ since K is a belief set. Then $\gamma \leq \beta$ for all $\gamma \in \mathcal{L}$ by (AE4). In particular $\alpha \leq \beta$ as desired.

In considering the former case, we can suppose $K \not\vdash \alpha \wedge \beta$. Now since $\alpha \notin K^{\oplus_\leq}_{\neg\alpha\vee\neg\beta}$ then by Condition (C$\oplus$) $\alpha \notin K$ and either $\neg(\neg\alpha\vee\neg\beta) \in K$ or $(\neg\alpha\vee\neg\beta) \rightarrow \neg\alpha \not< (\neg\alpha\vee\neg\beta) \rightarrow \alpha$. The former case is not possible since it contradicts our supposition that $K \not\vdash \alpha \wedge \beta$. In the latter case, $(\neg\alpha \vee \neg\beta) \rightarrow \alpha \leq (\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha$ by Lemma 2.2.17 (i). Therefore, using the equivalences $\vdash \alpha \leftrightarrow [(\neg\alpha \vee \neg\beta) \rightarrow \alpha]$ and $\vdash (\alpha \rightarrow \beta) \leftrightarrow [(\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha]$ we get $\alpha \leq \alpha \rightarrow \beta$. So $\alpha \wedge \alpha \leq (\alpha \rightarrow \beta) \wedge \alpha$ by Lemma 2.2.18 (x). Now $(\alpha \rightarrow \beta) \wedge \alpha \vdash \beta$ so $(\alpha \rightarrow \beta) \wedge \alpha \leq \beta$ by (SEE2) and, similarly, $\vdash (\alpha \wedge \alpha) \leftrightarrow \alpha$ so $\alpha \leq \alpha \wedge \alpha$. Putting these together we get $\alpha \leq \alpha \wedge \alpha \leq (\alpha \rightarrow \beta) \wedge \alpha \leq \beta$. Therefore, through (SEE1), $\alpha \leq \beta$ as desired.

(Only If)

Suppose $\alpha \leq \beta$. We need to show $\alpha \leq_{\oplus_{\leq}} \beta$.

If $K \vdash \alpha \wedge \beta$, then $\alpha \leq_{\oplus_{\leq}} \beta$ directly by Condition (C$\leq$). So, assume $K \nvdash \alpha \wedge \beta$. Clearly then $K \neq K_{\perp}$. If $\alpha \in K$, then, since $\alpha \leq \beta$ and by (AE4), $\beta \in K$. It would follow that $K \vdash \alpha \wedge \beta$ contradicting our assumption. Therefore $\alpha \notin K$. Now $\beta \vdash \alpha \rightarrow \beta$ so, by (SEE2), $\beta \leq \alpha \rightarrow \beta$ and consequently we get $\alpha \leq \alpha \rightarrow \beta$ by (SEE1). Using the equivalences in the proof above we get $(\neg\alpha \vee \neg\beta) \rightarrow \alpha \leq (\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha$ and therefore, by Lemma 2.2.17 (i), $(\neg\alpha \vee \neg\beta) \rightarrow \neg\alpha \nleq (\neg\alpha \vee \neg\beta) \rightarrow \alpha$. This together with the fact that $\alpha \notin K$ above, gives $\alpha \notin K^{\oplus_{\leq}}_{\neg\alpha\vee\neg\beta}$ by Condition (C$\oplus$). Now, we also have $K \nvdash \alpha \wedge \beta$. Therefore, Condition (C$\leq$) gives $\alpha \leq_{\oplus_{\leq}} \beta$ as desired. ■

Theorem 5.3.20 *Let $K \in \mathcal{K}$ be a consistent belief set. If $\leq$ is an abductive entrenchment ordering for K and $\mathcal{IS}$ an internal system of spheres centred within $[K]$, then an abductive expansion function determined from $\leq$ by condition $(C\oplus)$ and one from $\mathcal{IS}$ via $f_{\mathcal{IS}}$ are the same if and only if condition $(AE\mathcal{IS})$ is satisfied.*

Proof:

Let $K \in \mathcal{K}$ be consistent, $\leq$ an abductive entrenchment for K and $\mathcal{IS}$ a system of spheres centred within $[K]$. We need to show that an abductive entrenchment function $\oplus_{\leq}$ determined from $\leq$ by condition (C$\oplus$) and one from $\mathcal{IS}$ via $f_{\mathcal{IS}}$ are the same (i.e., $K^{\oplus_{\leq}}_{\alpha} = th(f_{\mathcal{IS}}(\alpha))$) if and only if condition (AE$\mathcal{IS}$) is satisfied. We omit the subscript $\leq$ from the expansion function $\oplus_{\leq}$ in the proof unless it is required to remove confusion.

(If)

Let condition (AE$\mathcal{IS}$) be satisfied. We need to show $K^{\oplus_{\leq}}_{\alpha} = th(f_{\mathcal{IS}}(\alpha))$.

$K^{\oplus_{\leq}}_{\alpha} \subseteq th(f_{\mathcal{IS}}(\alpha))$

> Suppose $\beta \in K^{\oplus}_{\alpha}$. We need to show $\beta \in th(f_{\mathcal{IS}}(\alpha))$.
>
> By condition (C$\oplus$), either $\beta \in K$ or both $\neg\alpha \notin K$ and $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$.

In the former case, $[K] \subseteq [\beta]$. Now, there are two cases to consider: (i) $[K] \cap [\alpha] = \emptyset$ and (ii) $[K] \cap [\alpha] \neq \emptyset$. In case (i), $\neg\alpha \in K$ so, by condition (C$\oplus$) $K^{\oplus}_{\alpha} = K$ and $f_{\mathcal{IS}}(\alpha) = [K]$, meaning that $th(f_{\mathcal{IS}}(\alpha)) = [K] = th([K]) = K$ by Lemma 2.2.14 (i). Therefore, $\beta \in th(f_{\mathcal{IS}}(\alpha))$. In case (ii), $c_{\mathcal{IS}}(\alpha) \subseteq [\beta]$ since $[K]$ is the $\subseteq$-maximal sphere in $\mathcal{IS}$. Then, $c_{\mathcal{IS}}(\alpha) \cap [\alpha] \subseteq [\beta]$ and $th([\beta]) \subseteq th(c_{\mathcal{IS}}(\alpha) \cap [\alpha]) = th(f_{\mathcal{IS}}(\alpha))$. Using Lemma 2.2.14 (iii), $Cn(\beta) \subseteq th(f_{\mathcal{IS}}(\alpha))$. Hence $\beta \in th(f_{\mathcal{IS}}(\alpha))$ as required.

In the latter case, $\neg\alpha \notin K$ gives $[K] \cap [\alpha] \neq \emptyset$ and we can suppose $\beta \notin K$ as we have already dealt with this case. Therefore, $f_{\mathcal{IS}}(\alpha) = c_{\mathcal{IS}}(\alpha) \cap [\alpha]$. Now $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$ means, by definition, that $\alpha \rightarrow \neg\beta \leq \alpha \rightarrow \beta$ and $\alpha \rightarrow \beta \not\leq \alpha \rightarrow \neg\beta$. By condition (AE$\mathcal{IS}$), $c_{\mathcal{IS}}(\neg(\alpha \rightarrow \neg\beta)) \subseteq c_{\mathcal{IS}}(\neg(\alpha \rightarrow \beta))$ and $c_{\mathcal{IS}}(\neg(\alpha \rightarrow \beta)) \not\subseteq c_{\mathcal{IS}}(\neg(\alpha \rightarrow \neg\beta))$. That is, $c_{\mathcal{IS}}(\alpha \wedge \beta) \subseteq c_{\mathcal{IS}}(\alpha \wedge \neg\beta)$ and $c_{\mathcal{IS}}(\alpha \wedge \beta) \subset c_{\mathcal{IS}}(\alpha \wedge \neg\beta)$ by logical equivalences. Now, since $[\alpha \wedge \beta] = [\alpha] \cap [\beta]$, we have $c_{\mathcal{IS}}(\alpha) \subseteq c_{\mathcal{IS}}(\alpha \wedge \beta)$ and the latter inclusion above gives us $c_{\mathcal{IS}}(\alpha) \cap [\alpha] \subseteq [\beta]$. Therefore $\beta \in th(c_{\mathcal{IS}}(\alpha) \cap [\alpha]) = th(f_{\mathcal{IS}}(\alpha))$.

$th(f_{\mathcal{IS}}(\alpha)) \subseteq K^{\oplus\leq}_{\alpha}$

Suppose $\beta \in th(f_{\mathcal{IS}}(\alpha))$. We need to show $\beta \in K^{\oplus}_{\alpha}$.

We consider two cases.

(i) $[K] \cap [\alpha] = \emptyset$

Therefore, $f_{\mathcal{IS}}(\alpha) = [K]$ by definition and $th(f_{\mathcal{IS}}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (i). Hence $\beta \in K$ and $\beta \in K^{\oplus}_{\alpha}$ by condition (C$\oplus$).

(ii) $[K] \cap [\alpha] \neq \emptyset$

Then $\neg\alpha \notin K$ and $f_{\mathcal{IS}}(\alpha) = c_{\mathcal{IS}}(\alpha) \cap [\alpha]$ by definition. It follows that $c_{\mathcal{IS}}(\alpha) \cap [\alpha] \subseteq [\beta]$ by (Def th). Therefore, $c_{\mathcal{IS}}(\alpha \wedge \beta) \subset c_{\mathcal{IS}}(\alpha \wedge \neg\beta)$. Consequently, $c_{\mathcal{IS}}(\alpha \wedge \beta) \subseteq c_{\mathcal{IS}}(\alpha \wedge \neg\beta)$ and $c_{\mathcal{IS}}(\alpha \wedge \neg\beta) \not\subseteq c_{\mathcal{IS}}(\alpha \wedge \beta)$. By condition (AE $\mathcal{IS}$) and logical equivalences,

$\alpha \to \neg\beta \leq \alpha \to \beta$ and $\alpha \to \beta \not\leq \alpha \to \neg\beta$ respectively. Therefore $\alpha \to \neg\beta < \alpha \to \beta$. Hence $\beta \in K^{\oplus}_{\alpha}$ by condition (C$\oplus$).

That the antecedent is in fact possible (i.e., that condition (AE$\mathcal{IS}$) allows one to obtain $\leq$ from $\mathcal{IS}$and vice versa), is easily verifiable.

(Only If)

Let $K^{\oplus\leq}_{\alpha} = th(f_{\mathcal{IS}}(\alpha))$. We need to show that condition (AE $\mathcal{IS}$) is satisfied.

(If)

Let either $\beta \in K$ or both $\alpha \notin K$ and $c_{\mathcal{IS}}(\neg\alpha) \subseteq c_{\mathcal{IS}}(\neg\beta)$. We show $\alpha \leq \beta$.

If $\beta \in K$, then $\alpha \leq \beta$ for all $\mathcal{L}$ by (AE4). Suppose $\beta \notin K$, $\alpha \notin K$ and $c_{\mathcal{IS}}(\neg\alpha) \subseteq c_{\mathcal{IS}}(\neg\beta)$. We consider two cases.

(i) $(c_{\mathcal{IS}}(\neg\beta) \cap [\neg\beta]) \cap [\neg\alpha] \neq \emptyset$

Since $\beta \notin K$ then $[K] \cap [\neg\beta] \neq \emptyset$. Also $f_{\mathcal{IS}}(\neg\beta) = c_{\mathcal{IS}}(\neg\beta) \cap [\neg\beta]$ and therefore $\alpha \notin th(f_{\mathcal{IS}}(\neg\beta)) = th(c_{\mathcal{IS}}(\neg\beta) \cap [\neg\beta])$. Consequently $\alpha \notin K^{\oplus}_{\neg\beta}$. By condition (C$\oplus$), both $\alpha \notin K$ and either $\beta \in K$ or $\neg\beta \to \neg\alpha \not< \neg\beta \to \alpha$. Now $\beta \notin K$, therefore $\neg\beta \to \neg\alpha \not< \neg\beta \to \alpha$, then $\neg\beta \to \alpha \leq \neg\beta \to \neg\alpha$. Therefore $(\neg\beta \to \alpha) \wedge \alpha \leq (\neg\beta \to \neg\alpha) \wedge \alpha$ by Lemma 2.2.18 (*x*). Now $\vdash [(\neg\beta \to \alpha) \wedge \alpha] \leftrightarrow \alpha$ and $\vdash [(\neg\beta \to \neg\alpha) \wedge \alpha] \leftrightarrow (\alpha \wedge \beta)$. It follows that $\alpha \leq \alpha \wedge \beta$. Therefore $\alpha \leq \beta$ by Lemma 2.2.18 (*xii*). Hence $\alpha \leq \beta$ trivially.

(ii) $(c_{\mathcal{IS}}(\neg\beta) \cap [\neg\beta]) \cap [\neg\alpha] = \emptyset$

Since $c_{\mathcal{IS}}(\neg\alpha) \subseteq c_{\mathcal{IS}}(\neg\beta)$ it follows that $(c_{\mathcal{IS}}(\neg\alpha) \cap [\neg\beta]) \cap [\neg\alpha] = \emptyset$ and, rearranging, $(c_{\mathcal{IS}}(\neg\alpha) \cap [\neg\alpha]) \cap [\neg\beta] = \emptyset$. Now, since $[\neg\alpha \vee \neg\beta] = [\neg\alpha] \cup [\neg\beta]$, then $c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) = c_{\mathcal{IS}}(\neg\alpha)$. It follows that $(c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha \vee \neg\beta]) \cap [\neg\alpha] \neq \emptyset$. Also, since $\alpha,\ \beta \notin K$, then $f_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) = [\neg\alpha \vee \neg\beta] \cap c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta)$. Therefore, $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$. By condition (C$\oplus$), both $\alpha \notin K$ and either $\alpha \wedge \beta \in K$

or $(\neg\alpha \vee \neg\beta) \to \neg\alpha \not\leq (\neg\alpha \vee \neg\beta) \to \alpha$. Since $\alpha,\ \beta \notin K$, then $\alpha \wedge \beta \notin K$. Therefore, $(\neg\alpha \vee \neg\beta) \to \neg\alpha \not\leq (\neg\alpha \vee \neg\beta) \to \alpha$. That is, $(\neg\alpha \vee \neg\beta) \to \alpha \leq (\neg\alpha \vee \neg\beta) \to \neg\alpha$. By logical equivalences and since $(\neg\alpha \vee \neg\beta) \to \alpha \vdash \alpha \to \beta$, then $\alpha \leq \alpha \to \beta$. Therefore $\alpha \wedge \alpha \leq (\alpha \to \beta) \wedge \alpha$ by Lemma 2.2.18 (*x*). That is, $\alpha \leq \alpha \wedge \beta$. Hence $\alpha \leq \beta$ by Lemma 2.2.18 (*xii*).

(Only If)

Let $\alpha \leq \beta$. Suppose $\beta \notin K$. We need to show $K \nvdash \alpha$ and $c_{\mathcal{IS}}(\neg\alpha) \subseteq c_{\mathcal{IS}}(\neg\beta)$.

By Theorem 5.3.16, condition (C$\leq$) is satisfied. Therefore, either $\alpha \notin K^{\oplus}_{\neg\alpha\vee\neg\beta}$ or $K \vdash \alpha \wedge \beta$. The latter case is not possible since $\beta \notin K$. In the former case, $\alpha \notin th(f_{\mathcal{IS}}(\neg\alpha \vee \neg\beta))$. We consider two cases: (i) $[K] \cap [\neg\alpha \vee \neg\beta] = \emptyset$ and (ii) $[K] \cap [\neg\alpha \vee \neg\beta] \neq \emptyset$. in the former case, since $[\neg\alpha \vee \neg\beta] = [\neg\alpha] \cup [\neg\beta]$, then $\alpha,\ \beta \in K$. However, $\beta \notin K$, therefore this case is not possible. In the latter case, $f_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) = c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha \vee \neg\beta]$. By (Def th) and since $\alpha \notin th(f_{\mathcal{IS}}(\neg\alpha \vee \neg\beta))$, then $(c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha \vee \neg\beta]) \subseteq [\neg\alpha]$. Hence, again since $[\neg\alpha \vee \neg\beta] = [\neg\alpha] \cup [\neg\beta]$, we have $(c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap ([\neg\alpha] \cup [\neg\beta])) \subseteq [\neg\alpha]$ and therefore $(c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha]) \cup (c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap [\neg\beta]) \subseteq [\neg\alpha]$. Therefore $c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \cap [\neg\beta] \subseteq [\neg\alpha]$. Now $c_{\mathcal{IS}}(\neg\alpha \vee \neg\beta) \subseteq c_{\mathcal{IS}}(\neg\beta)$ and consequently $c_{\mathcal{IS}}(\neg\beta) \cap [\neg\alpha] \neq \emptyset$. Hence $c_{\mathcal{IS}}(\neg\alpha) \subseteq c_{\mathcal{IS}}(\neg\beta)$.

■

Appendix C

Proofs for Chapter 6

Note: This chapter contains the proofs for claims made in Chapter 6.

Lemma 6.1.6 *Let K be a belief set and $\alpha \in K$ non-tautological. The $\Delta_{K,\alpha}$-restriction sets for all maximally consistent sets Δ in $\mathcal{L}$ containing $\neg\alpha$ partition $K \perp\!\!\!\perp \alpha$.*

Proof:

Let K be a belief set and $\alpha \in K$ non-tautological. Need to show

(i) every $K' \in K \perp\!\!\!\perp \alpha$ belongs to some $\Delta_{K,\alpha}$-restriction set; and,

(ii) the $\Delta_{K,\alpha}$-restriction sets are disjoint (i.e., no K' belongs to more than one $\Delta_{K,\alpha}$-restriction set).

(i)

We do so by showing that $K' \subseteq \Delta$ for some maximally consistent Δ in $\mathcal{L}$ containing $\neg\alpha$. Since $K' \in K \perp\!\!\!\perp \alpha$ we know that $Cn(K' \cup \{\neg\alpha\})$ is maximally consistent in $\mathcal{L}$. Moreover, $\neg\alpha \in Cn(K' \cup \{\neg\alpha\})$ so $Cn(K' \cup \{\neg\alpha\})$ is a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. Now $K' \subseteq K' \cup \{\neg\alpha\}$ so by (Monotonicity), $K' = Cn(K') \subseteq Cn(K' \cup \{\neg\alpha\})$. Hence $Cn(K' \cup \{\neg\alpha\})$ is an appropriate Δ.

(ii)

Need to show that K' does not belong to more than one $\Delta_{K,\alpha}$-restriction set. Suppose for *reductio ad absurdum* that $K' \in K \perp\!\!\!\perp \mid \Delta$, $K \perp\!\!\!\perp \alpha \mid \Delta'$ for maximally consistent sets Δ, Δ' in $\mathcal{L}$ containing $\neg\alpha$ such that $\Delta \neq \Delta'$. Therefore $K' \subseteq \Delta$ and $K' \subseteq \Delta'$. Now $K' \cup \{\neg\alpha\} \subseteq \Delta \cup \{\neg\alpha\} = \Delta$ and $K' \cup \{\neg\alpha\} \subseteq \Delta' \cup \{\neg\alpha\} = \Delta'$. Using (Monotonicity) $Cn(K' \cup \{\neg\alpha\}) \subseteq Cn(\Delta) = \Delta$ and $Cn(K' \cup \{\neg\alpha\}) \subseteq Cn(\Delta') = \Delta'$. However, $Cn(K' \cup \{\neg\alpha\})$, Δ and Δ' are maximally consistent in $\mathcal{L}$. So $Cn(K' \cup \{\neg\alpha\}) = \Delta$ and $Cn(K' \cup \{\neg\alpha\}) = \Delta'$ Consequently, $\Delta = \Delta'$ contradicting our supposition above. Hence K' belongs to only one $\Delta_{K,\alpha}$-restriction set. ■

Lemma 6.1.7 *Let K be a belief set, $\alpha \in K$ non-tautological and Δ a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. If $K',\ K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$, then*

(*i*) *$K' \cap K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$ and moreover $K' \cap K''$ is the greatest lower bound of $\{K',\ K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.*

(*ii*) *$Cn(K' \cup K'') \in K \perp\!\!\!\perp \alpha \mid \Delta$ and moreover $Cn(K' \cup K'')$ is the least upper bound of $\{K',\ K''\}$ $Cn(K' \cup K'')$ is the least upper bound of $\{K',\ K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.*

Proof:

Let K be a belief set, $\alpha \in K$ non-tautological and Δ a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. Let $K',\ K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$.

(*i*) Need to show $K',\ K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$ and $K' \cap K''$ is the greatest lower bound of $\{K',\ K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.

Now $K' \cap K'' = Cn(K' \cap K'')$ since $K',\ K''$ are closed under Cn. Also, since $K', K'' \subseteq K$ it follows that $K' \cap K'' \subseteq K$. We now show that $Cn((K' \cap K'') \cup \{\neg\alpha\})$ is maximally consistent in $\mathcal{L}$. Now $Cn(K' \cup \{\neg\alpha\})$ and $Cn(K'' \cup \{\neg\alpha\})$ are maximally consistent in $\mathcal{L}$. So, for any $\beta \in \mathcal{L}$ both either $\beta \in Cn(K' \cup \{\neg\alpha\})$ or $\neg\beta \in Cn(K' \cup \{\neg\alpha\})$ and either $\beta \in Cn(K'' \cup \{\neg\alpha\})$ or $\neg\beta \in Cn(K'' \cup \{\neg\alpha\})$. We consider two cases (the other two are similar). Pick some $\beta \in \mathcal{L}$.

(a) $\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\beta \in Cn(K'' \cup \{\neg\alpha\})$.

By (Deduction) $\neg\alpha \to \beta \in Cn(K') = K'$ and $\neg\alpha \to \beta \in Cn(K'') = K''$. Therefore $\neg\alpha \to \beta \in K' \cap K''$ and, using (Deduction) again, $\beta \in Cn((K' \cap K'') \cup \{\neg\alpha\})$ as required.

(b) $\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\neg\beta \in Cn(K'' \cup \{\neg\alpha\})$.

By (Deduction) $\neg\alpha \to \beta \in Cn(K') = K'$ and $\neg\alpha \to \beta \in Cn(K'') = K''$. Now $K', K'' \subseteq \Delta$ so $\neg\alpha \to \beta,\ \neg\alpha \to \neg\beta \in \Delta$. But Δ is maximally consistent in $\mathcal{L}$ (and hence closed under Cn) so $\alpha \in \Delta$ (since $\vdash \alpha \leftrightarrow [(\neg\alpha \to \beta) \wedge (\neg\alpha \to \neg\beta)]$). However, this contradicts the fact that $\neg\alpha \in \Delta$ and Δ is consistent. So this case is not possible.

(c) $\neg\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\beta \in Cn(K'' \cup \{\neg\alpha\})$.

Similar to case (b).

(d) $\neg\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\neg\beta \in Cn(K'' \cup \{\neg\alpha\})$.

Similar to case (a).

It follows that either $\beta \in Cn((K' \cap K'') \cup \{\neg\alpha\})$ or $\neg\beta \in Cn(K' \cap (K'') \cup \{\neg\alpha\})$.

We now show $Cn((K' \cap K'') \cup \{\neg\alpha\})$ is consistent. Suppose for *reductio ad absurdum* that $\perp \in Cn(K' \cap (K'') \cup \{\neg\alpha\})$. Therefore $\neg\alpha \to \perp \in Cn(K' \cap K'') = K' \cap K''$. So $\alpha \in K' \cap K''$ by logical equivalence. Then $\alpha \in K'$ and $\alpha \in K''$ contradicting the fact that $Cn(K' \cup \{\neg\alpha\})$ and $Cn(K'' \cup \{\neg\alpha\})$

are consistent in $\mathcal{L}$. Therefore $Cn((K' \cap K'') \cup \{\neg\alpha\})$ is consistent. Hence $K' \cap K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$.

It remains to show that $K' \cap K''$ is the greatest lower bound of $\{K', K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$. Clearly $K' \cap K''$ is a lower bound of $\{K', K''\}$ with respect to $\subseteq$ for $K' \cap K'' \subseteq K'$ and $K' \cap K'' \subseteq K''$. Need to show that if $K^\# \in K \perp\!\!\!\perp \alpha \mid \Delta$ is a lower bound of $\{K', K''\}$, then $K^\# \subseteq K' \cap K''$. Suppose, for *reductio ad absurdum* that there is a lower bound of $\{K', K''\}$ with respect to $\subseteq$, $K^\#$ and that $K^\# \not\subseteq K' \cap K''$. That is, there is some $\beta \in K^\#$ and $\beta \notin K' \cap K''$. But $K^\#$ is a lower bound of $\{K', K''\}$ with respect to $\subseteq$ so $K^\# \subseteq K'$ and $K^\# \subseteq K''$. It follows that $\beta \in K'$ and $\beta \in K''$. Therefore $\beta \in K' \cap K''$ which contradicts our supposition above. Hence no such $K^\#$ exists and $K' \cap K''$ is the greatest lower bound of $\{K', K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.

(*ii*) Need to show $K', K'' \in K \perp\!\!\!\perp \alpha \mid \Delta$ and $Cn(K' \cup K'')$ is the greatest lower bound of $\{K', K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$.

Now obviously $Cn(Cn(K' \cup K'')) = Cn(K' \cup K'')$ by (Iteration). Also, since $K', K'' \subseteq K$ clearly $K' \cup K'' \subseteq K$ and by (Monotonicity) $Cn(K' \cup K'') \subseteq Cn(K) = K$. Now $Cn(K' \cup \{\neg\alpha\})$ and $Cn(K'' \cup \{\neg\alpha\})$ are maximally consistent in $\mathcal{L}$. So, for any $\beta \in \mathcal{L}$ both either $\beta \in Cn(K' \cup \{\neg\alpha\})$ or $\neg\beta \in Cn(K'' \cup \{\neg\alpha\})$. We consider two cases (the other two are similar). Pick some $\beta \in \mathcal{L}$.

Case(a) $\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\beta \in Cn(K'' \cup \{\neg\alpha\})$.

By (Deduction) $\neg\alpha \to \beta \in Cn(K') = K'$ and $\neg\alpha \to \beta \in Cn(K'') = K''$. Therefore $\neg\alpha \to \beta \in K' \cup \{\neg\alpha\}$ and by (Monotonicity) $\neg\alpha \to \beta \in Cn(K' \cup \{K''\})$. Using (Deduction) again $\beta \in Cn(Cn(K' \cup \{K''\}) \cup \{\neg\alpha\})$ as required.

Case(b) $\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\neg\beta \in Cn(K'' \cup \{\neg\alpha\})$.

By (Deduction) $\neg\alpha \to \beta \in Cn(K') = K'$ and $\neg\alpha \to \neg\beta \in Cn(K'') = K''$. New $K',\ K'' \subseteq \Delta$ so $\neg\alpha \to \beta,\ \neg\alpha \to \neg\beta \in \Delta$. But Δ is maximally consistent in $\mathcal{L}$ (and hence closed under Cn) so $\alpha \in \Delta$ (since $\vdash \alpha \leftrightarrow [(\neg\alpha \to \beta) \wedge (\neg\alpha \to \neg\beta)]$. However, this contradicts the fact that $\neg\alpha \in \Delta$ and Δ is consistent. So this case is not possible.

Case(c) $\neg\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\beta \in Cn(K'' \cup \{\neg\alpha\})$.

Similar to case (b).

Case(d) $\neg\beta \in Cn(K' \cup \{\neg\alpha\})$ and $\neg\beta \in Cn(K'' \cup \{\neg\alpha\})$.

Similar to case (d).

It follows that either $\beta \in Cn(Cn(K' \cup K'') \cup \{\neg\alpha\})$ or $\neg\beta \in Cn(Cn(K' \cup K'') \cup \{\neg\alpha\})$ we need to show $Cn(Cn(K' \cup K'') \cup \{\neg\alpha\})$ is consistent. Suppose for *reductio ad absurdum* that $\perp \in Cn(Cn(K' \cup K'') \cup \{\neg\alpha\})$. Then, by (Deduction) and (Iteration) $\neg\alpha \to \perp \in Cn(Cn(K' \cup K'')) = Cn(K' \cup K'')$. Now $K',\ K'' \subseteq \Delta$ so $K' \cup K'' \subseteq \Delta$ and by (Monotonicity) $Cn(K' \cup K'') \subseteq Cn(\Delta) = \Delta$. So $\neg\alpha \to \perp \in \Delta$. But $\vdash (\neg\alpha \to \perp) \leftrightarrow \alpha$. Therefore $\alpha \in \Delta$. This contradicts the fact that Δ is maximally consistent in $\mathcal{L}$. Therefore $Cn(Cn(K' \cup K'') \cup \{\neg\alpha\})$ is consistent. It follows that $Cn(K' \cup K'') \in K \perp\!\!\!\perp \alpha$ and as we have just seen, $Cn(K' \cup K'') \subseteq \Delta$. Consequently $Cn(K' \cup K'') \subseteq K \perp\!\!\!\perp \alpha \mid \Delta$.

It remains to show that $Cn(K' \cup K'')$ is a least upper bound of $\{K',\ K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$. Clearly $Cn(K' \cup K'')$ is an upper bound of $\{K',\ K''\}$ with respect to $\subseteq$ for $K',\ K'' \subseteq Cn(K' \cup K'')$. Need to show that if $K^{\#} \in K \perp\!\!\!\perp \alpha \mid \Delta$ is an upper bound of $\{K',\ K''\}$, then $Cn(K' \cup K'') \subseteq K^{\#}$. Suppose for *reductio ad absurdum* that there is an upper bound $K^{\#}$ of $\{K',\ K''\}$ with respect to $\subseteq$ and that $Cn(K' \cup K'') \not\subseteq K^{\#}$. That is, there is some $\beta \in Cn(K' \cup K'')$ and $\beta \notin K^{\#}$. Now, by (Compactness) there are some $\Gamma_1 \subseteq K'$ and $\Gamma_2 \subseteq K''$ (so $\Gamma_1 \cup \Gamma_2 \subseteq K' \cup K''$) such that $\beta \in Cn(\Gamma_1 \cup \Gamma_2)$. It follows that not all of $\Gamma_1 \cup \Gamma_2$ are elements of $K^{\#}$ otherwise $\beta \in Cn(K^{\#}) = K^{\#}$.

However, then $K' \cup K'' \not\subseteq K^{\#}$ so either $K' \not\subseteq K^{\#}$ or $K'' \not\subseteq K^{\#}$. This contradicts the supposition that $K^{\#}$ is an upper bound of $\{K',\ K''\}$ with respect to $\subseteq$. Therefore, no such $K^{\#}$ exists. Hence $Cn(K' \cup K'')$ is the least upper bound of $\{K',\ K''\}$ in $K \perp\!\!\!\perp \alpha \mid \Delta$ with respect to $\subseteq$. ■

Theorem 6.1.8 *Let K be a belief set, $\alpha \in K$ non-tautological and Δ a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. Then $K \perp\!\!\!\perp \alpha \mid \Delta$ is a lattice relative to $\subseteq$.*

Proof:

Let K be a belief set, $\alpha \in K$ non-tautological and Δ a maximally consistent set in $\mathcal{L}$ containing $\neg\alpha$. Clearly $\subseteq$ partially orders the elements of $K \perp\!\!\!\perp \alpha \mid \Delta$. It follows directly by Lemma 6.1.6 and Lemma 6.1.7 that $K \perp\!\!\!\perp \alpha \mid \Delta$ is a lattice relative to $\perp\!\!\!\perp$. ■

Theorem 6.1.9 *Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred on $[K]$ for some belief set $K \in \mathcal{K}$. If, for any $\alpha \in \mathcal{L}$, we define $K^{\ominus}_{\alpha}$ to be $th(g_{\mathcal{S}}(\alpha))$, then postulates* ($K^{\ominus}1$) — ($K^{\ominus}8$) *are satisfied.*

Proof:

Let $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha))$. We show that each of ($K^{\ominus}1$) — ($K^{\ominus}8$) is satisfied.

Recall that

$$\text{(Def } g_{\mathcal{S}}\text{)} \qquad g_{\mathcal{S}}(\alpha) = \begin{cases} c_{\mathcal{S}}(\neg\alpha) & \text{whenever } [\alpha] \neq \mathcal{M}_{\mathcal{L}} \\ [\mathrm{K}] & \text{otherwise} \end{cases}$$

($K^{\ominus}1$) $K^{\ominus}_{\alpha}$ is a belief set

Directly from the definition of th (since $g_{\mathcal{S}} : \mathcal{L} \to 2^{\mathcal{M}_{\mathcal{L}}}$).

(K$^\ominus$2) $K^\ominus_\alpha \subseteq K$

Suppose $[\alpha] = \mathcal{M}_\mathcal{L}$ (i.e., $\vdash \alpha$). Then $g_\mathcal{S}(\alpha) = [K]$. Now $K^\ominus_\alpha = th(g_\mathcal{S}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (i). So $K^\ominus_\alpha \subseteq K$ trivially.

Otherwise $[\alpha] \neq \mathcal{M}_\mathcal{L}$ (i.e., $\nvdash \alpha$). Then $g_\mathcal{S}(\alpha) = c_\mathcal{S}(\neg\alpha)$. Now $[K] \subseteq c_\mathcal{S}(\neg\alpha)$ since $[K]$ is the smallest sphere by definition. Therefore $th(c_\mathcal{S}(\neg\alpha)) \subseteq th([K])$ by Lemma 2.2.14 (iv). However, $th(c_\mathcal{S}(\neg\alpha)) = th(g_\mathcal{S}(\alpha)) = K^\ominus_\alpha$. Hence $K^\ominus_\alpha \subseteq K$ by Lemma 2.2.14 (i).

(K$^\ominus$3) If $\nvdash \alpha$ then $\alpha \notin K^\ominus_\alpha$

Let $\nvdash \alpha$. It follows that $[\alpha] \neq \mathcal{M}_\mathcal{L}$. Therefore $K^\ominus_\alpha = th(g_\mathcal{S}(\alpha)) = th(c_\mathcal{S}(\neg\alpha))$. By definition of $c_\mathcal{S}(\neg\alpha)$, $[\neg\alpha] \cap c_\mathcal{S}(\neg\alpha) \neq \emptyset$. Hence by (Def th) $\alpha \notin th(c_\mathcal{S}(\neg\alpha)) = K^\ominus_\alpha$.

(K$^\ominus$4) If $\vdash \alpha \leftrightarrow \beta$, then $K^\ominus_\alpha = K^\ominus_\beta$

Let $\alpha \leftrightarrow \beta$. Then $[\alpha] = [\beta]$. Suppose $[\alpha] = \mathcal{M}_\mathcal{L}$. Clearly then $[\beta] = \mathcal{M}_\mathcal{L}$. Then $g_\mathcal{S}(\alpha) = [K]$ and $g_\mathcal{S}(\beta) = [K]$. Therefore, $K^\ominus_\alpha = th(g_\mathcal{S}(\alpha)) = th([K]) = K$ and $K^\ominus_\beta = th(g_\mathcal{S}(\beta)) = th([K]) = K$. Hence $K^\ominus_\alpha = K^\ominus_\beta$ $(= K)$.

Suppose $[\alpha] \neq \mathcal{M}_\mathcal{L}$. Then $[\beta] \neq \mathcal{M}_\mathcal{L}$. Now $g_\mathcal{S}(\alpha) = c_\mathcal{S}(\neg\alpha)$ and $g_\mathcal{S}(\beta) = c_\mathcal{S}(\neg\beta)$. However, $c_\mathcal{S}(\neg\alpha) = c_\mathcal{S}(\neg\beta)$ so $th(g_\mathcal{S}(\alpha)) = th(c_\mathcal{S}(\neg\alpha)) = th(c_\mathcal{S}(\neg\beta)) = th(g_\mathcal{S}(\beta))$ by Lemma 2.2.14 (iv). Hence $K^\ominus_\alpha = K^\ominus_\beta$.

(K$^\ominus$5) If $\alpha \notin K$, then $K^\ominus_\alpha = K$

Let $\alpha \notin K$. Then $[\neg\alpha] \cap [K] \neq \emptyset$ and $[\alpha] \neq \mathcal{M}_\mathcal{L}$. Now $g_\mathcal{S}(\alpha) = c_\mathcal{S}(\neg\alpha)$. Since $[\neg\alpha] \cap [K] \neq \emptyset$ ($K \neq K_\perp$ since $\alpha \notin K$) and $[K]$ is the $\subseteq$-minimal sphere by definition, then $c_\mathcal{S}(\neg\alpha) = [K]$. Therefore $K^\ominus = th(g_\mathcal{S}(\alpha)) = th(c_\mathcal{S}(\neg\alpha)) = th([K]) = K$ by Lemma 2.2.14 (iv) and (i).

(K$^{\ominus}$6) If $\vdash \alpha$, then $K^{\ominus}_{\alpha} = K$

Let $\vdash \alpha$. Then $[\alpha] = \mathcal{M}_{\mathcal{L}}$. Now $g_{\mathcal{S}}(\alpha) = [K]$ by definition. Therefore, $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (iv) and (i).

(K$^{\ominus}$7) $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha\wedge\beta}$

If $\vdash \alpha$, then, as in (K$^{\ominus}$6), $K^{\ominus}_{\alpha} = K$ and $\vdash (\alpha \wedge \beta) \leftrightarrow \beta$. Therefore, as in (K$^{\ominus}$4), $K^{\ominus}_{\alpha\wedge\beta} = K^{\ominus}_{\beta}$. Now, as in (K$^{\ominus}$2), $K^{\ominus}_{\beta} \subseteq K$. Together we have $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} = K^{\ominus}_{\beta} = K^{\ominus}_{\alpha\wedge\beta}$. Hence $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha\wedge\beta}$ trivially.

Similarly for $\vdash \beta$.

Suppose $\nvdash \alpha$ and $\nvdash \beta$. Then clearly $\nvdash \alpha \wedge \beta$. It follows that $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$, $[\beta] \neq \mathcal{M}_{\mathcal{L}}$ and $[\alpha \wedge \beta] \neq \mathcal{M}_{\mathcal{L}}$. Therefore $g_{\mathcal{S}}(\alpha) = c_{\mathcal{S}}(\neg\alpha)$, $g_{\mathcal{S}}(\beta) = c_{\mathcal{S}}(\neg\beta)$ and $g_{\mathcal{S}}(\alpha \wedge \beta) = c_{\mathcal{S}}(\neg(\alpha \wedge \beta)) = c_{\mathcal{S}}(\neg\alpha \vee \neg\beta)$. Now $[\neg\alpha \vee \neg\beta] = [\alpha] \cup [\beta]$. Consequently, either $c_{\mathcal{S}}(\neg\alpha\vee\neg\beta) \subseteq c_{\mathcal{S}}(\neg\alpha)$ or $c_{\mathcal{S}}(\neg\alpha\vee\neg\beta) \subseteq c_{\mathcal{S}}(\neg\beta)$. (These can, of course, be strengthened to equivalence but $\subseteq$ will suffice to demonstrate the result.) Therefore, $th(c_{\mathcal{S}}(\neg\alpha)) \subseteq th(c_{\mathcal{S}}(\neg\alpha \vee \neg\beta))$ or $th(c_{\mathcal{S}}(\neg\beta)) \subseteq th(c_{\mathcal{S}}(\neg\alpha\vee\neg\beta))$. It follows that $th(c_{\mathcal{S}}(\neg\alpha))\cap th(c_{\mathcal{S}}(\neg\beta)) \subseteq th(c_{\mathcal{S}}(\neg\alpha\vee\neg\beta))$. That is, $th(g_{\mathcal{S}}(\alpha)) \cap th(g_{\mathcal{S}}(\beta)) \subseteq th(g_{\mathcal{S}}(\alpha \wedge \beta))$. Hence $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha\wedge\beta}$.

(K$^{\ominus}$8) If $\alpha \notin K^{\ominus}_{\alpha\wedge\beta}$, then $K^{\ominus}_{\alpha\wedge\beta} \subseteq K^{\ominus}_{\alpha}$

Let $\alpha \notin K^{\ominus}_{\alpha\wedge\beta}$. Then $\nvdash \alpha$ and $\nvdash \alpha \wedge \beta$. It follows that $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$ and $[\alpha \wedge \beta] \neq \mathcal{M}_{\mathcal{L}}$. Consequently, $g_{\mathcal{S}}(\alpha) = c_{\mathcal{S}}(\neg\alpha)$ and $g_{\mathcal{S}}(\alpha \wedge \beta) = c_{\mathcal{S}}(\neg(\alpha \wedge \beta)) = c_{\mathcal{S}}(\neg\alpha \vee \neg\beta)$. Now, since $\alpha \notin K^{\ominus}_{\alpha\wedge\beta} = th(g_{\mathcal{S}}(\alpha \wedge \beta)) = th(c_{\mathcal{S}}(\neg\alpha \vee \neg\beta)$, then $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha] \neq \emptyset$. Therefore $c_{\mathcal{S}}(\neg\alpha) \subseteq c_{\mathcal{S}}(\neg\alpha \vee \neg\beta)$ and $th(c_{\mathcal{S}}(\neg\alpha\vee\neg\beta)) \subseteq th(c_{\mathcal{S}}(\neg\alpha))$ by Lemma 2.2.14 (iv). Then $th(g_{\mathcal{S}}(\alpha\wedge\beta)) \subseteq th(g_{\mathcal{S}}(\alpha))$. Hence $K^{\ominus}_{\alpha\wedge\beta} \subseteq K^{\ominus}_{\alpha}$.

∎

Theorem 6.1.10 *Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_{\mathcal{L}}$ centred on $[K]$ for some belief set $K \in \mathcal{K}$. If, for any $\alpha \in \mathcal{L}$, we define $K^{\ominus}_{\alpha}$ to be $th(g'_{\mathcal{S}}(\alpha))$, then postulates* (K$^{\ominus}$1) — (K$^{\ominus}$8) *are satisfied.*

Proof:

Let $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha))$. We show that each of (K$^{\ominus}$1) — (K$^{\ominus}$8) is satisfied.

To simplify the proofs, we denote by $C_{\mathcal{S}}(\alpha)$ those worlds in the innermost "band" intersecting $[\alpha]$. That is, $C_{\mathcal{S}}(\alpha) = \{m \in c_{\mathcal{S}}(\alpha) \mid m \notin U$ for any $U \in \mathcal{S}$ such that $U \subset c_{\mathcal{S}}(\alpha)\}$whenever $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$. In other words,

$$g'_{\mathcal{S}}(\alpha) = \begin{cases} [K] \cup C_{\mathcal{S}}(\neg\alpha) & \text{whenever } [\alpha] \neq \mathcal{M}_{\mathcal{L}} \\ \text{[K]} & \text{otherwise} \end{cases}$$

(K$^{\ominus}$1) $K^{\ominus}_{\alpha}$ is a belief set

Directly from the definition of th (since $g'_{\mathcal{S}} : \mathcal{L} \to 2^{\mathcal{M}_{\mathcal{L}}}$).

(K$^{\ominus}$2) $K^{\ominus}_{\alpha} \subseteq K$

Suppose $[\alpha] = \mathcal{M}_{\mathcal{L}}$ (i.e., $\vdash \alpha$). Then $g'_{\mathcal{S}}(\alpha) = [K]$. Now $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (i). So $K^{\ominus}_{\alpha} \subseteq K$ trivially.

Otherwise $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$ (i.e., $\not\vdash \alpha$). Then $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha)$. Now $[K] \subseteq [K] \cup C_{\mathcal{S}}(\neg\alpha)$. Therefore $th([K] \cup C_{\mathcal{S}}(\neg\alpha)) \subseteq th([K])$ by Lemma 2.2.14 (iv). However, $th([K] \cup C_{\mathcal{S}}(\neg\alpha)) = th(g'_{\mathcal{S}}(\alpha)) = K^{\ominus}_{\alpha}$. Hence $K^{\ominus}_{\alpha} \subseteq K$ by Lemma 2.2.14 (i).

(K$^{\ominus}$3) If $\not\vdash \alpha$ then $\alpha \notin K^{\ominus}_{\alpha}$

Let $\not\vdash \alpha$. It follows that $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$. Therefore $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th([K] \cup C_{\mathcal{S}}(\neg\alpha))$. By definition of $C_{\mathcal{S}}(\neg\alpha)$, $[\neg\alpha] \cap C_{\mathcal{S}}(\neg\alpha) \neq \emptyset$. Hence by (Def th) $\alpha \notin th([K] \cup C_{\mathcal{S}}(\neg\alpha)) = K^{\ominus}_{\alpha}$.

(K$^{\ominus}$4) If $\vdash \alpha \leftrightarrow \beta$, then $K^{\ominus}_{\alpha} = K^{\ominus}_{\beta}$

Let $\alpha \leftrightarrow \beta$. Then $[\alpha] = [\beta]$. Suppose $[\alpha] = \mathcal{M}_{\mathcal{L}}$. Clearly then $[\beta] = \mathcal{M}_{\mathcal{L}}$. Then $g'_{\mathcal{S}}(\alpha) = [K]$ and $g'_{\mathcal{S}}(\beta) = [K]$. Therefore, $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th([K]) = K$ and $K^{\ominus}_{\beta} = th(g'_{\mathcal{S}}(\beta)) = th([K]) = K$. Hence $K^{\ominus}_{\alpha} = K^{\ominus}_{\beta}$ ($= K$).

Suppose $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$. Then $[\beta] \neq \mathcal{M}_{\mathcal{L}}$. Now $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha)$ and $g'_{\mathcal{S}}(\beta) = [K] \cup C_{\mathcal{S}}(\neg\beta)$. However, $c_{\mathcal{S}}(\neg\alpha) = c_{\mathcal{S}}(\neg\beta)$ and consequently $C_{\mathcal{S}}(\neg\alpha) = C_{\mathcal{S}}(\neg\beta)$ by definition. Therefore, $[K] \cup C_{\mathcal{S}}(\neg\alpha) = [K] \cup C_{\mathcal{S}}(\neg\beta)$ and it follows that $g'_{\mathcal{S}}(\alpha) = g'_{\mathcal{S}}(\beta)$. Hence $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha)) = th(g'_{\mathcal{S}}(\beta)) = K^{\ominus}_{\beta}$ by Lemma 2.2.14 (iv).

($K^{\ominus}5$) If $\alpha \notin K$, then $K^{\ominus}_{\alpha} = K$

Let $\alpha \notin K$. Then $[\neg\alpha] \cap [K] \neq \emptyset$ and $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$. Now $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha)$. Since $[\neg\alpha] \cap [K] \neq \emptyset$ ($K \neq K_{\perp}$ since $\alpha \notin K$) and $[K]$ is the $\subseteq$-minimal sphere by definition, then $c_{\mathcal{S}}(\neg\alpha) = [K]$. It follows, by definition, that $C_{\mathcal{S}}(\neg\alpha) = c_{\mathcal{S}}(\neg\alpha) = [K]$. Therefore $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th([K] \cup C_{\mathcal{S}}(\neg\alpha)) = th([K]) = K$ by Lemma 2.2.14 (iv) and (i).

($K^{\ominus}6$) If $\vdash \alpha$, then $K^{\ominus}_{\alpha} = K$

Let $\vdash \alpha$. Then $[\alpha] = \mathcal{M}_{\mathcal{L}}$. Now $g'_{\mathcal{S}}(\alpha) = [K]$ by definition. Therefore, $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (iv) and (i).

($K^{\ominus}7$) $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha\wedge\beta}$

If $\vdash \alpha$, then, as in ($K^{\ominus}6$), $K^{\ominus}_{\alpha} = K$ and $\vdash (\alpha \wedge \beta) \leftrightarrow \beta$. Therefore, as in ($K^{\ominus}4$), $K^{\ominus}_{\alpha\wedge\beta} = K^{\ominus}_{\beta}$. Now, as in ($K^{\ominus}2$), $K^{\ominus}_{\beta} \subseteq K$. Together we have $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} = K^{\ominus}_{\beta} = K^{\ominus}_{\alpha\wedge\beta}$. Hence $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha\wedge\beta}$ trivially.

Similarly for $\vdash \beta$.

Suppose $\nvdash \alpha$ and $\nvdash \beta$. Then clearly $\nvdash \alpha \wedge \beta$. It follows that $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$, $[\beta] \neq \mathcal{M}_{\mathcal{L}}$ and $[\alpha \wedge \beta] \neq \mathcal{M}_{\mathcal{L}}$. Therefore $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha)$, $g'_{\mathcal{S}}(\beta) =$

$[K] \cup c_{\mathcal{S}}(\neg\beta)$ and $g'_{\mathcal{S}}(\alpha \wedge \beta) = [K] \cup C_{\mathcal{S}}(\neg(\alpha \wedge \beta)) = C_{\mathcal{S}}(\neg\alpha \vee \neg\beta)$. Now $[\neg\alpha \vee \neg\beta] = [\alpha] \cup [\beta]$. Consequently, either $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = c_{\mathcal{S}}(\neg\alpha)$ or $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = c_{\mathcal{S}}(\neg\beta)$. This, by definition, gives either $C_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = C_{\mathcal{S}}(\neg\alpha)$ or $C_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = C_{\mathcal{S}}(\neg\beta)$. Therefore, either $[K] \cup C_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = [K] \cup C_{\mathcal{S}}(\neg\alpha)$ or $[K] \cup C_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = [K] \cup C_{\mathcal{S}}(\neg\beta)$. In other words, either $g'_{\mathcal{S}}(\alpha \wedge \beta) = g'_{\mathcal{S}}(\alpha)$ or $g'_{\mathcal{S}}(\alpha \wedge \beta) = g'_{\mathcal{S}}(\beta)$. Then $th(g'_{\mathcal{S}}(\alpha \wedge \beta)) = th(g'_{\mathcal{S}}(\alpha))$ or $th(g'_{\mathcal{S}}(\alpha \wedge \beta)) = th(g'_{\mathcal{S}}(\beta))$ by Lemma 2.2.14 (iv). Consequently, either $K^{\ominus}_{\alpha\wedge\beta} = K^{\ominus}_{\alpha}$ or $K^{\ominus}_{\alpha\wedge\beta} = K^{\ominus}_{\beta}$. Hence, in either case, $K^{\ominus}_{\alpha} \cap K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha\wedge\beta}$.

($K^{\ominus}8$) If $\alpha \notin K^{\ominus}_{\alpha\wedge\beta}$, then $K^{\ominus}_{\alpha\wedge\beta} \subseteq K^{\ominus}_{\alpha}$

Let $\alpha \notin K^{\ominus}_{\alpha\wedge\beta}$. Then $\not\vdash \alpha$ and $\not\vdash \alpha \wedge \beta$. It follows that $[\alpha] \neq \mathcal{M}_{\mathcal{L}}$ and $[\alpha \wedge \beta] \neq \mathcal{M}_{\mathcal{L}}$. Consequently, $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha)$ and $g'_{\mathcal{S}}(\alpha \wedge \beta) = [K] \cup C_{\mathcal{S}}(\neg(\alpha\wedge\beta)) = [K] \cup C_{\mathcal{S}}(\neg\alpha \vee \neg\beta)$. Now, since $\alpha \notin K^{\ominus}_{\alpha\wedge\beta} = th(g'_{\mathcal{S}}(\alpha\wedge\beta)) = th([K] \cup C_{\mathcal{S}}(\neg\alpha\vee\neg\beta)$, then $([K] \cup C_{\mathcal{S}}(\neg\alpha\vee\neg\beta)) \cap [\neg\alpha] \neq \emptyset$. Therefore, either $[K] \cap [\neg\alpha] \neq \emptyset$ or $C_{\mathcal{S}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha] \neq \emptyset$ (i.e., $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha] \neq \emptyset$). In the first case, $c_{\mathcal{S}}(\neg\alpha) = [K]$ and $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = [K]$. Consequently, $C_{\mathcal{S}}(\neg\alpha) = [K]$ and $C_{\mathcal{S}}(\neg\alpha\vee\neg\beta) = [K]$. Therefore, $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha) = [K] \cup [K] = [K]$ and $g'_{\mathcal{S}}(\alpha \wedge \beta) = [K] \cup C_{\mathcal{S}}(\neg(\alpha \wedge \beta)) = [K] \cup [K] = [K]$. Then $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th([K]) = K$ and $K^{\ominus}_{\alpha\wedge\beta} = th(g'_{\mathcal{S}}(\alpha \wedge \beta)) = th([K]) = K$. As a result, $K^{\ominus}_{\alpha\wedge\beta} \subseteq K^{\ominus}_{\alpha}$. In the latter case, and using the fact that $[\neg\alpha \vee \neg\beta] = [\neg\alpha] \cup [\neg\beta]$, it can be seen that $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = c_{\mathcal{S}}(\neg\alpha)$ (otherwise $c_{\mathcal{S}}(\neg\alpha \vee \neg\beta) \cap [\neg\alpha] = \emptyset$ because the $\neg\beta$-worlds that are not $\neg\alpha$-worlds are in an inner sphere). Consequently $C_{\mathcal{S}}(\neg\alpha \vee \neg\beta) = C_{\mathcal{S}}(\neg\alpha)$. Therefore, $g'_{\mathcal{S}}(\alpha) = [K] \cup C_{\mathcal{S}}(\neg\alpha) = [K] \cup C_{\mathcal{S}}(\neg(\alpha \wedge \beta)) = g'_{\mathcal{S}}(\alpha \wedge \beta)$. Then $K^{\ominus}_{\alpha} = th(g'_{\mathcal{S}}(\alpha)) = th(g'_{\mathcal{S}}(\alpha \wedge \beta)) = K^{\ominus}_{\alpha\wedge\beta}$. Again, $K^{\ominus}_{\alpha\wedge\beta} \subseteq K^{\ominus}_{\alpha}$. Hence, in either case, $K^{\ominus}_{\alpha\wedge\beta} \subseteq K^{\ominus}_{\alpha}$.

Note, the reason we have equality instead of inclusion in both cases is because we have actually demonstrated a stronger result that follows from the postulates for value-based Levi-contraction over K.

■

Observation 6.1.11 *Postulate* (K$^\ominus$9) *implies postulate* (K$^\ominus$7) *in the presence of postulates* (K$^\ominus$1) — (K$^\ominus$6).

Proof:

Suppose both α and β are in $K^\ominus_{\alpha\wedge\beta}$. That is, $\alpha,\ \beta \in K^\ominus_{\alpha\wedge\beta}$. By (K$^\ominus$1), $\alpha \wedge \beta \in K^\ominus_{\alpha\wedge\beta}$. Then $\vdash \alpha \wedge \beta$ by (K$^\ominus$3) and $K^\ominus_{\alpha\wedge\beta} = K$ by (K$^\ominus$6). Now, by (K$^\ominus$2), $K^\ominus_\alpha \subseteq K$ and $K^\ominus_\beta \subseteq K$. Hence $K^\ominus_\alpha \cap K^\ominus_\beta \subseteq K = K_{\alpha\wedge\beta}$.

Suppose one of α or β is not in $K^\ominus_{\alpha\wedge\beta}$. Without loss of generality, assume $\alpha \notin K^\ominus_{\alpha\wedge\beta}$. By (K$^\ominus$9), $K^\ominus_{\alpha\wedge\beta} \subseteq K^\ominus_\alpha$. Clearly $\not\vdash \alpha$ by (K$^\ominus$1). Therefore $\alpha \notin K^\ominus_\alpha$ by (K$^\ominus$3) and consequently $\alpha \wedge \beta \notin K^\ominus_\alpha$ by (K$^\ominus$1). Then $K^\ominus_\alpha \subseteq K^\ominus_{\alpha\wedge\beta}$ by (K$^\ominus$9). Hence $K^\ominus_\alpha \cap K^\ominus_\beta \subseteq K_{\alpha\wedge\beta}$. ■

(5.3) Either $K^\ominus_\alpha \subseteq K^\ominus_\beta$ or $K^\ominus_\beta \subseteq K^\ominus_\alpha$

Proof:

Now, either $\alpha \notin K^\ominus_\beta$ or $\alpha \in K^\ominus_\beta$. In the former case, (K$^\ominus$9) gives $K^\ominus_\beta \subseteq K^\ominus_\alpha$. In the latter, if $\not\vdash \alpha$ (K$^\ominus$10) gives $K^\ominus_\alpha \subseteq K^\ominus_\beta$. If $\vdash \alpha$, then $K^\ominus_\alpha = K$ by (K$^\ominus$6) and so, by (K$^\ominus$2) $K^\ominus_\beta \subseteq K = K^\ominus_\alpha$. Hence, either $K^\ominus_\alpha \subseteq K^\ominus_\beta$ or $K^\ominus_\beta \subseteq K^\ominus_\alpha$. ■

Lemma 6.1.12 *Let $\mathcal{S}$ be any system of spheres in $\mathcal{M}_\mathcal{L}$ centred on $[K]$ for some belief set $K \in \mathcal{K}$. If we define, for any $\alpha \in \mathcal{L}$, $K^\ominus_\alpha$ to be $th(g_\mathcal{S}(\alpha))$, the postulates* (K$^\ominus$9) *and* (K$^\ominus$10) *are satisfied.*

Proof:

Let $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha))$. We show that each of (K$^{\ominus}$9) and (K$^{\ominus}$10) is satisfied.

Recall that

$$\text{(Def } g_{\mathcal{S}}\text{)} \qquad g_{\mathcal{S}}(\alpha) = \begin{cases} c_{\mathcal{S}}(\neg\alpha) & \text{whenever } [\alpha] \neq \mathcal{M}_{\mathcal{L}} \\ \text{[K]} & \text{otherwise} \end{cases}$$

(K$^{\ominus}$9) If $\alpha \notin K^{\ominus}_{\beta}$, then $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$

Let $\alpha \notin K^{\ominus}_{\beta}$.

If $\vdash \beta$, then $g_{\mathcal{S}}(\beta) = [K]$ and $th(g_{\mathcal{S}}(\beta)) = th([K]) = K$ using Lemma 2.2.14 (i). Since $\alpha \notin K^{\ominus}_{\beta}$, then $\alpha \notin K$. Therefore $[K] \cap [\neg\alpha] \neq \emptyset$. Now clearly $\nvdash \alpha$ so $g_{\mathcal{S}}(\alpha) = c_{\mathcal{S}}(\neg\alpha) = [K]$ since $[K]$ is the $\subseteq$-minimal sphere. Consequently, $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha)) = th([K]) = K$. Hence $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$ trivially.

Suppose $\nvdash \beta$. Then $g_{\mathcal{S}}(\beta) = c_{\mathcal{S}}(\neg\beta)$. Since $K^{\ominus}_{\beta} = th(g_{\mathcal{S}}(\beta)) = th(c_{\mathcal{S}}(\neg\beta))$ and $\alpha \notin K^{\ominus}_{\beta}$, then $c_{\mathcal{S}}(\neg\beta) \cap [\neg\alpha] \neq \emptyset$. Therefore $c_{\mathcal{S}}(\neg\alpha) \subseteq c_{\mathcal{S}}(\neg\beta)$. Consequently $g_{\mathcal{S}}(\alpha) \subseteq g_{\mathcal{S}}(\beta)$ and $th(g_{\mathcal{S}}(\beta)) \subseteq th(g_{\mathcal{S}}(\alpha))$ by Lemma 2.2.14 (iv). Hence $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$.

(K$^{\ominus}$10) If $\nvdash \alpha$ and $\alpha \in K^{\ominus}_{\beta}$, then $K^{\ominus}_{\alpha} \subseteq K^{\ominus}_{\beta}$

Let $\nvdash \alpha$ and $\alpha \in K^{\ominus}_{\beta}$.

If $\vdash \beta$, then $g_{\mathcal{S}}(\beta) = [K]$ and $th(g_{\mathcal{S}}(\beta)) = th([K]) = K$ using Lemma 2.2.14 (i). By (K$^{\ominus}$2) which is satisfied, $\alpha \in K$. Then $[K] \subseteq [\alpha]$ and since $[K]$ is the $\subseteq$-minimal sphere, then $[K] \subseteq g_{\mathcal{S}}(\alpha)$. Therefore $th(g_{\mathcal{S}}(\alpha)) \subseteq th([K])$ by Lemma 2.2.14 (iv). Hence $K^{\ominus}_{\alpha} \subseteq K^{\ominus}_{\beta}$.

Suppose $\nvdash \beta$. Then $g_{\mathcal{S}}(\beta) = c_{\mathcal{S}}(\neg\beta)$ and $K^{\ominus}_{\beta} = th(g_{\mathcal{S}}(\beta)) = th(c_{\mathcal{S}}(\neg\beta))$. Now $\alpha \in K^{\ominus}_{\beta} = th(c_{\mathcal{S}}(\neg\beta))$. Therefore $c_{\mathcal{S}}(\neg\beta) \subseteq [\alpha]$. It follows that $c_{\mathcal{S}}(\neg\beta) \subseteq c_{\mathcal{S}}(\neg\alpha)$ and, by Lemma 2.2.14 (iv), $th(c_{\mathcal{S}}(\neg\alpha)) \subseteq th(c_{\mathcal{S}}(\neg\beta))$. Since $\nvdash \alpha$, then $g_{\mathcal{S}}(\alpha) = c_{\mathcal{S}}(\neg\alpha)$. Therefore $th(g_{\mathcal{S}}(\alpha)) \subseteq th(g_{\mathcal{S}}(\beta))$. Hence $K^{\ominus}_{\alpha} \subseteq K^{\ominus}_{\beta}$.

■

Theorem 6.1.13 *Let* $\ominus : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ *be any function satisfying postulates* (K$^\ominus$1) — (K$^\ominus$10)*. Then for any belief set* $K \in \mathcal{K}$ *there is a system of spheres on* $\mathcal{M}_{\mathcal{L}}$*, say* $\mathcal{S}$*, centred on* $[K]$ *and satisfying* $K^\ominus_\alpha = th(g_{\mathcal{S}}(\alpha))$ *for any* $\alpha \in \mathcal{L}$*.*

Proof:

Let $\ominus : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ be a function satisfying postulates (K$^\ominus$1) — (K$^\ominus$10). Let $K \in \mathcal{K}$ be a belief set.

Let $\mathcal{S}'$ be the class of all nonempty subsets U of $\mathcal{M}_{\mathcal{L}}$ such that

$$[K^\ominus_\alpha] = U \text{ for some } \alpha \in \mathcal{L}.$$

Let $\mathcal{S}$ be the system of spheres $\mathcal{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ if $K \neq K_\perp$ and $\mathcal{S}' \cup \{\mathcal{M}_{\mathcal{L}},\ \emptyset\}$ otherwise.

It is straightforward to show that $\mathcal{S}$ is a system of spheres centred on $[K]$. Condition ($\mathcal{S}$1) follows directly from property (5.3) which is a consequence of postulates (K$^\ominus$1) — (K$^\ominus$10), ($\mathcal{S}$3) follows from postulate (K$^\ominus$2) and ($\mathcal{S}$2) and ($\mathcal{S}$4) follow directly from our finite language assumption.

It remains to verify that, for all $\alpha \in \mathcal{L}$, $K^\ominus_\alpha = th(g_{\mathcal{S}}(\alpha))$. If $\vdash \alpha$, then $K^\ominus_\alpha = K$ by (K$^\ominus$6). Also, $[\neg\alpha] = \mathcal{M}_{\mathcal{L}}$ and consequently $g_{\mathcal{S}}(\alpha) = [K]$. Therefore $K^\ominus_\alpha = th(g_{\mathcal{S}}(\alpha)) = th([K]) = K$ by Lemma 2.2.14 (i).

Suppose $\not\vdash \alpha$. Then $g_{\mathcal{S}}(\alpha) = c_{\mathcal{S}}(\neg\alpha)$ and we need to find such a sphere $c_{\mathcal{S}}(\neg\alpha)$ and show that is the same as $[K^\ominus_\alpha]$. By Lemma 2.2.14 (iv) it would then follow that $K^\ominus_\alpha = th(g_{\mathcal{S}}(\alpha))$. Since $\mathcal{S}$ is a system of spheres it can easily be seen that $c_{\mathcal{S}}(\neg\alpha)$ (the smallest sphere intersecting $\neg\alpha$) is given by $\cap\{U \in \mathcal{S} :\ U \cap [\neg\alpha] \neq \emptyset\}$. Now, by (K$^\ominus$3), $\alpha \notin K^\ominus_\alpha$. Therefore $[K^\ominus_\alpha] \cap [\neg\alpha] \neq \emptyset$. Hence clearly $\cap\{U \in \mathcal{S} :\ U \cap [\neg\alpha] \neq \emptyset\} \subseteq [K^\ominus_\alpha]$. We need to show equality. We can do so by ensuring that there is no sphere $V \in \mathcal{S}$ such that $V \cap [\neg\alpha] \neq \emptyset$ and $V \subset [K^\ominus_\alpha]$. Suppose, for *reductio ad absurdum*, that such a V

exists. That is, there is a $V = [K^{\ominus}_{\beta}]$ (by construction) such that $[K^{\ominus}_{\beta}] \cap [\neg\alpha] \neq \emptyset$. V is clearly a sphere by our construction. Moreover, since $[K^{\ominus}_{\beta}] \cap [\neg\alpha] \neq \emptyset$, then $\alpha \notin K^{\ominus}_{\beta}$. It follows from (K$^{\ominus}$9) that $K^{\ominus}_{\beta} \subseteq K^{\ominus}_{\alpha}$. Therefore $[K^{\ominus}_{\alpha}] \subseteq [K^{\ominus}_{\beta}]$ by Lemma 2.2.14 (v). That is, $[K^{\ominus}_{\alpha}] \subseteq V$. This contradicts our supposition above. Therefore no such sphere exists. $[K^{\ominus}_{\alpha}] = \cap\{U \in \mathcal{S} : U \cap [\neg\alpha] \neq \emptyset\}$ and since $\mathcal{S}$ is a system of spheres, the right hand side is a sphere. Hence $K^{\ominus}_{\alpha} = th(g_{\mathcal{S}}(\alpha))$. ■

Appendix D

Proofs for Chapter 7

Note: This chapter contains the proofs for claims made in Chapter 7.

Observation 7.0.1 *Let $K \in \mathcal{K}$ be a belief set. If $\ominus$ is a Levi-contraction function over K satisfying postulates (K$^{\ominus}$1) — (K$^{\ominus}$6), then there is a revision equivalent AGM contraction function $-$ over K satisfying postulates (K^{-}1) — (K^{-}6) and vice versa.*

Proof:

Let $K \in \mathcal{K}$ be a belief set.

Suppose $\ominus$ is a Levi-contraction function over K satisfying postulates (K$^{\ominus}$1) — (K$^{\ominus}$6), Clearly $\ominus$ is a withdrawal function. The result follows directly from the observation in Makinson [71] (p. 389). This tells us that there is a unique AGM contraction function $-$ over K (i.e., satisfying postulates (K^{-}1) — (K^{-}6)) which is revision equivalent to $\ominus$.

To see the converse, suppose $-$ is an AGM contraction function over K satisfying postulates (K^{-}1) — (K^{-}6). By Makinson's observation, $-$ is the greatest element (in terms of set inclusion) of an equivalence class of withdrawal functions. It suffices to find one which satisfies the postulate of failure (K$^{\ominus}$6) for it will be a Levi-contraction function. However, $-$ is a withdrawal function satisfying the postulate of failure (this can be obtained from postulates (K^{-}1) — (K^{-}5)) and therefore a suitable candidate (i.e., there may be others). Note here that this means that every AGM contraction function is a Levi-contraction function which is obvious from the fact that any AGM contraction function satisfies (K$^{\ominus}$1) — (K$^{\ominus}$6) and should also be evident given the relationship between $K\perp\alpha$ and $K\perp\!\!\!\perp\alpha$ in the last chapter. ■

Theorem 7.1.1 *Let $K \in \mathcal{K}$ be a belief set. Let $\ominus$ be a contraction function satisfying postulates (K$^{\ominus}$1) — (K$^{\ominus}$6) over K and $\oplus$ an abductive expansion function satisfying postulates (K$^{\oplus}$1) — (K$^{\oplus}$6) over K. Then the abductive revision function $\otimes$ obtained through (Def $*$) satisfies postulates (K$^{\otimes}$1), (K$^{\otimes}$2) and (K$^{\otimes}$4) — (K$^{\otimes}$6) over K.*

.

Proof:

Let $K \in \mathcal{K}$ be a belief set. Let $\ominus$ be a contraction function satisfying postulates (K$^{\ominus}$1) — (K$^{\ominus}$6) over K, $\oplus$ an abductive expansion function satisfying postulates (K$^{\oplus}$1) — (K$^{\oplus}$6) over K and $\otimes$ an abductive revision function obtained through (Def $*$). We need to show $\otimes$ satisfies postulates (K$^{\otimes}$1), (K$^{\otimes}$2) and (K$^{\otimes}$4) — (K$^{\otimes}$6) over K.

(K$^{\otimes}$1) $K^{\otimes}_{\alpha}$ is a belief set

Directly by (K$^{\ominus}$1) and (K$^{\oplus}$1).

(K$^{\otimes}$2) If $\nvdash \neg\alpha$, then $\alpha \in K^{\otimes}_{\alpha}$

Let $\nvdash \neg\alpha$. By (K$^{\ominus}$4), $\neg\alpha \notin K^{\ominus}_{\neg\alpha}$. Therefore $\alpha \in (K^{\ominus}_{\neg\alpha})^{\oplus}_{\alpha}$ by (K$^{\oplus}$2). Hence $\alpha \in K^{\otimes}_{\alpha}$ via (Def $*$).

(K$^{\otimes}$4) If $\neg\alpha \notin K$, then $K^{\oplus}_{\alpha} = K^{\otimes}_{\alpha}$

Let $\neg\alpha \notin K$, then $K^{\ominus}_{\neg\alpha} = K$ by (K$^{\ominus}$5). Therefore $(K^{\ominus}_{\neg\alpha})^{\oplus}_{\alpha} = K^{\oplus}_{\alpha}$. Hence $K^{\oplus}_{\alpha} = K^{\otimes}_{\alpha}$ via the (Def $*$).

(K$^{\otimes}$5) $K^{\otimes}_{\alpha} = K_{\perp}$ if and only if $\vdash \neg\alpha$ and $K = K_{\perp}$

(If)

Let $\vdash \neg\alpha$ and $K = K_\perp$. By $(K^\ominus 6)$, $K^\ominus_{\neg\alpha} = K$. Therefore $(K^\ominus_{\neg\alpha})^\oplus_\alpha = K^\oplus_\alpha$. However, $\neg\alpha \in K = K^\ominus_{\neg\alpha}$ since K is a belief set. So $K^\oplus_\alpha = K$ by ($K^\oplus$4). Hence, by (Def $*$), $K^\otimes_\alpha = K = K_\perp$.

(Only If)

Let $K^\otimes_\alpha = K_\perp$. That is, by (Def $*$), $(K^\ominus_{\neg\alpha})^\oplus_\alpha = K_\perp$. By (5.1) and (5.2) which follow from the postulates ($K^\oplus$1) — ($K^\oplus$5), $\neg\alpha \in K^\ominus_{\neg\alpha}$ and $K^\ominus_{\neg\alpha} = K_\perp$ (apply the contrapositive). Therefore, by ($K^\ominus$3) $\vdash \neg\alpha$ and by ($K^\ominus$6), $K^\ominus_{\neg\alpha} = K$. Hence $\vdash \neg\alpha$ and $K = K_\perp$.

($K^\otimes$6) If $\vdash \alpha \leftrightarrow \beta$, then $K^\otimes_\alpha = K^*_\beta$

Let $\vdash \alpha \leftrightarrow \beta$. It follows that $\vdash \neg\alpha \leftrightarrow \neg\beta$ (by contraposition). Therefore $K^\ominus_{\neg\alpha} = K^\ominus_{\neg\beta}$ by ($K^\ominus$4) and $(K^\ominus_{\neg\alpha})^\oplus_\alpha = (K^\ominus_{\neg\beta})^\oplus_\beta$ by ($K^\oplus$6). Hence $K^\otimes_\alpha = K^\otimes_\beta$ by (Def $*$).

■