Abstract

We prove that rooted divergence-preserving branching bisimilarity is a congruence for the
process specification language consisting of $0$, action prefix, choice, and the recursion construct
$\mu X$.  

Keywords: Process algebra; Recursion; Branching bisimulation; Divergence; Congruence.

1 Introduction

Branching bisimilarity [13] is a behavioural equivalence on processes that is compatible with ab-
straction from internal activity, while at the same time preserving the branching structure of
processes in a strong sense [9]. Branching bisimilarity abstracts to a large extent from divergence
(i.e., infinite internal activity). For instance, it identifies a process, say $P$, that may perform some
internal activity after which it returns to its initial state (i.e., $P$ has a $\tau$-loop) with a process,
say $P'$, that admits the same behaviour as $P$ except that it cannot perform the internal activity
leading to the initial state (i.e., $P'$ is $P$ without the $\tau$-loop).

In situations where fairness principles apply, abstraction from divergence is often desirable.
But there are circumstances in which abstraction from divergence is undesirable: A behavioural
equivalence that abstracts from divergence is not compatible with any temporal logic featuring an
eventually modality: for any desired state that $P'$ will eventually reach, the mentioned internal
activity of $P$ may be performed forever, and thus prevent $P$ from reaching this desired state. It is
also generally not compatible with a process-algebraic priority operator (cf. [23, pp. 130–132]) or
sequencing operator [3]. Since a divergence may be exploited to simulate recursively enumerable
branching in a computable transition system [21], a divergence-insensitive behavioural equivalence
may be considered too coarse for a theory that integrates computability and concurrency [2].

Preservation of divergence is widely considered an important correctness criterion when studying
the relative expressiveness of process calculi [14, 25, 6].

The notion of branching bisimilarity with explicit divergence, also stemming from [13], is a suit-
able refinement of branching bisimilarity that is compatible with the well-known branching-time
temporal logic CTL$^*$ without the nexttime operator $X$ (which is known to be incompatible with
abstraction from internal activity). In fact, in [12] we have proved that it is the coarsest seman-
tic equivalence on labelled transition systems with silent moves that is a congruence for parallel
composition (as found in process algebras like CCS, CSP or ACP) and only equates processes sat-
sifying the same CTL$^*_{-X}$ formulas. In [2], for stylistic reasons, branching bisimilarity with explicit
divergence was named divergence-preserving branching bisimilarity; we shall henceforth use this
term.

Divergence-preserving branching bisimilarity is the finest behavioural equivalence in the linear
time — branching time spectrum [8]. It is the principal behavioural equivalence underlying the the-
ory of executability [1, 9, 16, 17]. Reduction modulo divergence-preserving branching bisimilarity
is a part of methods for formal verification and analysis of the behaviour of systems [13, 24, 22, 26]. In [5] a game-based characterisation of divergence-preserving branching bisimilarity is presented.

Processes are usually specified in some process specification language. For compositional reasoning it is then important that the behavioural equivalence used is a congruence with respect to the constructs of that language. Following Milner [19], we consider the language basic CCS with recursion, i.e., the language consisting of 0, action prefix, and choice, extended with the recursion construct $\mu X$. This language precisely allows the specification of finite-state behaviours. It is easy to see that divergence-preserving branching bisimilarity is not a congruence for that language; in fact, it is not a congruence for any language that includes choice. The goal of this paper is to prove that adding the usual root condition suffices to obtain a congruence—and, in fact, the coarsest congruence—for the language under consideration that is included in divergence-preserving branching bisimilarity.

Recently, a congruence format was proposed for (rooted) divergence-preserving branching bisimilarity [4]. The operational rules for action prefix and choice are in this format. Unfortunately, however, this format does not support the recursion construct $\mu X$. Interestingly, as far as we know, the recursion construct has not been covered at all in the rich literature on congruence formats, with the recent exception of [10]. (The article [10] differentiates between lean and full congruences for recursion; in this article we consider the full congruence.)

The congruence result obtained in this paper should serve as a stepping stone towards a complete axiomatisation of divergence-preserving branching bisimilarity for basic CCS with recursion. Such work, inspired by Milner’s complete axiomatisation of weak bisimilarity [19], would combine the adaptations of [7] to branching bisimilarity, and of [15] to several divergence-sensitive variants of weak bisimilarity.

## 2 Rooted divergence-preserving branching bisimilarity

Let $\mathcal{A}$ be a non-empty set of actions, and let $\tau$ be a special action not in $\mathcal{A}$. Let $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$. Furthermore, let $\mathcal{V}$ be a set of recursion variables. The set of process expressions $\mathcal{E}$ is generated by the following grammar:

$$E ::= 0 \mid X \mid \alpha.E \mid \mu X.E \mid E + E \quad (\alpha \in \mathcal{A}_\tau, X \in \mathcal{V}).$$

An occurrence of a recursion variable $X$ in a process expression $E$ is bound if it is in the scope of a $\mu X$-abstraction, and otherwise it is free. We denote by $\text{FV}(E)$ the set of variables with a free occurrence in $E$. If $\bar{X} = X_0, \ldots, X_n$ is a sequence of variables, and $\bar{F} = F_0, \ldots, F_n$ is a sequence of process expressions of the same length, then we write $E[\bar{F}/\bar{X}]$ for the process expression obtained from $E$ by replacing all free occurrences of $X_i$ in $E$ by $F_i$ ($i = 0, \ldots, n$), applying $\alpha$-conversion to $E$ if necessary to avoid capture.

On $\mathcal{E}$ we define an $\mathcal{A}_\tau$-labelled transition relation $\rightarrow \subseteq \mathcal{E} \times \mathcal{A}_\tau \times \mathcal{E}$ as the least ternary relation satisfying the following rules for all $\alpha \in \mathcal{A}_\tau$, $X \in \mathcal{V}$, and process expressions $E, E', F$ and $F'$:

$$
\begin{align*}
1. & \quad \alpha.E \xrightarrow{\alpha} E \\
2. & \quad E[\mu X.E/X] \xrightarrow{\alpha} E' \\
3. & \quad E + F \xrightarrow{\alpha} E' \\
4. & \quad E + F \xrightarrow{\alpha} E' \\
\end{align*}
$$

We write $E \xrightarrow{\alpha} E'$ for $(E, \alpha, E') \in \rightarrow$ and we abbreviate the statement ‘$E \xrightarrow{\alpha} E'$ or (\(\alpha = \tau\) and $E = E'$)’ by $E \xrightarrow{\tau} E'$. Furthermore, we write $\xrightarrow{\tau}$ for the reflexive-transitive closure of $\xrightarrow{\tau}$, i.e., $E \xrightarrow{\tau} E'$ if there exist $E_0, E_1, \ldots, E_n \in \mathcal{E}$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} E_n = E'$.

A process expression is closed if it contains no free occurrences of recursion variables; we denote by $\mathcal{P}$ the subset of $\mathcal{E}$ consisting of all closed process expressions. It is easy to check that if $P$ is a closed process expression and $P \xrightarrow{\alpha} E$, then $E$ is a closed process expression too. Hence, the transition relation restricts in a natural way to closed process expressions, and thus associates with every closed process expression a behaviour. We proceed to define when two process expressions may be considered to represent the same behaviour.
Definition 1. A symmetric binary relation $R$ on $P$ is a branching bisimulation if it satisfies the following condition for all $P, Q \in P$ and $\alpha \in A_r$:

\[ (T) \text{ if } P R Q \text{ and } P \xrightarrow{\alpha} P' \text{ for some closed process expression } P', \text{ then there exist closed process expressions } Q' \text{ and } Q'' \text{ such that } Q \xrightarrow{(\alpha)} Q', P R Q'' \text{ and } P' R Q'. \]

We write $P \leftrightarrow_b Q$ if there exists a branching bisimulation $R$ such that $P R Q$. The relation $\leftrightarrow_b$ is referred to as branching bisimilarity.

We say that a branching bisimulation $R$ preserves (internal) divergence if

\[ (D) \text{ if } P R Q \text{ and there is an infinite sequence of closed process expressions } (P_k)_{k \in \omega} \text{ such that } P = P_0, P_k \xrightarrow{\tau} P_{k+1} \text{ and } P_k R Q \text{ for all } k \in \omega, \text{ then there is an infinite sequence of closed process expressions } (Q_{\ell})_{\ell \in \omega} \text{ such that } Q = Q_0, Q_{\ell} \xrightarrow{\tau} Q_{\ell+1} \text{ and } P_k R Q_{\ell} \text{ for all } k, \ell \in \omega. \]

We write $P \leftrightarrow^b Q$ if there exists a divergence-preserving branching bisimulation $R$ such that $P R Q$. The relation $\leftrightarrow^b$ was introduced in [13] under the name branching bisimilarity with explicit divergence and is here referred to as divergence-preserving branching bisimilarity.

The relation $\leftrightarrow^b$ was studied in detail in [11]; we recap some of the facts established ibidem.

First, the relation $\leftrightarrow^b$ is an equivalence relation. Second, the relation $\leftrightarrow^b$ satisfies the condition (D), with the following generalisation as a straightforward consequence.

Lemma 2. Let $P$ and $Q$ be closed process expressions. If $P \leftrightarrow^b Q$ and $P \xrightarrow{\alpha} P'$ for some closed process expressions $P'$ and $P''$, then there exist closed process expressions $Q'$ and $Q''$ such that $Q \xrightarrow{(\alpha)} Q', P' \leftrightarrow^b Q''$ and $P'' \leftrightarrow^b Q'$. Third, $\leftrightarrow^b$ also satisfies (D). In [11] several alternative definitions of divergence preservation are studied, which, in the end, all give rise to the same notion of divergence-preserving branching bisimilarity. In particular, the following alternative relational characterisations will be useful tools in the remainder.

Proposition 3. Let $P$ and $Q$ be closed process expressions. Then

- $P \leftrightarrow^b Q$ if, and only if, $P$ and $Q$ are related by some branching bisimulation $R$ satisfying

\[ (D') \text{ if } P R Q \text{ and there is an infinite sequence of closed process expressions } (P_k)_{k \in \omega} \text{ such that } P = P_0 \text{ and } P_k \xrightarrow{\tau} P_{k+1}, \text{ then there is an infinite sequence of closed process expressions } (Q_{\ell})_{\ell \in \omega} \text{ and a mapping } \sigma : \omega \to \omega \text{ such that } Q = Q_0, Q_{\ell} \xrightarrow{\tau} Q_{\ell+1} \text{ and } P_\sigma(\ell) R Q_{\ell} \text{ for all } \ell \in \omega; \text{ and} \]

- $P \leftrightarrow^b Q$ if, and only if, $P$ and $Q$ are related by some branching bisimulation $R$ satisfying

\[ (D'') \text{ if } P R Q \text{ and there is an infinite sequence of closed process expressions } (P_k)_{k \in \omega} \text{ such that } P = P_0 \text{ and } P_k \xrightarrow{\tau} P_{k+1}, \text{ then there exists a closed process expression } Q' \text{ such that } Q \xrightarrow{\tau} Q' \text{ and } P_k R Q' \text{ for some } k \in \omega. \]

Moreover, $\leftrightarrow^b$ itself satisfies (D) and (D').

Proof. See [11]; condition (D') is (D3) and condition (D'') is (D2). \[ \square \]

And finally, it was proved in [11] that $\leftrightarrow^b$ satisfies the following so-called stuttering property.

Proposition 4. Let $P$ be a closed process expression and let $Q_0, \ldots, Q_k$ be closed process expressions such that $Q_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} Q_k$. If $P \leftrightarrow^b_0 Q_0$ and $P \leftrightarrow^b_k Q_k$, then $P \leftrightarrow^b_0 Q_i$ for all $0 \leq i \leq k$.

The relation $\leftrightarrow^b$ is not compatible with $+ (0 \leftrightarrow^b \tau.0 \text{ but } 0 + a.0 \not\leftrightarrow^b \tau.0 + a.0)$, and hence not a congruence for the language we are considering. We proceed to define a relation for which we shall prove that it is the coarsest congruence for our language that is contained in $\leftrightarrow^b$. \[ 3 \]
Definition 5. Let $P$ and $Q$ be closed process expressions. We say that $P$ and $Q$ are rooted divergence-preserving branching bisimilar (notation: $P \overset{\Delta}{\leftrightarrow} b Q$) if for all $\alpha \in A_\tau$ the following holds:

(R1) if $P \xrightarrow{\alpha} P'$, then there exists a $Q'$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \overset{\Delta}{\leftrightarrow} b Q'$; and

(R2) if $Q \xrightarrow{\alpha} Q'$, then there exists a $P'$ such that $P \xrightarrow{\alpha} P'$ and $P' \overset{\Delta}{\leftrightarrow} b Q'$.

The following proposition is a straightforward consequence of the fact that $\overset{\Delta}{\leftrightarrow} b$ is an equivalence.

**Proposition 6.** The relation $\overset{\Delta}{\leftrightarrow} b$ is an equivalence relation on $\mathcal{P}$. □

Moreover, it is easy to verify that $\overset{\Delta}{\leftrightarrow} b \subseteq \overset{\Delta}{\leftrightarrow}$. We have defined the notions of $\overset{\Delta}{\leftrightarrow} b$ and $\overset{\Delta}{\leftrightarrow}$ on closed process expressions because those are thought of as directly representing behaviour. Due to the presence of the binding construct $\mu X$, it is, however, convenient to lift these notions to expressions with free variables even if the goal is simply to establish behavioural equivalence of closed process expressions.

Definition 7. Let $E$ and $F$ be process expressions, and let the sequence $\vec{X}$ of variables at least include all the variables with a free occurrence in $E$ or $F$. We write $E \overset{\Delta}{\leftrightarrow} b F$ (or $E \overset{\Delta}{\leftrightarrow} b$) if $E[\vec{P}/\vec{X}] \overset{\Delta}{\leftrightarrow} b F[\vec{P}/\vec{X}]$ for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{X}$.

It is clear from the definition above that, since $\overset{\Delta}{\leftrightarrow} b$ is an equivalence relation on $\mathcal{P}$, its lifted version is an equivalence relation on $\vec{E}$. We shall prove that it is also compatible with the constructs of the syntax, i.e., if $E \overset{\Delta}{\leftrightarrow} b F$, then $\alpha.E \overset{\Delta}{\leftrightarrow} b \alpha.F$ for all $\alpha \in A_\tau$, $\mu X.E \overset{\Delta}{\leftrightarrow} b \mu X.F$ for all $X \in \mathcal{V}$, $E + H \overset{\Delta}{\leftrightarrow} b H + F$ and $H + E \overset{\Delta}{\leftrightarrow} b F + H$ for all process expressions $H$. To prove that $\overset{\Delta}{\leftrightarrow} b$ is compatible with $\alpha$, and $+$ is straightforward, but for $\mu X$ this is considerably more work.

### 3 The congruence proof

Our proof that $\overset{\Delta}{\leftrightarrow} b$ is compatible with $\mu X$ relies on the following observation: If $\vec{Y}$ is some sequence of variables and $\vec{P}$ is a sequence of closed terms of the same length, then, on the one hand, $E \overset{\Delta}{\leftrightarrow} b F$ implies $E[\vec{P}/\vec{Y}] \overset{\Delta}{\leftrightarrow} b F[\vec{P}/\vec{Y}]$, and, on the other hand, if $X$ does not occur in $\vec{Y}$, then from $\mu X.E[\vec{P}/\vec{Y}] \overset{\Delta}{\leftrightarrow} b \mu X.F[\vec{P}/\vec{Y}]$ it follows that $\mu X.E[\vec{P}/\vec{Y}] \overset{\Delta}{\leftrightarrow} b (\mu X.F)[\vec{P}/\vec{Y}]$. Therefore, it is enough to establish that $E \overset{\Delta}{\leftrightarrow} b F$ implies $\mu X.E \overset{\Delta}{\leftrightarrow} b \mu X.F$ in the special case that $E$ and $F$ are process expressions with no other free variables than $X$; such process expressions will be called $X$-closed.

The rest of this section is organised as follows.

We shall first characterise, in Section 3.1, the relation $\overset{\Delta}{\leftrightarrow} b$ on $X$-closed process expressions in terms of the transition relation on $X$-closed process expressions.

Then, in Section 3.2, we shall present a suitable notion of rooted divergence-preserving branching bisimulation up to $\overset{\Delta}{\leftrightarrow} b$, and we shall prove that every pair of rooted divergence-preserving branching bisimilar $X$-closed process expressions $(E, F)$ gives rise to a relation $\mathcal{R}_X^\mu$ of which we can show that it is a rooted divergence-preserving branching bisimulation up to $\overset{\Delta}{\leftrightarrow} b$. The relation $\mathcal{R}_X^\mu$ will be defined in such a way that it relates $\mu X.E$ and $\mu X.F$ and thus allows us to conclude that these process expressions are rooted divergence-preserving bisimilar.

In Section 3.3, we shall then put the pieces together and prove $\overset{\Delta}{\leftrightarrow} b$ is the coarsest congruence contained in $\overset{\Delta}{\leftrightarrow}$ for basic CCS with recursion.

#### 3.1 $\overset{\Delta}{\leftrightarrow} b$ on $X$-closed process expressions

We say that a process expression $E$ is $X$-closed if $FV(E) \subseteq \{X\}$; the set of all $X$-closed process expressions is denoted by $\mathcal{P}_X$. Note that if $E$ is $X$-closed and $E \xrightarrow{\alpha} E'$, then $E'$ is $X$-closed too, and so the $A_\tau$-labelled transition relation restricts in a natural way to $X$-closed process expressions.
Definition 8. We define when $X$ is exposed in a (not necessarily $X$-closed) process expression $E$ with induction on the structure of $E$:

i. if $E = X$, then $X$ is exposed in $E$;

ii. if $E = \mu Y. E'$, then $X$ is a recursion variable distinct from $X$ and $X$ is exposed in $E'$, then $X$ is exposed in $E$;

iii. if $E = E_1 + E_2$ and $X$ is exposed in $E_1$ or $E_2$, then $X$ is exposed in $E$.

Note that the variable $X$ is exposed in $E$ if, and only if, $E$ has an unguarded occurrence of $X$ in the sense of $[9]$.

We establish a relationship between the transitions of a closed process expression $E[P/X]$ that is obtained by substituting a closed process expression $P$ for the variable $X$ in an $X$-closed process expression $E$, and the transitions of $E$ and $P$.

Lemma 9. Let $E$ be an $X$-closed process expression, and let $P$ be a closed process expression.

1. If $E \xrightarrow{\alpha} E'$, then $E[P/X] \xrightarrow{\alpha} E'[P/X]$, and if $X$ is exposed in $E$ and $P \xrightarrow{\alpha} P'$, then $E[P/X] \xrightarrow{\alpha} P'$.

2. If $E[P/X] \xrightarrow{\alpha} P'$ for some closed process expression $P'$, then either there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or $X$ is exposed in $E$, $P \xrightarrow{\alpha} P'$ and every derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.

Proof. Statement 1 of the lemma is established with straightforward inductions on a derivation of $E \xrightarrow{\alpha} E'$ and on the structure of $E$.

We proceed to establish with induction on a derivation of $E[P/X] \xrightarrow{\alpha} P'$ that there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or $X$ is exposed in $E$, $P \xrightarrow{\alpha} P'$ and a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$.

This implies statement 2.

We distinguish cases according to the structure of $E$:

- Clearly, $E$ cannot be $0$, for if $E = 0$, then $E[P/X] = 0$, and $0$ does not admit any transitions.

- If $E = X$, then $X$ is exposed in $E$ and $P = E[P/X] \xrightarrow{\alpha} P'$. It is then also immediate that the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.

- If $E = \beta.E'$ for some $\beta \in \mathcal{A}_r$ and some $X$-closed process expression $E'$, then $\beta = \alpha$ and $E \xrightarrow{\beta} E'$. Since $E[P/X] = \beta.(E'[P/X])$, rule 1 is the last rule applied in the derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$, so $P' = E'[P/X]$.

- If $E = \mu Y.F$ for some process expression $F$ with $FV(F) \subseteq \{X, Y\}$, then there are two subcases:

  On the one hand, if $Y = X$, then, since $X$ has no free occurrence in $E$, we have $E = E[P/X] \xrightarrow{\alpha} P'$. Furthermore, since $P'$ is closed we have that $P' = P'[P/X]$.

  On the other hand, if $Y \neq X$, then $E[P/X] = \mu Y.(F[P/X])$, and therefore the last rule applied in the considered derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$ is rule 2. Consequently, the considered derivation has a proper subderivation of the transition $F[P/X][\mu Y.(F[P/X])][Y] \xrightarrow{\alpha} P'$. Note that $F[P/X][\mu Y.(F[P/X])][Y] = (F[\mu Y.F][Y])[P/X]$. Hence, by the induction hypothesis, either there exists an $E'$ such that $F[\mu Y.F][Y] \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or $X$ is exposed in $F[\mu Y.F][Y]$, $P \xrightarrow{\alpha} P'$, and the derivation of $F[\mu Y.F][Y][P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \rightarrow P'$ as a subderivation. In the first case, it follows from $F[\mu Y.F][Y] \xrightarrow{\alpha} E'$, by rule 2, that $E = \mu Y.F \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$. In the second case, it suffices to note that $X$ is exposed in $F$, hence also in $E$, and that a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$.
If $E = E_1 + E_2$, then $[E][P/X] = E_1[P/X] + E_2[P/X]$.

The last rule applied in the considered derivation of the transition $[E][P/X] \xrightarrow{\alpha} P'$ is either rule 5 or rule 4.

If it is rule 4 then $E_1[P/X] \xrightarrow{\alpha} P'$, and since this transition has a derivation that is a proper subderivation of the considered derivation of $[E][P/X] \xrightarrow{\alpha} P'$, by the induction hypothesis it follows that either $E_1 \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or $X$ is exposed in $E_1$, $P \xrightarrow{\alpha} P'$, and a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation the derivation of $E_1[P/X] \xrightarrow{\alpha} P'$.

In the first case, it remains to note that then also $E \xrightarrow{\alpha} E'$, and in the second case, it remains to note that $X$ is also exposed in $E$.

If the last rule applied in the considered derivation is rule 4, then the proof is analogous.

**Corollary 10.** Let $E$ be an $X$-closed process expression. If $E[\mu X.E/X] \xrightarrow{\alpha} P'$ for some closed process expression $P'$, then there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P' = E'[\mu X.E/X]$.

**Proof.** Consider a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P'$ that is minimal in the sense that it does not have a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P'$ as proper subderivation. Let $P = \mu X.E$. Since every derivation of $P \xrightarrow{\alpha} P'$ has a derivation of $E[P/X] \xrightarrow{\alpha} P'$ as a proper subderivation (see the operational rules, and rule 2 in particular), it follows that the considered derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P'$ does not have a subderivation of $P \xrightarrow{\alpha} P'$. Hence, by Lemma 9.2 there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P' = E'[\mu X.E/X]$. □

**Corollary 11.** Let $G_0$ and $E$ be $X$-closed process expressions. If there is an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $G_0[\mu X.E/X] = P_0$ and $P_k \xrightarrow{\alpha} P_{k+1}$ for all $k \in \omega$, then there is an infinite sequence of $X$-closed process expressions $(G_k)_{k \in \omega}$ such that $P_k = G_k[\mu X.E/X]$ and, for all $k \in \omega$, either $G_k \xrightarrow{\alpha} G_{k+1}$ or $X$ is exposed in $G_k$ and $E \xrightarrow{\alpha} G_{k+1}$.

**Proof.** We construct $(G_k)_{k \in \omega}$ with induction on $k$. Suppose that $G_k$ with $G_k[\mu X.E/X] = P_k$ has already been constructed. Since $P_k \xrightarrow{\alpha} P_{k+1}$, by Lemma 9.2 there are two cases: either there is a $G_{k+1}$ with $G_k \xrightarrow{\alpha} G_{k+1}$ and $P_{k+1} = G_{k+1}[\mu X.E/X]$, in which case we are done, or $X$ is exposed in $G_k$ and $\mu X.E \xrightarrow{\alpha} P_{k+1}$. In the latter case $E[\mu X.E/X] \xrightarrow{\alpha} P_{k+1}$ (see the operational rules, and rule 2 in particular). By Corollary 10 there exists an $X$-closed process expression $G_{k+1}$ such that $G_k \xrightarrow{\alpha} G_{k+1}$ and $P_{k+1} = G_{k+1}[\mu X.E/X]$. □

Let $E$ and $E'$ be process expressions. We write $E \xrightarrow{\Delta} E'$ if there exists an $\alpha \in A_\tau$ such that $E \xrightarrow{\alpha} E'$, and denote by $\xrightarrow{\Delta}^*$ the reflexive-transitive closure of $\xrightarrow{\Delta}$. If $E \xrightarrow{\Delta}^* E'$, then we say that $E'$ is reachable from $E$.

**Proposition 12** ([7, Proposition 1]). If $E$ is a process expression, then the set of all expressions reachable from $E$ is finite.

On $E$ we define an $V \uplus A_\tau$-labelled transition relation $\xrightarrow{\Delta} \subseteq E \times (V \uplus A_\tau) \times E$ as the least ternary relation satisfying, besides the four rules of Section 2, also the rule

$$X \xrightarrow{\alpha} 0$$

for each $X \in V$. Intuitively, the $V \uplus A_\tau$-labelled transition relation treats a process expression $E$ as the closed term obtained from $E$ by replacing all free occurrences of the variable $X$ by the closed process expression $X.0$ in which $X$ is interpreted as an action instead of as a recursion variable. Note that a variable $X$ is exposed in an expression $E$ according to Definition 3 if $E \xrightarrow{\Delta} F$, which is the case iff $E \xrightarrow{\Delta} 0$. Now let $\xrightarrow{\Delta}^\tau_{\uplus X}$ and $\xrightarrow{\Delta}^\tau_{\uplus X}$ be defined exactly like $\xrightarrow{\Delta}^\tau$ and $\xrightarrow{\Delta}^\tau$, but using the $V \uplus A_\tau$-labelled transition relation instead of the $A_\tau$-labelled one, and applying all definitions directly to expressions with free variables, instead of applying the lifting of Definition 7.
We proceed to show that on X-closed process expressions \( \equiv_{\Delta X} \) coincides with \( \equiv_{\Delta} \), and \( \equiv_{\Delta X} \) with \( \equiv_{\Delta} \). This characterisation, for weak and branching bisimilarity without preservation of divergence, stems from [19] and [7]. Here we use it solely to obtain Corollaries [15] and [16].

**Lemma 13.** The relation

\[ B = \{(E[P/X], F[P/X]) \mid E, F \text{ are } X \text{-closed, } E \equiv_{\Delta X} F, P \text{ is closed}\} \]

is a branching bisimulation satisfying \( \mathcal{P}' \) of Proposition [3].

**Proof.** It is immediate from its definition that \( B \) is symmetric.

We show it satisfies (T). Suppose, \( E, F \) are \( X \)-closed, \( E \equiv_{\Delta X} F \) and \( P \) closed. Let \( E[P/X] \xrightarrow{a} P' \) for some \( a \in \mathcal{A}_r \). By Lemma [9] either there exists an \( X \)-closed process expression \( E' \) such that \( E \xrightarrow{a} E' \) and \( P' = E'[P/X] \), or \( X \) is exposed in \( E \) and \( P \xrightarrow{a} P' \). In the first case, since \( E \equiv_{\Delta X} F \), there exist process expressions \( F' \) and \( F'' \) such that \( F \xrightarrow{a} F' \) and \( E \equiv_{\Delta X} F'' \). By Lemma [11] \( F[P/X] \xrightarrow{a} F'[P/X] \). Furthermore, \( E[P/X] \equiv F'[P/X] \) and \( P' = F'[P/X] \). In the second case, since \( X \) is exposed in \( E \), we have that \( E \xrightarrow{X} 0 \) and hence, since \( E \equiv_{\Delta X} F \), there exist process expressions \( F' \) and \( F'' \) such that \( F \xrightarrow{a} F' \) and \( E \equiv_{\Delta X} F'' \). By Lemma [11] \( X \xrightarrow{a} F' \). Furthermore, \( B \equiv F''[P/X] \).

It remains to show \( B \) satisfies \( \mathcal{P}' \). Suppose, \( E, F \) are \( X \)-closed, \( E \equiv_{\Delta X} F \) and \( P \) closed, and there is an infinite sequence of closed process expressions \( \alpha \) such that \( E[P/X] \xrightarrow{\alpha} P_k \) and \( P_k \xrightarrow{\alpha} P_{k+1} \). By Lemma [9] either there exists an infinite sequence of \( X \)-closed process expressions \( \alpha \equiv \beta \) for some \( k \in \omega \) such that \( E_0 = E \), \( E_k \xrightarrow{\alpha} E_{k+1} \), and \( P_k = E_{k+1}[P/X] \) for all \( k \in \omega \), or there exists a finite sequence of \( X \)-closed process expressions \( \alpha \equiv \beta \), for some \( k \in \omega \) such that \( E_0 = E \), \( E_i \xrightarrow{\alpha} E_{i+1} \), and \( P_i = E_{i+1}[P/X] \). In the first case, since \( E \equiv_{\Delta X} F \), using \( \alpha \equiv \beta \), there exist a process expression \( F' \) such that \( F \xrightarrow{\alpha} F' \) and \( E \equiv_{\Delta X} F' \) for some \( k \in \omega \). By Lemma [11] \( F[P/X] \xrightarrow{\alpha} F'[P/X] \), and \( E_k \equiv F'[P/X] \). Furthermore, \( E_{k+1}[P/X] \equiv F'[P/X] \). In the second case, since \( E \equiv_{\Delta X} F \), with induction on \( i \) there exists a sequence \( F_0, \ldots, F_{k+1} \) for all \( i \leq k \) and \( F \equiv_{\Delta X} F_{k+1} \), and \( F_\rho \xrightarrow{\rho} F_{\rho(i)} \) for all \( i \leq k \). Let \( \rho : \{0, \ldots, m\} \rightarrow \{0, \ldots, k\} \) and \( \rho(m) = m \) such that \( F = F_0 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} F_m \xrightarrow{\alpha} F_{m+1} \) and \( F \equiv_{\Delta X} F_{m+1} \). Furthermore, \( F_{m+1} \equiv F_{k+1} \).

For every \( \alpha \in \mathcal{A}_r \) and \( \alpha \in \omega \), we define the closed process expression \( \alpha \) inductively by \( \alpha_0 = 0 \) and \( \alpha^{n+1} = \alpha \cdot \alpha^n \). Note that, if \( \alpha \neq \tau \), then \( \alpha \equiv_{\Delta X} \alpha \) implies \( i = j \). Recall that we have assumed that \( \Delta \) is non-empty; we now fix, for the remainder of this section, a particular action \( \alpha \in \mathcal{A} \).

**Proposition 14.** Let \( E \) and \( F \) be \( X \)-closed expressions. Then \( E \equiv_{\Delta X} F \) iff \( E \equiv_{\Delta b} F \) and \( E \equiv_{\Delta X} F \) iff \( E \equiv_{\Delta b} F \).

**Proof.** We need to show that \( E \equiv_{\Delta X} F \) iff \( E[P/X] \equiv_{\Delta X} F[P/X] \) for each closed process expression \( P \), and likewise \( E \equiv_{\Delta X} F \) iff \( E[P/X] \equiv_{\Delta X} F[P/X] \) for each closed process expression \( P \).

“Only if”: Lemma [13] immediately yields that \( E \equiv_{\Delta X} F \) implies \( E[P/X] \equiv_{\Delta X} F[P/X] \) for each closed process expression \( P \). Now let \( E \equiv_{\Delta X} F \) and \( E[P/X] \equiv_{\Delta X} F'[P/X] \) for each closed process expression \( P \). We have that \( E \equiv_{\Delta X} F \) and \( E \equiv_{\Delta X} F' \). By Lemma [11] \( E \equiv_{\Delta X} F' \). Furthermore, \( \equiv_{\Delta X} F' \). In the second case, since \( X \) is exposed in \( E \), we have that \( E \xrightarrow{X} 0 \) and hence, since \( E \equiv_{\Delta X} F \), there exists a process expression \( F' \) such that \( F \xrightarrow{X} F' \). By Lemma [11] \( F[P/X] \xrightarrow{X} F'[P/X] \). Furthermore, \( P \equiv_{\Delta X} P' \). The other clause follows by symmetry, thus yielding \( E[P/X] \equiv_{\Delta X} F'[P/X] \).

“If”: Let \( E \) and \( F \) be \( X \)-closed process expressions. Since by Proposition [13] the set of all process expressions reachable from \( E \) and \( F \) is finite, there exists a natural number \( n \in \omega \) such that for all \( G \) reachable from \( E \) or \( F \) it holds that \( G \equiv_{\Delta} a^n \), and thus \( G[a^{n+1}/X] \equiv_{\Delta b} a^n \). Let

\[ \mathcal{R} = \{(E', F') \mid E \xrightarrow{*} E', F \xrightarrow{*} F', E'[a^{n+1}/X] \equiv_{\Delta b} F'[a^{n+1}/X]\} \]
Claim: The symmetric closure of $R$ is a branching bisimulation satisfying \( \Delta'' \) w.r.t. the $V \uplus A-$labelled transition relation.

Proof of the claim: To prove that $R$ satisfies condition \( \Delta'' \) of Definition \( \Delta \), let $E'$ and $F'$ be such that $E' R F'$, and suppose that $E' \xrightarrow{\Delta''} F''$. Then $E'[a^{n+1}/X] \xrightarrow{\Delta''} F'[a^{n+1}/X]$ and, using Lemma \( \text{[11]} \), $E'[a^{n+1}/X] \xrightarrow{\alpha} E''[a^{n+1}/X]$. Since $E'[a^{n+1}/X] \xrightarrow{\Delta''} F'[a^{n+1}/X]$ there exist closed process expressions $Q''$ and $Q'''$ such that $F'[a^{n+1}/X] \xrightarrow{\alpha} Q''$ and $E'[a^{n+1}/X] \xrightarrow{\Delta''} Q''$. By Lemma \( \text{[12]} \) using that $a \neq \tau$, there exists a $X$-closed process expression $F''$ such that $F' \xrightarrow{\alpha} F''$ and $Q'' = F''[a^{n+1}/X]$; moreover, either there exists an $X$-closed process expression $F''''$ such that $F'' \xrightarrow{\alpha} F'''$ and $Q''' = F'''[a^{n+1}/X]$, or $X$ is exposed in $F''$ and $a^{n+1} \xrightarrow{\Delta''} Q'''$. In the latter case we would have $E''[a^{n+1}/X] \xrightarrow{\Delta''} Q''' = a^n$, which is impossible by the choice of $n$. So the former case applies: we have $F' \xrightarrow{\alpha} F''$ and $E' R F''$. The case that $F' \xrightarrow{\alpha} F''$ proceeds by symmetry, so the symmetric closure of $R$ satisfies condition \( \Delta'' \).

To show that $R$ (and its symmetric closure) satisfies \( \Delta''' \), let $(E_k)_{k \in \omega}$ be an infinite sequence of $X$-closed process expressions such that $E_k \xrightarrow{\alpha} E_{k+1}$ for all $k \in \omega$, and let $F_0$ be such that $E_0 R F_0$.

Then $E_0[a^{n+1}/X] \xrightarrow{\Delta''} F_0[a^{n+1}/X]$ and by Lemma \( \text{[11]} \), $E_k[a^{n+1}/X] \xrightarrow{\alpha} E_{k+1}[a^{n+1}/X]$ for all $k \in \omega$. Using \( \text{[17]} \), there exist a process expression $Q'$ such that $F_0 \xrightarrow{\Delta''} Q'$ and $E_k[a^{n+1}/X] \xrightarrow{\Delta''} Q'$ for some $k \in N$. By Lemma \( \text{[12]} \) using that $a \neq \tau$, there exists a $X$-closed process expression $F'$ such that $F_0 \xrightarrow{\Delta''} F'$ and $Q' = F'[a^{n+1}/X]$. Furthermore, $E_k R F'$. Application of the claim: Let $E[P/X] \xrightarrow{\Delta''} F[P/X]$ for each closed process expression $P$. Then $E[a^{n+1}/X] \xrightarrow{\Delta''} F[a^{n+1}/X]$. The claim yields $E \xrightarrow{\Delta''} F$.

Now let $E'[P/X] \xrightarrow{\Delta''} F'[P/X]$ for each closed $P$. Then $E[a^{n+1}/X] \xrightarrow{\Delta''} F[a^{n+1}/X]$. Suppose that $E \xrightarrow{\Delta''} E'$ with $\alpha \in \mathcal{A}_t$. Then $E[a^{n+1}/X] \xrightarrow{\alpha} E'[a^{n+1}/X]$ by Lemma \( \text{[11]} \). So there exists a $Q'$ with $F[a^{n+1}/X] \xrightarrow{\alpha} Q'$ and $E'[a^{n+1}/X] \xrightarrow{\alpha} Q'$. By Lemma \( \text{[12]} \) either there exists an $X$-closed process expression $F'$ such that $E \xrightarrow{\alpha} F'$ and $Q' = F'[a^{n+1}/X]$, or $X$ is exposed in $F$ and $a^{n+1} \xrightarrow{\alpha} Q'$. In the latter case we would have $E'[a^{n+1}/X] \xrightarrow{\alpha} a^n$, which is impossible by the choice of $n$. So the former case applies, and $E' R F'$. The claim yields $E' \xrightarrow{\Delta''} F'$. The other clause follows by symmetry, so $E \xrightarrow{\Delta''} F$.

The following is an immediate corollary of Propositions \( \text{[14]} \), \( \text{[8]} \) and \( \text{[3]} \).

**Corollary 15.** Let $E$ and $F$ be $X$-closed process expressions such that $E \xrightarrow{\Delta''} F$.

1. If $E \xrightarrow{\alpha} E'$, then there exist $X$-closed process expressions $F_0, \ldots, F_n$ and $F'$ such that $F = F_0 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} F_n \xrightarrow{\alpha} F'$ such that $E \xrightarrow{\Delta''} F_i (0 \leq i \leq n)$ and $E' \xrightarrow{\Delta''} F'$.

2. If $X$ is exposed in $E$, then there exist $k \geq 0$ and $X$-closed process expressions $F_0, \ldots, F_k$ such that $F = F_0 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} F_k, E \xrightarrow{\Delta''} F_i (0 \leq i \leq k)$, and $X$ is exposed in $F_k$.

3. If there is an infinite sequence of $X$-closed process expressions $(E_k)_{k \in \omega}$ such that $E = E_0$ and $E_k \xrightarrow{\alpha} E_{k+1}$, then there exists an $X$-closed process expression $F'$ such that $F \xrightarrow{\alpha} F'$ and $E_k \xrightarrow{\Delta''} F'$ for some $k \in \omega$.

Similarly, by combining Propositions \( \text{[14]} \) and Definition \( \text{[5]} \) we get the following corollary.

**Corollary 16.** Let $E$ and $F$ be $X$-closed process expressions such that $E \xrightarrow{\Delta''} F$. If $E \xrightarrow{\alpha} E'$, then there exists an $X$-closed process expression $F'$ such that $F \xrightarrow{\alpha} F'$ and $E' \xrightarrow{\Delta''} F'$.

### 3.2 Rooted divergence-preserving branching bisimulation up to $\xrightarrow{\Delta''}$

As was already illustrated by Milner [20], a suitable up-to relation is a crucial tool in the proof that a behavioural equivalence is compatible with the recursion construct. In [7], Milner’s notion of weak bisimulation up to weak bisimilarity is adapted to branching bisimulation up to branching bisimilarity. Here we make two further modifications. Not only do we add a divergence condition; we also incorporate rootedness into the relation.
Definition 17. Let \( \mathcal{R} \) be a symmetric binary relation on \( \mathcal{P} \), and denote by \( \mathcal{R}^u \) the relation \( \bowtie_{\Delta}^u : \mathcal{R} \bigtriangleup \bowtie_{\Delta}^u \). We say that \( \mathcal{R} \) is a rooted divergence-preserving branching bisimulation up to \( \bowtie_{\Delta}^u \) if for all \( P, Q \in \mathcal{P} \) such that \( P \mathcal{R} Q \) the following three conditions are satisfied:

(U1) \( P \xrightarrow{\alpha} P' \), then there exist \( Q' \) such that \( Q \xrightarrow{\alpha} Q' \), and \( P' \mathcal{R}^u Q' \).

(U2) \( P \xrightarrow{\alpha} P' \), then there exist \( Q' \) and \( Q'' \) such that \( Q \xrightarrow{\alpha} Q'' \), \( Q'' \xrightarrow{\alpha} Q' \), \( P'' \mathcal{R}^u Q'' \) and \( P' \mathcal{R}^u Q' \).

(U3) if there exists an infinite sequence of closed process expressions \( (P_k)_{k \in \omega} \) such that \( P = P_0 \), and \( P_k \xrightarrow{\tau} P_{k+1} \) for all \( k \in \omega \), then there also exists an infinite sequence of closed process expressions \( (Q_k)_{k \in \omega} \) and a mapping \( \sigma : \omega \rightarrow \omega \) such that \( Q = Q_0 \), and \( Q_k \xrightarrow{\tau} Q_{\alpha(k)} \) and \( P_{\sigma(k)} \mathcal{R}^u Q_k \) for all \( k \in \omega \).

Proposition 18. Let \( P \) and \( Q \) be closed process expressions and let \( \mathcal{R} \) be a rooted divergence-preserving branching bisimulation up to \( \bowtie_{\Delta}^u \). If \( P \mathcal{R} Q \), then \( P \bowtie_{\Delta}^u b Q \).

Proof. If \( P \mathcal{R} Q \) and \( P \xrightarrow{\alpha} P' \), then since \( \mathcal{R} \) satisfies condition (U1) of Definition 17 there exists a \( Q' \) such that \( Q \xrightarrow{\alpha} Q' \) and \( P' \mathcal{R}^u Q' \). Furthermore, since \( \mathcal{R} \) is symmetric, whenever \( P \mathcal{R} Q \) also \( Q \mathcal{R} P \), so if \( Q \xrightarrow{\alpha} Q' \), then by condition (U1) of Definition 17 there exists a \( P' \) such that \( P \xrightarrow{\alpha} P' \) and \( Q' \mathcal{R}^u P' \). It remains to establish that \( P \bowtie_{\Delta}^u Q' \), and for this, it suffices by Proposition 3 to prove that \( \mathcal{R}^u \) is a branching bisimulation satisfying (U3).

Note that, since \( \bowtie_{\Delta}^u \) and \( \mathcal{R} \) are both symmetric, also \( \mathcal{R}^u \) is symmetric.

To prove that \( \mathcal{R}^u \) satisfies (U3), let \( F_0, P_1, Q_0 \) and \( Q_1 \) be closed process expressions such that \( P_1 \bowtie_{\Delta}^u P_0 \mathcal{R} Q_0 \bowtie_{\Delta}^u Q_1 \), and suppose that \( P_1 \xrightarrow{\alpha} P_1' \). Since \( P_1 \bowtie_{\Delta}^u P_0 \) and \( \bowtie_{\Delta}^u \) satisfies (U1), there exist \( P_0', P_0'' \) such that \( P_0 \xrightarrow{\alpha} P_0' \), \( P_1 \bowtie_{\Delta}^u P_0'' \), and \( P_1' \bowtie_{\Delta}^u P_0'' \). So it follows by condition (U2) of Definition 17 that there exist \( Q_0' \) and \( Q_0'' \) such that \( Q_0 \xrightarrow{\alpha} Q_0' \), \( Q_0' \bowtie_{\Delta}^u Q_0'' \), \( P_0'' \mathcal{R}^u Q_0'' \), and \( P_0' \mathcal{R}^u Q_0' \). Hence, since \( Q_0 \bowtie_{\Delta}^u Q_1 \), by Lemma 2 there exist closed process expressions \( Q_1' \) and \( Q_1'' \) such that \( Q_1 \xrightarrow{\alpha} Q_1' \), \( Q_1'' \bowtie_{\Delta}^u Q_1' \), \( Q_0' \bowtie_{\Delta}^u Q_1'' \), and \( Q_0'' \bowtie_{\Delta}^u Q_1' \). Note, moreover, that \( P_1 \bowtie_{\Delta}^u P_0'' \mathcal{R}^u Q_0'' \bowtie_{\Delta}^u Q_1'' \) whence \( P_1 \mathcal{R}^u Q_1'' \), and \( P_1' \bowtie_{\Delta}^u P_0' \mathcal{R}^u Q_0' \bowtie_{\Delta}^u Q_1' \) whence \( P_1' \mathcal{R}^u Q_1' \).

It remains to prove that \( \mathcal{R}^u \) satisfies (U3) of Proposition 3. To this end, let \( F_0, P_1, Q_0 \) and \( Q_1 \) be closed process expressions such that \( P_1 \bowtie_{\Delta}^u P_0 \mathcal{R} Q_0 \bowtie_{\Delta}^u Q_1 \), and suppose that there exists an infinite sequence of closed process expressions \( (P_k)_{k \in \omega} \) such that \( P_1 = P_{1,0} \) and \( P_{1,k} \xrightarrow{\tau} P_{1,k+1} \). Then, since \( P_1 \bowtie_{\Delta}^u P_0 \), by Proposition 3 there exists an infinite sequence of closed process expressions \( (P_{0,k})_{k \in \omega} \) and a mapping \( \sigma_P : \omega \rightarrow \omega \) such that \( P_0 = P_{0,0} \), \( P_{0,t} \xrightarrow{\tau} P_{0,t+1} \), and \( P_{1,\sigma_P(t)} \bowtie_{\Delta}^u P_{0,t} \) for all \( t \in \omega \). Hence, since \( P_0 \mathcal{R} Q_0 \) and \( \mathcal{R} \) is a divergence-preserving branching bisimulation up to \( \bowtie_{\Delta}^u \), there exists an infinite sequence of closed process expressions \( (Q_{0,m})_{m \in \omega} \) and a mapping \( \sigma_{P,Q} : \omega \rightarrow \omega \) such that \( Q_0 = Q_{0,0} \), \( Q_{0,m} \xrightarrow{\tau} Q_{0,m+1} \), \( P_{0,\sigma_{P,Q}(m)} \mathcal{R}^u Q_{0,m} \) for all \( m \in \omega \). Hence, since \( Q_0 \bowtie_{\Delta}^u Q_1 \), by Proposition 3 there exists an infinite sequence of closed process expressions \( (Q_{1,n})_{n \in \omega} \) and a mapping \( \sigma_Q : \omega \rightarrow \omega \) such that \( Q_1 = Q_{1,0} \), \( Q_{1,n} \xrightarrow{\tau} Q_{0,n+1} \) and \( Q_{0,\sigma_Q(n)} \bowtie_{\Delta}^u Q_{1,n} \) for all \( n \in \omega \). We define

\[
\sigma = \sigma_P \circ \sigma_{P,Q} \circ \sigma_Q,
\]
and then we have that \( P_{1,\sigma(n)} \bowtie_{\Delta}^u ; \mathcal{R}^u ; \bowtie_{\Delta}^u Q_{1,n} \), and hence \( P_{1,\sigma(n)} \mathcal{R}^u Q_{1,n} \) for all \( n \in \omega \).
Lemma 19. For all $X$-closed process expressions $G$, if $G\mu X.E/X \xrightarrow{\alpha} P$, then there exists a $Q$ such that $G\mu X.E/X \xrightarrow{\alpha} Q$ and $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q$.

Proof. Let $G$ be an $X$-closed process expression, and suppose that $G\mu X.E/X \xrightarrow{\alpha} P$; we proceed with induction on a derivation of this transition. By Lemma 19 there are two cases: either the transition under consideration stems directly from $G$, i.e., there exists a $G'$ such that $G \xrightarrow{\alpha} G'$ and $P = G'\mu X.E/X$, or $X$ is exposed in $G$, $\mu X.E \xrightarrow{\alpha} P$ and every derivation of $G\mu X.E/X \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation.

In the first case, Lemma 19 implies $G\mu X.E/X \xrightarrow{\alpha} G'\mu X.E/X$ and $P = G'\mu X.E/X$, and according to the definition of $\mathcal{R}_{E,F}$, $G'\mu X.E/X$, so, since $\Delta^{\alpha}$ is reflexive, also $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} G'\mu X.E/X$.

In the second case, since the considered derivation of the transition $G\mu X.E/X \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation, and the last rule applied in this subderivation must be $\beta$, it follows that the considered derivation of $G\mu X.E/X \xrightarrow{\alpha} P$ has a derivation of $E\mu X.E/X \xrightarrow{\alpha} P$ as a proper subderivation. So by the induction hypothesis there exists a $Q$ such that $E\mu X.E/X \xrightarrow{\alpha} Q$ and $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q$. Furthermore, since $E \Delta^{\alpha} F$, and $E\mu X.E/X \xrightarrow{\alpha} F\mu X.E/X$, it follows that there exists a $R$ such that $F\mu X.E/X \xrightarrow{\alpha} R$ and $Q \Delta^{\alpha} R$. It follows from $F\mu X.E/X \xrightarrow{\alpha} R$ that $\mu X.E \xrightarrow{\alpha} R$. Since $X$ is exposed in $G$, Lemma 19 yields $G\mu X.E/X \xrightarrow{\alpha} R$. From $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q$ and $Q \Delta^{\alpha} R$ it follows that $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} R$. □

As an immediate corollary to Lemma 19 we get that if $E \Delta^{\alpha} F$, then $\mathcal{R}_{E,F}$ satisfies the first condition of rooted divergence-preserving branching bisimulations up to $\Delta^{\alpha}$.

Corollary 20. $\mathcal{R}_{E,F}$ satisfies condition (E1) of Definition 17.

With a little more work, Lemma 19 can also be used to derive that $\mathcal{R}_{E,F}$ satisfies the second condition of rooted divergence-preserving branching bisimulations up to $\Delta^{\alpha}$. To this end, we first prove the following lemma.

Lemma 21. Let $P$ and $Q$ be closed process expressions. If $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q$ and $P \xrightarrow{\alpha} P'$, then there exist $Q'$ and $Q''$ such that $Q \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$, $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q''$ and $P' \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q'$. 

Proof. Suppose that $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q$ and $P \xrightarrow{\alpha} P'$. Then there exists an $R$ such that $P \mathcal{R}_{E,F} R \Delta^{\alpha} Q$, and according to the definition of $\mathcal{R}_{E,F}$ there exists an $X$-closed process expression $G$ such that either $P = G\mu X.E/X$ and $R = G\mu X.E/X$ or $P = G\mu X.E/X$ and $R = G\mu X.E/X$. There is clearly no loss of generality in assuming that $P = G\mu X.E/X$ and $R = G\mu X.E/X$. By Lemma 19 there exists an $R'$ such that $R \xrightarrow{\alpha} R'$ and $P' \mathcal{R}_{E,F} \vdash \Delta^{\alpha} R'$. Hence, since $R \Delta^{\alpha} Q$, there exist $Q'$ and $Q''$ such that $Q \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$, $R \Delta^{\alpha} Q''$ and $R' \Delta^{\alpha} Q'$. It follows that $P \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q''$ and $P' \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q'$, so the proof of the lemma is complete. □

Applying Lemma 21 with induction on the length of a transition sequence that gives rise to $P \xrightarrow{\alpha} P'$, it is straightforward to establish the following corollary.

Corollary 22. $\mathcal{R}_{E,F}$ satisfies condition (E2) of Definition 17.

Proof. Let $P_0, \ldots, P_n, P'_0, Q, \in \mathcal{P}$, such that $P_n \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q$, $P_{i+1} \xrightarrow{\alpha} P_i$ for all $0 \leq i < n$, and $P_0 \xrightarrow{\alpha} P_n$; we prove with induction on $n$ that there exists $Q'$ and $Q''$ such that $Q \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$, $P_0 \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q''$, and $P' \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q'$. Note that, since $\Delta^{\alpha}$ is reflexive, it then follows that $\mathcal{R}_{E,F}$ satisfies (E2) of Definition 17.

If $n = 0$, then we distinguish two cases: If $\alpha = \tau$ and $P_0 \neq P'$, then we can take $Q'' = Q' = Q$. If $\alpha \neq \tau$ or $P_0 = P'$, and the result follows from Lemma 21.

Suppose that $n > 0$. Then $P_n \xrightarrow{\alpha} P_{n-1}$, so by Lemma 21 there exist $Q_n$ and $Q_{n-1}$ such that $Q \xrightarrow{\alpha} Q_n \xrightarrow{\alpha} Q_{n-1}$, $P_n \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q_n$, and $P_{n-1} \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q_{n-1}$. Furthermore, by the induction hypothesis, there exist $Q'$ and $Q''$ such that $Q_{n-1} \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$, $P_{n-1} \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q''$, and $P' \mathcal{R}_{E,F} \vdash \Delta^{\alpha} Q'$. Clearly, we then also have that $Q \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$. □
It remains to establish that \( R_{E,F} \) satisfies the third condition of rooted divergence-preserving branching bisimulations up to \( \Delta_b \).

**Lemma 23.** Let \( G \) and \( H \) be \( X \)-closed process expressions such that \( G \triangleleft_{\Delta_b} H \). If there exists an infinite sequence of closed process expressions \( (P_k)_{k \in \omega} \) such that \( G[\mu X.F/X] = P_0 \) and \( P_k \rightsquigarrow P_{k+1} \) for all \( k \in \omega \), then there also exists an infinite sequence of closed process expressions \( (Q_\ell)_{\ell \in \omega} \) and a mapping \( \sigma : \omega \to \omega \) such that \( H[\mu X.F/X] = Q_0, Q_\ell \rightsquigarrow Q_{\ell+1} \), and \( P_{\sigma(\ell)} R_{E,F} Q_\ell \) for all \( \ell \in \omega \).

**Proof.** Suppose that there exists an infinite sequence of closed process expressions \( (P_k)_{k \in \omega} \) such that \( G[\mu X.E/X] = P_0 \) and \( P_k \rightsquigarrow P_{k+1} \) for all \( k \in \omega \). By Corollary 11, there is an infinite sequence of \( X \)-closed process expressions \( (G_k)_{k \in \omega} \) such that \( P_k = G_k[\mu X.E/X] \) and either \( G_k \rightsquigarrow G_{k+1} \) or \( E \rightsquigarrow_{\Delta} G_{k+1} \) for all \( k \in \omega \). We shall define simultaneously, with induction on \( \ell \), an infinite sequence of \( X \)-closed process expressions \( (H_\ell)_{\ell \in \omega} \) with \( H_0 = H \) and \( H_\ell[\mu X.F/X] \rightsquigarrow H_{\ell+1}[\mu X.F/X] \), and a mapping \( \sigma : \omega \to \omega \) such that \( G_{\sigma(\ell)} \triangleleft_{\Delta_b} H_\ell \). This will suffice, because for all \( \ell \in \omega \), defining \( Q_\ell \) as \( H_\ell[\mu X.F/X] \) we obtain \( Q_\ell \rightsquigarrow Q_{\ell+1} \) and \( P_{\sigma(\ell)} R_{E,F} Q_\ell \). Suppose, by the way of induction hypothesis, that \( H_\ell \) and \( \sigma(\ell) \) have been defined already, such that \( G_{\sigma(\ell)} \triangleleft_{\Delta_b} H_\ell \). By Corollary 11 there are two cases:

1. \( G_{\sigma(\ell)+k} \rightsquigarrow G_{\sigma(\ell)+k+1} \) for all \( k \in \omega \). Then, since \( G_{\sigma(\ell)} \triangleleft_{\Delta_b} H_\ell \), by Corollary 15 there exists an \( X \)-closed process expression \( H' \) such that \( H_\ell \rightsquigarrow H' \) and \( G_{\sigma(\ell)+k} \triangleleft_{\Delta_b} H' \) for some \( k \in \omega \). We define \( H_{\ell+1} = H' \) and \( \sigma(\ell+1) = \sigma(\ell) + k \). Now \( H_{\ell}[\mu X.F/X] \rightsquigarrow_{\Delta_b} H_{\ell+1}[\mu X.F/X] \) by Lemma 11 and \( G_{\sigma(\ell+1)} \triangleleft_{\Delta_b} H_{\ell+1} \).

2. There is a \( k \in \omega \) such that \( G_{\sigma(\ell)+i} \rightsquigarrow G_{\sigma(\ell)+i+1} \) for all \( i < k \), \( X \) is exposed in \( G_{\sigma(\ell)+k} \) and \( E \rightsquigarrow_{\Delta} G_{\sigma(\ell)+k+1} \). Then, since \( G_{\sigma(\ell)} \triangleleft_{\Delta_b} H_\ell \), by Corollary 15 and with induction on \( \ell \) there exists a sequence \( H'_0, \ldots, H'_m \) and a mapping \( \rho : \{0, \ldots, m\} \to \{0, \ldots, k\} \) with \( \rho(m) = k \) such that \( H_\ell = H'_0 \rightsquigarrow \cdots \rightsquigarrow H'_m \) and \( G_{\sigma(\ell)+\rho(i)} \triangleleft_{\Delta_b} H'_i \). Using Corollary 15, we may furthermore assume that \( X \) is exposed in \( H'_m \).

If \( m > 0 \), then we define \( H_{\ell+1} = H'_1 \) and \( \sigma(\ell+1) = \sigma(\ell) + \rho(1) \). Now \( H_{\ell}[\mu X.F/X] \rightsquigarrow_{\Delta_b} H_{\ell+1}[\mu X.F/X] \) by Lemma 11 and \( G_{\sigma(\ell+1)} \triangleleft_{\Delta_b} H_{\ell+1} \).

So it remains to consider the case that \( m = 0 \). Since \( E \triangleleft_{\Delta_b} F \), there exists, by Corollary 16, an \( X \)-closed process expression \( F' \) such that \( F \rightsquigarrow_{\Delta} F' \) and \( G_{\sigma(\ell)+k+1} \triangleleft_{\Delta_b} F' \). We now define \( H_{\ell+1} = F' \) and \( \sigma(\ell+1) = \sigma(\ell) + k + 1 \). We then have that \( G_{\sigma(\ell+1)} = G_{\sigma(\ell)+k+1} \triangleleft_{\Delta_b} H_{\ell+1} \), and \( F'[\mu X.F/X] \rightsquigarrow_{\Delta_b} H_{\ell+1}[\mu X.F/X] \) by Lemma 11. So \( \mu X.F \rightsquigarrow_{\Delta_b} H_{\ell+1}[\mu X.F/X] \) by rule 2 and Lemma 11 yields \( H_{\ell}[\mu X.F/X] \rightsquigarrow_{\Delta_b} H_{\ell+1}[\mu X.F/X] \), using that \( X \) is exposed in \( H_\ell \).

From Lemma 26, with \( G = H \) we immediately get the following corollary.

**Corollary 24.** \( R_{E,F} \) satisfies condition \( (U5) \) of Definition 17.

The relation \( R_{E,F} \) is symmetric by definition and we have now also proved that it satisfies conditions \( (U1), (U2) \) and \( (U3) \), so we have established the following result.

**Corollary 25.** \( R_{E,F} \) is a rooted divergence-preserving branching bisimulation up to \( \Delta_b \).

### 3.3 The main results

We can now establish that \( \triangleleft_b \) is compatible with \( \alpha \), \( \mu X. \), and \( + \).

**Proposition 26.** If \( E \triangleleft_{\Delta_b} F \), then \( \alpha.E \triangleleft_{\Delta_b} \alpha.F \) for all \( \alpha \in A \), \( E + H \triangleleft_{\Delta_b} F + H \) and \( H + E \triangleleft_{\Delta_b} H + F \) for all process expressions \( H \), and \( \mu X.E \triangleleft_{\Delta_b} \mu X.F \).
Proof. To prove that $\equiv^\Delta_{rb}$ is compatible with $\alpha$ and $+$ is straightforward. (First, establish the property for closed terms, and then use that substitution distributes over $\alpha$ and $+$.)

It remains to prove that $\equiv^\Delta_{rb}$ is compatible with $\mu X.\_\_$ i.e., that $E \equiv^\Delta_{rb} F$ implies $\mu X.E \equiv^\Delta_{rb} \mu X.F$. Note that in the special case that $E$ and $F$ are $X$-closed this immediately follows from Corollary 25 and Proposition 18. Now, for the general case, let $E$ and $F$ be process expressions and suppose that $E \equiv^\Delta_{rb} F$. Let $X, \bar{Y}$ be a sequence of variables that at least includes the variables with a free occurrence in $E$ or $F$, and such that $X$ does not occur in $\bar{Y}$. Then, according to the definition of $\equiv^\Delta_{rb}$ on process expressions with free variables (Definition 7), we have that, for every closed process expression $P$ and for every sequence of closed process expressions $\bar{P}$ of the same length as $\bar{Y}$, $E[\bar{P}/X, \bar{Y}] \equiv^\Delta_{rb} F[\bar{P}/X, \bar{Y}]$. So, clearly, also $E[\bar{P}/\bar{Y}] \equiv^\Delta_{rb} F[\bar{P}/\bar{Y}]$, and since $E[\bar{P}/\bar{Y}]$ and $F[\bar{P}/\bar{Y}]$ are $X$-closed, it follows that $\mu X.E[\bar{P}/\bar{Y}] \equiv^\Delta_{rb} \mu X.F[\bar{P}/\bar{Y}]$. Since $X$ is not among the $\bar{Y}$, we may conclude that $(\mu X.E)[\bar{P}/\bar{Y}] \equiv^\Delta_{rb} (\mu X.F)[\bar{P}/\bar{Y}]$ for every sequence of closed process expressions $\bar{P}$ of the same length as $\bar{Y}$, and hence $\mu X.E \equiv^\Delta_{rb} \mu X.F$. \hfill $\square$

We have now obtained our main result that $\equiv^\Delta_{rb}$ is a congruence. In fact, it is the coarsest contained in $\equiv^\Delta_{rb}$.

**Theorem 27.** The relation $\equiv^\Delta_{rb}$ is the coarsest congruence contained in $\equiv^\Delta_{rb}$.

Proof. By Propositions 8 and 20 the relation $\equiv^\Delta_{rb}$ is a congruence. To prove that it is coarsest, it suffices to prove that for every relation $R \subseteq \equiv^\Delta_{rb}$ that is compatible with $+$ we have that $R \subseteq \equiv^\Delta_{rb}$. Let $P$ and $Q$ be closed process expressions, and suppose that $P R Q$.

Since by Proposition 12 the set of closed process expressions reachable from $P$ and $Q$ is finite and $\Delta$ is non-empty, there exists a natural number $n \in \omega$ such that for all $R$ reachable from $P$ or $Q$ it holds that $R \equiv^\Delta_{rb} a^n$. This implies that for all $R$ reachable from $P$ or $Q$ it holds that $R \equiv^\Delta_{rb} P + a^{n+1}$ and $R \equiv^\Delta_{rb} Q + a^{n+1}$.

Since $R$ is compatible with $+$, we have that $P + a^{n+1} R Q + a^{n+1}$, and hence $P + a^{n+1} \equiv^\Delta_{rb} Q + a^{n+1}$. To prove (R1), suppose that $P \xrightarrow{\alpha} P'$. Then $P + a^{n+1} \xrightarrow{\alpha} P'$, so by Lemma 2 there exist closed process expressions $Q'$ and $Q''$ such that $Q + a^{n+1} \xrightarrow{\alpha} Q''$, $P + a^{n+1} \xrightarrow{\alpha} Q''$, and $P' \equiv^\Delta_{rb} Q'$. Since $a \neq \tau$, we have that $Q'' = Q + a^{n+1}$, for otherwise $Q''$ is reachable from $Q$ and $Q'' \equiv^\Delta_{rb} P + a^{n+1}$. Moreover, $Q' \xrightarrow{\alpha} Q'$, for otherwise $P' \xrightarrow{\alpha} Q' = Q'' = Q + a^{n+1}$. Condition (R2) follows by symmetry. \hfill $\square$

**References**


