Rooted Divergence-Preserving Branching Bisimilarity is a Congruence

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Abstract

We prove that rooted divergence-preserving branching bisimilarity is a congruence for the process specification language consisting of 0, action prefix, choice, and the recursion construct μX.

Keywords: Process algebra; Recursion; Branching bisimulation; Divergence; Congruence.

1 Introduction

Branching bisimilarity [13] is a behavioural equivalence on processes that is compatible with abstraction from internal activity, while at the same time preserving the branching structure of processes in a strong sense [9]. Branching bisimilarity abstracts to a large extent from divergence (i.e., infinite internal activity). For instance, it identifies a process, say $P$, that may perform some internal activity after which it returns to its initial state (i.e., $P$ has a $\tau$-loop) with a process, say $P'$, that admits the same behaviour as $P$ except that it cannot perform the internal activity leading to the initial state (i.e., $P'$ is $P$ without the $\tau$-loop).

In situations where fairness principles apply, abstraction from divergence is often desirable. But there are circumstances in which abstraction from divergence is undesirable: A behavioural equivalence that abstracts from divergence is not compatible with any temporal logic featuring an eventually modality: for any desired state that $P'$ will eventually reach, the mentioned internal activity of $P$ may be performed forever, and thus prevent $P$ from reaching this desired state. It is also generally not compatible with a process-algebraic priority operator (cf. [23, pp. 130–132]) or sequencing operator [3]. Since a divergence may be exploited to simulate recursively enumerable branching in a computable transition system [21], a divergence-insensitive behavioural equivalence may be considered too coarse for a theory that integrates computability and concurrency [2]. Preservation of divergence is widely considered an important correctness criterion when studying the relative expressiveness of process calculi [14, 25, 6].

The notion of branching bisimilarity with explicit divergence, also stemming from [13], is a suitable refinement of branching bisimilarity that is compatible with the well-known branching-time temporal logic CTL* without the nexttime operator $X$ (which is known to be incompatible with abstraction from internal activity). In fact, in [14] we have proved that it is the coarsest semantic equivalence on labelled transition systems with silent moves that is a congruence for parallel composition (as found in process algebras like CCS, CSP or ACP) and only equates processes satisfying the same $\text{CTL}_X$ formulas. In [2], for stylistic reasons, branching bisimilarity with explicit divergence was named divergence-preserving branching bisimilarity; we shall henceforth use this term.

Divergence-preserving branching bisimilarity is the finest behavioural equivalence in the linear time — branching time spectrum [8]. It is the principal behavioural equivalence underlying the theory of executability [1, 2, 16, 17]. Reduction modulo divergence-preserving branching bisimilarity
is a part of methods for formal verification and analysis of the behaviour of systems \cite{13 24 22 26}. In \cite{5} a game-based characterisation of divergence-preserving branching bisimilarity is presented.

Processes are usually specified in some process specification language. For compositional reasoning it is then important that the behavioural equivalence used is a congruence with respect to the constructs of that language. Following Milner \cite{19}, we consider the language basic CCS with recursion, i.e., the language consisting of $0$, action prefix, and choice, extended with the recursion construct $\mu X$.\textendash $\tau$. This language precisely allows the specification of finite-state behaviours. As for other weak behavioural equivalences, divergence-preserving branching bisimilarity is not a congruence for that language; in fact, it is not a congruence for any language that includes choice. The goal of this paper is to prove that adding the usual root condition suffices to obtain a congruence for those languages; in fact, it is not a congruence for any language that includes choice.

Recently, a congruence format was proposed for (rooted) divergence-preserving branching bisimilarity \cite{4}. The operational rules for action prefix and choice are in this format. Unfortunately, however, this format does not support the recursion construct $\mu X$. Interestingly, as far as we know, the recursion construct has not been covered at all in the rich literature on congruence formats, with the recent exception of \cite{10}. (The article \cite{10} differentiates between lean and full congruences for recursion; in this article we consider the full congruence.)

The congruence result obtained in this paper should serve as a stepping stone towards a complete axiomatisation of divergence-preserving branching bisimilarity for basic CCS with recursion. Such work, inspired by Milner’s complete axiomatisation of weak bisimilarity \cite{19}, would combine the adaptations of \cite{7} to branching bisimilarity, and of \cite{15} to several divergence-sensitive variants of weak bisimilarity.

We originally thought that congruence for recursion could be obtained in the same spirit as Milner’s ingenious proof in \cite{20} for strong bisimilarity, which cleverly makes use of an up-to technique. The proofs for weak and branching bisimilarity essentially reuse this idea \cite{20 7}, but requiring the use of a weak step in the antecedent of the transfer condition. We were not able to generalise the idea to divergence-preserving branching bisimilarity until we included the root condition in up-to technique.

We believe that the proofs of Corollaries \ref{cor:10} and \ref{cor:11}, Propositions \ref{prop:14} and \ref{prop:18}, and Lemma \ref{lem:23} contain novel twists. Although the other proofs are either routine or adaptations of the ones in \cite{20}, we have included them for the convenience of the reader.

### 2 Rooted divergence-preserving branching bisimilarity

Let $\mathcal{A}$ be a non-empty set of actions, and let $\tau$ be a special action not in $\mathcal{A}$. Let $\mathcal{A}_r = \mathcal{A} \cup \{\tau\}$. Furthermore, let $\mathcal{V}$ be a set of recursion variables. The set of process expressions $\mathcal{E}$ is generated by the following grammar:

\[
E ::= 0 \mid X \mid \alpha.E \mid \mu X.E \mid E + E \quad (\alpha \in \mathcal{A}_r, \ X \in \mathcal{V}).
\]

An occurrence of a recursion variable $X$ in a process expression $E$ is \textit{bound} if it is in the scope of a $\mu X$-construction and otherwise it is \textit{free}. We denote by $FV(E)$ the set of variables with a free occurrence in $E$. If $\vec{X} = X_0, \ldots, X_n$ is a sequence of variables, and $\vec{F} = F_0, \ldots, F_n$ is a sequence of process expressions of the same length, then we write $E[\vec{F/\vec{X}}]$ for the process expression obtained from $E$ by replacing all free occurrences of $X_i$ in $E$ by $F_i$ ($i = 0, \ldots, n$), applying $\alpha$-conversion to $E$ if necessary to avoid capture.

On $\mathcal{E}$ we define an $\mathcal{A}_r$-labelled transition relation $\xrightarrow{} \subseteq \mathcal{E} \times \mathcal{A}_r \times \mathcal{E}$ as the least ternary relation satisfying the following rules for all $\alpha \in \mathcal{A}_r$, $X \in \mathcal{V}$, and process expressions $E$, $E'$, $F$, and $F'$:

\[
\begin{align*}
1 & \quad \alpha.E \xrightarrow{\alpha} E \\
2 & \quad E[\mu X.E/X] \xrightarrow{\mu X.E} E' \\
3 & \quad E \xrightarrow{\alpha} E' \\
4 & \quad E + F \xrightarrow{\alpha} E' \\
5 & \quad E + F \xrightarrow{\alpha} F'
\end{align*}
\]
We write $E \xrightarrow{\alpha} E'$ for $(E, \alpha, E') \in \rightarrow$ (as we already did in the rules above) and we abbreviate the statement `$E \xrightarrow{\alpha} E'$ or $(\alpha = \tau$ and $E = E')$ by $E \xrightarrow{\alpha}{\tau} E'$. Furthermore, we write $\rightarrow$ for the reflexive-transitive closure of $\rightarrow$, i.e., $E \rightarrow E'$ if there exist $E_0, E_1, \ldots, E_n \in \mathcal{E}$ such that $E = E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n = E'$.

A process expression is closed if it contains no free occurrences of recursion variables; we denote by $\mathcal{P}$ the subset of $\mathcal{E}$ consisting of all closed process expressions. It is easy to check that if $P$ is a closed process expression and $P \xrightarrow{\alpha} E$, then $E$ is a closed process expression too. Hence, the transition relation restricts in a natural way to closed process expressions, and thus associates with every closed process expression a behaviour. We proceed to define when two process expressions may be considered to represent the same behaviour.

**Definition 1.** A symmetric binary relation $\mathcal{R}$ on $\mathcal{P}$ is a branching bisimulation if it satisfies the following condition for all $P, Q \in \mathcal{P}$ and $\alpha \in A_r$:

(T) if $P \mathcal{R} Q$ and $P \xrightarrow{\alpha} P'$ for some closed process expression $P'$, then there exist closed process expressions $Q'$ and $Q''$ such that $Q \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$, $P \mathcal{R} Q''$ and $P' \mathcal{R} Q'$.

We say that a branching bisimulation $\mathcal{R}$ preserves (internal) divergence if

(D) if $P \mathcal{R} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $P = P_0, P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \mathcal{R} Q$ for all $k \in \omega$, then there is an infinite sequence of closed process expressions $(Q_\ell)_{\ell \in \omega}$ such that $Q = Q_0, Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_k \mathcal{R} Q_\ell$ for all $k, \ell \in \omega$.

We write $P \equiv_{\Delta}^\mathcal{R} Q$ if there exists a divergence-preserving branching bisimulation $\mathcal{R}$ such that $P \mathcal{R} Q$. The relation $\equiv_{\Delta}^\mathcal{R}$ was introduced in [13] under the name branching bisimilarity with explicit divergence and is here referred to as divergence-preserving branching bisimilarity.

The relation $\equiv_{\Delta}^\mathcal{R}$ was studied in detail in [11]; we recap some of the facts established *ibidem*.

First, the relation $\equiv_{\Delta}^\mathcal{R}$ is an equivalence relation. Second, the relation $\equiv_{\Delta}^\mathcal{R}$ satisfies the condition (1), with the following generalisation as a straightforward consequence.

**Lemma 2.** Let $P$ and $Q$ be closed process expressions. If $P \equiv_{\Delta}^\mathcal{R} Q$ and $P \xrightarrow{\alpha} P'$ for some closed process expressions $P'$ and $P''$, then there exist closed process expressions $Q'$ and $Q''$ such that $Q \xrightarrow{\alpha} Q'' \xrightarrow{\alpha} Q'$, $P' \equiv_{\Delta}^\mathcal{R} Q''$ and $P' \equiv_{\Delta}^\mathcal{R} Q'$.

Third, $\equiv_{\Delta}^\mathcal{R}$ also satisfies (1). In [11] several alternative definitions of divergence preservation are studied, which, in the end, all give rise to the same notion of divergence-preserving branching bisimilarity. In particular, the following alternative relational characterisations will be useful tools in the remainder.

**Proposition 3.** Let $P$ and $Q$ be closed process expressions. Then

- $P \equiv_{\Delta}^\mathcal{R} Q$ if, and only if, $P$ and $Q$ are related by some branching bisimulation $\mathcal{R}$ satisfying
  
  $(D')$ if $P \mathcal{R} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$, then there is an infinite sequence of closed process expressions $(Q_\ell)_{\ell \in \omega}$ and a mapping $\sigma : \omega \rightarrow \omega$ such that $Q = Q_0, Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_\sigma(\ell) \mathcal{R} Q_\ell$ for all $\ell \in \omega$; and

- $P \equiv_{\Delta}^\mathcal{R} Q$ if, and only if, $P$ and $Q$ are related by some branching bisimulation $\mathcal{R}$ satisfying
  
  $(D'')$ if $P \mathcal{R} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$, then there exists a closed process expression $Q'$ such that $Q \xrightarrow{\tau} Q'$ and $P_k \mathcal{R} Q'$ for some $k \in \omega$.

Moreover, $\equiv_{\Delta}^\mathcal{R}$ itself satisfies (1) and (1').

**Proof.** See [11]; condition $(D')$ is (D3) and condition $(D'')$ is (D2).
And finally, it was proved in [11] that $\equiv^\Delta$ satisfies the following so-called stuttering property.

**Proposition 4.** Let $P$ be a closed process expression and let $Q_0, \ldots, Q_k$ be closed process expressions such that $Q_0 \tau \rightarrow \cdots \tau \rightarrow Q_k$. If $P \equiv^\Delta Q_0$ and $P \equiv^\Delta Q_k$, then $P \equiv^\Delta Q_i$ for all $0 \leq i \leq k$.

As for all variants of bisimilarity that take some form of abstraction from internal activity into account, the relation $\equiv^\Delta$ is not compatible with $+ (0 \equiv^\Delta \tau.0$ but $0 + a.0 \not\equiv^\Delta \tau.0 + a.0)$, and hence not a congruence for the language we are considering. In contrast to its divergence-insensitive variant, divergence-preserving branching bisimilarity is not compatible with the recursion construct either, as we will argue below. Similarly as for the divergence-insensitive variant of branching bisimilarity, it suffices to add a root condition to obtain the coarsest congruence for our language that is contained in $\equiv^\Delta$, as we shall prove in the remainder of this paper.

**Definition 5.** Let $P$ and $Q$ be closed process expressions. We say that $P$ and $Q$ are rooted divergence-preserving branching bisimilar (notation: $P \equiv^\Delta_\triangleright Q$) if for all $\alpha \in A_\triangleright$ the following holds:

(R1) if $P \triangleright\rightarrow P'$, then there exists a $Q'$ such that $Q \triangleright\rightarrow Q'$ and $P' \equiv^\Delta Q'$; and

(R2) if $Q \triangleright\rightarrow Q'$, then there exists a $P'$ such that $P \triangleright\rightarrow P'$ and $P' \equiv^\Delta Q'$.

The following proposition is a straightforward consequence of the fact that $\equiv^\Delta$ is an equivalence.

**Proposition 6.** The relation $\equiv^\Delta_\triangleright$ is an equivalence relation on $P$.

Moreover, it is immediate that $\equiv^\Delta_\triangleright \subseteq \equiv^\Delta$.

We have defined the notions of $\equiv^\Delta$ and $\equiv^\Delta_\triangleright$ on closed process expressions because those are thought of as directly representing behaviour. Due to the presence of the binding construct $\mu X.$ it is, however, necessary to lift these notions to expressions with free variables even if the goal is simply to establish behavioural equivalence of closed process expressions.

**Definition 7.** Let $E$ and $F$ be process expressions, and let the sequence $\vec{X}$ of variables at least include all the variables with a free occurrence in $E$ or $F$. We write $E \equiv^\Delta \triangleright F$ ($E \equiv^\Delta_\triangleright F$) if $E[\vec{P}/\vec{X}] \equiv^\Delta \triangleright F[\vec{P}/\vec{X}]$ ($E[\vec{P}/\vec{X}] \equiv^\Delta_\triangleright F[\vec{P}/\vec{X}]$) for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{X}$.

It is clear from the definition above that, since $\equiv^\Delta$ and $\equiv^\Delta_\triangleright$ are equivalence relations on $P$, their lifted versions are equivalence relations on $E$. Note that $\equiv^\Delta$ is not compatible with the recursion construct: we have that $X \equiv^\Delta \triangleright \tau.X$, whereas $\mu X.X \not\equiv^\Delta \triangleright \mu X.\tau.X$. We shall prove that its rooted variant $\equiv^\Delta_\triangleright$ is, however, compatible with all the constructs of the syntax, i.e., if $E \equiv^\Delta_\triangleright F$, then $\alpha.E \equiv^\Delta_\triangleright \alpha.F$ for all $\alpha \in A_\triangleright$, $\mu X.E \equiv^\Delta_\triangleright \mu X.F$ for all $X \in \mathcal{V}$, $E + H \equiv^\Delta_\triangleright F + H$ and $H + E \equiv^\Delta_\triangleright H + F$ for all process expressions $H$. To prove that $\equiv^\Delta_\triangleright$ is compatible with $\alpha$ and $+$ is straightforward, but for $\mu X.$ this is considerably more work.

### 3 The congruence proof

Our proof that $\equiv^\Delta_\triangleright$ is compatible with $\mu X.$ relies on the following observation: If $\vec{Y}$ is some sequence of variables and $\vec{P}$ is a sequence of closed terms of the same length, then, on the one hand, $E \equiv^\Delta_\triangleright F$ implies $E[\vec{P}/\vec{Y}] \equiv^\Delta_\triangleright F[\vec{P}/\vec{Y}]$ by the definition of $\equiv^\Delta_\triangleright$ on $E$, and, on the other hand, if $X$ does not occur in $\vec{Y}$, then from $\mu X.E[\vec{P}/\vec{Y}] \equiv^\Delta_\triangleright \mu X.F[\vec{P}/\vec{Y}]$ it follows that $(\mu X.E)[\vec{P}/\vec{Y}] \equiv^\Delta_\triangleright (\mu X.F)[\vec{P}/\vec{Y}]$ by the definition of substitution. Therefore, as formalised in the proof of Proposition 26, it is enough to establish that $E \equiv^\Delta_\triangleright F$ implies $\mu X.E \equiv^\Delta_\triangleright \mu X.F$ in the special case that $E$ and $F$ are process expressions with no other free variables than $\vec{X}$; such process expressions will be called $X$-closed.

The rest of this section is organised as follows.
We shall first characterise, in Section 3.1, the relation $\equiv_{\Delta}^\Delta$ on $X$-closed process expressions in terms of the transition relation on $X$-closed process expressions.

Then, in Section 3.2, we shall present a suitable notion of rooted divergence-preserving branching bisimulation up to $\equiv_{\Delta}^\Delta$, and we shall prove that every pair of rooted divergence-preserving branching bisimilar $X$-closed process expressions $(E, F)$ gives rise to a relation $R^*$ of which we can show that it is a rooted divergence-preserving branching bisimulation up to $\equiv_{\Delta}^\Delta$. The relation $R^*$ will be defined in such a way that it relates $\mu X.E$ and $\mu X.F$ and thus allows us to conclude that these process expressions are rooted divergence-preserving bisimilar.

In Section 3.3, we shall then put the pieces together and prove $\equiv_{\Delta}^\Delta$ is the coarsest congruence contained in $\equiv_{\Delta}^\Delta$ for basic CCS with recursion.

### 3.1 $\equiv_{\Delta}^\Delta$ on $X$-closed process expressions

We say that a process expression $E$ is $X$-closed if $\text{FV}(E) \subseteq \{X\}$; the set of all $X$-closed process expressions is denoted by $\mathcal{P}_X$. Note that if $E$ is $X$-closed and $E \xrightarrow{\alpha} E'$, then $E'$ is $X$-closed too, and so the $\mathcal{A}_r$-labelled transition relation restricts in a natural way to $X$-closed process expressions.

**Definition 8.** We define when $X$ is exposed in a (not necessarily $X$-closed) process expression $E$ with induction on the structure of $E$:

i. if $E = X$, then $X$ is exposed in $E$;

ii. if $E = \mu Y.E'$, $Y$ is a recursion variable distinct from $X$ and $X$ is exposed in $E'$, then $X$ is exposed in $E$;

iii. if $E = E_1 + E_2$ and $X$ is exposed in $E_1$ or $E_2$, then $X$ is exposed in $E$.

Note that the variable $X$ is exposed in $E$ if, and only if, $E$ has an unguarded occurrence of $X$ in the sense of [19].

We establish a relationship between the transitions of a closed process expression $E[P/X]$ that is obtained by substituting a closed process expression $P$ for the variable $X$ in an $X$-closed process expression $E$, and the transitions of $E$ and $P$.

**Lemma 9.** Let $E$ be an $X$-closed process expression, and let $P$ be a closed process expression.

1. If $E \xrightarrow{\alpha} E'$, then $E[P/X] \xrightarrow{\alpha} E'[P/X]$, and if $X$ is exposed in $E$ and $P \xrightarrow{\alpha} P'$, then $E[P/X] \xrightarrow{\alpha} P'$.

2. If $E'[P/X] \xrightarrow{\alpha} P'$ for some (closed) process expression $P'$, then either there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or $X$ is exposed in $E$, $P \xrightarrow{\alpha} P'$ and every derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.

**Proof.** Statement 1 of the lemma is established with straightforward inductions on a derivation of $E \xrightarrow{\alpha} E'$ and on the structure of $E$.

We proceed to establish with induction on a derivation of $E[P/X] \xrightarrow{\alpha} P'$ that there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or $X$ is exposed in $E$, $P \xrightarrow{\alpha} P'$ and a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$. This implies statement 2.

We distinguish cases according to the structure of $E$:

- Clearly, $E$ cannot be $0$, for if $E = 0$, then $E[P/X] = 0$, and $0$ does not admit any transitions.

- If $E = X$, then $X$ is exposed in $E$ and $P = E[P/X] \xrightarrow{\alpha} P'$. It is then also immediate that the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.
• If \( E = \beta E' \) for some \( \beta \in A_P \) and some \( X \)-closed process expression \( E' \), then \( \beta = \alpha \) and \( E \xrightarrow{\alpha} E' \). Since \( E[P/X] = \beta.(E'[P/X]) \), rule 1 is the last rule applied in the derivation of the transition \( E[P/X] \xrightarrow{\alpha} P' \), so \( P' = E'[P/X] \).

• If \( E = \mu Y.F \) for some process expression \( F \) with \( \text{FV}(F) \subseteq \{X,Y\} \), then there are two subcases:

  On the one hand, if \( Y = X \), then \( E[P/X] = \mu Y.(F[P/X]) \), and therefore the last rule applied in the considered derivation of the transition \( E[P/X] \xrightarrow{\alpha} P' \) is rule 2. Consequently, the considered derivation has a proper subderivation of the transition \( F[P/X][\mu Y.(F[P/X])]/Y \xrightarrow{\alpha} P' \). Note that \( F[P/X][\mu Y.(F[P/X])]/Y = (F[\mu Y.F/Y])[P/X] \). Hence, by the induction hypothesis, either there exists an \( E' \) such that \( F[\mu Y.F/Y] \xrightarrow{\alpha} E' \) and \( P' = E'[P/X] \), or \( X \) is exposed in \( F[\mu Y.F/Y] \), \( P \xrightarrow{\alpha} P' \), and the derivation of \( F[\mu Y.F/Y][P/X] \xrightarrow{\alpha} P' \) has a derivation of \( P \xrightarrow{\alpha} P' \) as a subderivation. In the first case, it follows from \( F[\mu Y.F/Y] \xrightarrow{\alpha} E' \), by rule 2, that \( E = \mu Y.F \xrightarrow{\alpha} E' \) and \( P' = E'[P/X] \). In the second case, it suffices to note that \( X \) is exposed in \( F \), hence also in \( E \), and that a derivation of \( P \xrightarrow{\alpha} P' \) appears as a subderivation of the considered derivation of \( E[P/X] \xrightarrow{\alpha} P' \).

  On the other hand, if \( Y \neq X \), then \( E[P/X] = \mu Y.(F[Y/X]) \), and therefore the last rule applied in the considered derivation of the transition \( E[P/X] \xrightarrow{\alpha} P' \) is either rule 2 or rule 3.

  If it is rule 2, then \( E_1[P/X] \xrightarrow{\alpha} P' \), and since this transition has a derivation that is a proper subderivation of the considered derivation of \( E[P/X] \xrightarrow{\alpha} P' \), by the induction hypothesis it follows that either \( E_1 \xrightarrow{\alpha} E' \) and \( P' = E'[P/X] \), or \( X \) is exposed in \( E_1 \), \( P \xrightarrow{\alpha} P' \), and a derivation of \( P \xrightarrow{\alpha} P' \) appears as a subderivation of the derivation \( E_1[P/X] \xrightarrow{\alpha} P' \).

  In the first case, it remains to note that then also \( E \xrightarrow{\alpha} E' \), and in the second case, it remains to note that \( X \) is also exposed in \( E \).

If the last rule applied in the considered derivation is rule 3, then the proof is analogous. □

**Corollary 10.** Let \( E \) be an \( X \)-closed process expression. If \( E[\mu X.E/X] \xrightarrow{\alpha} P' \) for some (closed) process expression \( P' \), then there exists an \( X \)-closed process expression \( E' \) such that \( E \xrightarrow{\alpha} E' \) and \( P' = E'[\mu X.E/X] \).

**Proof.** Consider a derivation of \( E[\mu X.E/X] \xrightarrow{\alpha} P' \) that is minimal in the sense that it does not have a derivation of \( E[\mu X.E/X] \xrightarrow{\alpha} P' \) as proper subderivation. Let \( P = \mu X.E \). Since every derivation of \( P \xrightarrow{\alpha} P' \) has a derivation of \( E[P/X] \xrightarrow{\alpha} P' \) as a proper subderivation (see the operational rules, and rule 2 in particular), it follows that the considered derivation of \( E[\mu X.E/X] \xrightarrow{\alpha} P' \) does not have a subderivation of \( P \xrightarrow{\alpha} P' \). Hence, by Lemma 9.2 there exists an \( X \)-closed process expression \( E' \) such that \( E \xrightarrow{\alpha} E' \) and \( P' = E'[\mu X.E/X] \).

□

**Corollary 11.** Let \( G_0 \) and \( E \) be \( X \)-closed process expressions. If there is an infinite sequence of closed process expressions \( \{P_k \}_{k \in \omega} \) such that \( G_0[\mu X.E/X] = P_0 \) and \( P_k \xrightarrow{\tau} P_{k+1} \) for all \( k \in \omega \), then there is an infinite sequence of \( X \)-closed process expressions \( \{G_k \}_{k \in \omega} \) such that \( P_k = G_k[\mu X.E/X] \) and, for all \( k \in \omega \), either \( G_k \xrightarrow{\tau} G_{k+1} \) or \( X \) is exposed in \( G_k \) and \( E \xrightarrow{\tau} G_{k+1} \).

**Proof.** We construct \( \{G_k \}_{k \in \omega} \) with induction on \( k \). Suppose that \( G_k \) with \( G_k[\mu X.E/X] = P_k \) has already been constructed. Since \( P_k \xrightarrow{\tau} P_{k+1} \), by Lemma 9.2 there are two cases: either there is a \( G_{k+1} \) with \( G_k \xrightarrow{\tau} G_{k+1} \) and \( P_{k+1} = G_{k+1}[\mu X.E/X] \), in which case we are done, or \( X \) is exposed in \( G_k \) and \( \mu X.E \xrightarrow{\tau} P_{k+1} \). In the latter case \( E[\mu X.E/X] \xrightarrow{\alpha} P_{k+1} \) (see the operational rules, and rule 2 in particular). By Corollary 10 there exists an \( X \)-closed process expression \( G_{k+1} \) such that \( E \xrightarrow{\tau} G_{k+1} \) and \( P_{k+1} = G_{k+1}[\mu X.E/X] \). □
Let $E$ and $E'$ be process expressions. We write $E \rightarrow E'$ if there exists an $\alpha \in \mathbb{A}_\tau$ such that $E \xrightarrow{\alpha} E'$, and denote by $\rightarrow^*$ the reflexive-transitive closure of $\rightarrow$. If $E \rightarrow^* E'$, then we say that $E'$ is reachable from $E$.

**Proposition 12 ([2] Proposition 1).** If $E$ is a process expression, then the set of all expressions reachable from $E$ is finite.

We now characterise the relation $\xrightarrow{\alpha} \mathcal{E}$ on $\mathcal{E}$ from Definition [4] in the same style as Definition [1] but on an enriched transition system. To this end, we first define on $\mathcal{E}$ a $\mathbb{V} \sqcup \mathbb{A}_\tau$-labelled transition relation $\rightarrow \subseteq \mathcal{E} \times (\mathbb{V} \sqcup \mathbb{A}_\tau) \times \mathcal{E}$ as the least ternary relation satisfying, besides the four rules of Section 2, also the rule

$$X \xrightarrow{\alpha} 0$$

for each $X \in \mathbb{V}$. Intuitively, the $\mathbb{V} \sqcup \mathbb{A}_\tau$-labelled transition relation treats a process expression $E$ as the closed term obtained from $E$ by replacing all free occurrences of the variable $X$ by the closed process expression $X.0$ in which $X$ is interpreted as an action instead of as a recursion variable. Note that a variable $X$ is exposed in an expression according to Definition 6 if $\exists F. E \xrightarrow{\alpha} F$, which is the case iff $E \xrightarrow{X} 0$. Now let $\xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau}$ and $\xrightarrow{\mathbb{A}_\tau}$ be defined exactly like $\xrightarrow{\mathbb{V} \sqcup A}$ and $\xrightarrow{\mathbb{A}_\tau}$ but using the $\mathbb{V} \sqcup \mathbb{A}_\tau$-labelled transition relation instead of the $\mathbb{A}_\tau$-labelled one, and applying all definitions directly to expressions with free variables, instead of applying the lifting of Definition [4]. We proceed to show that on $\mathcal{E}$-closed process expressions $\xrightarrow{\mathbb{A}_\tau}$ coincides with $\xrightarrow{\mathbb{A}_\tau}$, and $\xrightarrow{\mathbb{A}_\tau}$ with $\xrightarrow{\mathbb{A}_\tau}$. This characterisation, for weak and branching bisimilarity without preservation of divergence, stems from [10] and [11]. Here we use it solely to obtain Corollaries 15 and 16.

**Lemma 13.** The relation

$$\mathcal{B} = \{(E[P/X], F[P/X]) \mid E, F \text{ are \textit{X}-closed, } E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F, \text{ P is closed}\}$$

is a branching bisimulation satisfying (D') of Proposition 3.

**Proof.** It is immediate from its definition that $\mathcal{B}$ is symmetric.

We show it satisfies (T). Suppose $E, F$ are $\mathcal{E}$-closed, $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F$ and $P$ closed. Let $E[P/X] \xrightarrow{\alpha} P'$ for some $\alpha \in \mathbb{A}_\tau$. By Lemma 12 either there exists an $X$-closed process expression $E'$ such that $E \xrightarrow{\alpha} E'$ and $P = E'[P/X]$, or $X$ is exposed in $E$ and $P \xrightarrow{\alpha} P'$. In the first case, since $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F$, there exist process expressions $F'$ and $F''$ such that $F \xrightarrow{} F'' \xrightarrow{\alpha} F'$, $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F''$ and $E' \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F'$. By Lemma 14, $F[P/X] \xrightarrow{} F'[P/X] \xrightarrow{\alpha} F''[P/X]$. Furthermore, $E[P/X] \mathcal{B} F''[P/X]$ and $P'' = E''[P/X] \mathcal{B} F''[P/X]$. In the second case, since $X$ is exposed in $E$, we have that $E \xrightarrow{X} 0$ and hence, since $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F$, there exist process expressions $F'$ and $F''$ such that $F \xrightarrow{} F'' \xrightarrow{X} F'$, $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F'$ and $0 \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F''$. Moreover, since $F'' \xrightarrow{X} F'$, $X$ is exposed in $F''$, so by Lemma 11, $F[P/X] \xrightarrow{} F''[P/X] \xrightarrow{X} P'$. Furthermore, $E[P/X] \mathcal{B} F''[P/X]$ and $P' \mathcal{B} P''$.

It remains to show that $\mathcal{B}$ satisfies (D'). Suppose $E$ are $\mathcal{E}$-closed, $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F$ and $P$ closed, and there is an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $E[P/X] = P_0$ and $P_k \xrightarrow{X} P_{k+1}$. By Lemma 12, either there exists an infinite sequence of $\mathcal{E}$-closed process expressions $(E_k)_{k \in \omega}$ such that $E_0 = E$, $E_k \xrightarrow{X} E_{k+1}$ and $P_{k+1} = E_{k+1}[P/X]$ for all $k \in \omega$, or there exists a finite sequence of $\mathcal{E}$-closed process expressions $(E_i)_{i \leq k}$ for some $k \in \omega$ such that $E_0 = E$, $E_i \xrightarrow{X} E_{i+1}$ and $P_{i+1} = E_{i+1}[P/X]$ for all $i < k$, $E_k \xrightarrow{X} 0$ and $P \xrightarrow{X} P_{k+1}$. In the first case, since $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F$, using (D'), there exist a process expression $F'$ such that $F \xrightarrow{} F'$ and $E_k \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F'$ for some $k \in N$. By Lemma 11, $F[P/X] \xrightarrow{} F'[P/X]$. Furthermore, $E_k[P/X] \mathcal{B} F'[P/X]$. In the second case, since $E \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F$, with induction on $i$ there exists a sequence $F_0, \ldots, F_m, F_{m+1}$ and a mapping $\rho : \{0, \ldots, m\} \rightarrow \{0, \ldots, k\}$ with $\rho(m) = k$ such that $F = F_0 \xrightarrow{X} \cdots \xrightarrow{X} F_m \xrightarrow{X} F_{m+1}$ and $E_{\rho(j)} \xrightarrow{\mathbb{V} \sqcup A \mathbb{A}_\tau} F_j$ for all $j = 0, \ldots, m$. If $m = 0$, then $X$ is exposed in $F$, so by Lemma 11, $F[P/X] \xrightarrow{} P_{k+1}$. Furthermore, $P_{k+1} \mathcal{B} P_{k+1}$. If $m > 0$, then let $F' = F_1$. By Lemma 11, $F[P/X] \xrightarrow{} F'[P/X]$. Furthermore, $E_{\rho(1)}[P/X] \mathcal{B} F'[P/X]$. $\blacksquare$
For every $\alpha \in \mathcal{A}_r$ and $n \in \omega$, we define the closed process expression $\alpha^n$ inductively by $\alpha^0 = 0$ and $\alpha^{n+1} = \alpha \cdot \alpha^n$. Note that, if $\alpha \neq \tau$, then $\alpha^n = \alpha^0$ implies $i = j$. Recall that we have assumed that $\mathcal{A}$ is non-empty; we now fix, for the remainder of this section, a particular action $\alpha \in \mathcal{A}$.

**Proposition 14.** Let $E$ and $F$ be $X$-closed process expressions. Then $E \triangleleft_X F$ iff $E \triangleleft_{\alpha^n} F$, and $E \triangleleft_{\alpha^n} F$ iff $E \triangleleft_{\alpha^n} F$.

Proof. We need to show that $E \triangleleft_{\alpha^n} F$ iff $E[P/X]\triangleleft_{\alpha^n} F[P/X]$ for each closed process expression $P$, and likewise $E \triangleleft_{\alpha^n} F$ iff $E[P/X]\triangleleft_{\alpha^n} F[P/X]$ for each closed process expression $P$.

"Only if": Lemma 12 immediately yields that $E \triangleleft_{\alpha^n} F$ implies $E[P/X]\triangleleft_{\alpha^n} F[P/X]$ for each closed process expression $P$. Now let $E \triangleleft_{\alpha^n} F$ and let $P \triangleleft_{\alpha^n} F$. By Lemma 10 either there exists an $X$-closed process expression $E'$ such that $E \longrightarrow E'$ and $E' = E'[P/X]$, or $X$ is exposed in $E$ and $P \longrightarrow P'$. In the first case, since $E \triangleleft_{\alpha^n} F$, there exists a process expression $E''$ such that $E \longrightarrow E''$ and $E'' \triangleleft_{\alpha^n} F$. By Lemma 12 $F[P/X] \longrightarrow F'[P/X]$. Furthermore, by Lemma 13 $E'' \triangleleft_{\alpha^n} F''$. The second case, since $X$ is exposed in $E$ we have $E \longrightarrow_{\alpha^n} 0$, and hence, since $E \triangleleft_{\alpha^n} F$, there exists a process expression $F'$ such that $F \longrightarrow F'$. By Lemma 12 $F[P/X] \longrightarrow F'$. Furthermore, $F' \triangleleft_{\alpha^n} P'$. The other clause follows by symmetry, thus yielding $E[P/X]\triangleleft_{\alpha^n} F[P/X]$.

"If": Let $E$ and $F$ be $X$-closed process expressions. Since by Proposition 12 the set of all process expressions reachable from $E$ and $F$ is finite, there exists a natural number $n \in \omega$ such that for all $G$ reachable from $E$ or $F$ it holds that $G \triangleleft_{\alpha^n} \alpha^n$, and thus $G[a^{n+1}/X] \triangleleft_{\alpha^n} \alpha^n$. Let

$$R = \{(E', F') \mid E' \longrightarrow^n E', F' \longrightarrow^n F', E'[a^{n+1}/X] \triangleleft_{\alpha^n} F'[a^{n+1}/X]\}.$$

Claim: The symmetric closure of $R$ is a branching bisimulation satisfying $\Box \mathcal{M}$ w.r.t. the $\forall \mathcal{A}_r$-labelled transition relation.

Proof of the claim: To prove that $R$ satisfies condition (4) of Definition 4, let $E'$ and $F'$ be such that $E' \in R F'$, and suppose that $E' \not\longrightarrow E''$. Then $E'[a^{n+1}/X] \triangleleft_{\alpha^n} F'[a^{n+1}/X]$ and, using Lemma 13 $E'[a^{n+1}/X] \longrightarrow E''[a^{n+1}/X]$. Since $E'[a^{n+1}/X] \triangleleft_{\alpha^n} F'[a^{n+1}/X]$ there exist closed process expressions $Q''$ and $Q''$ such that $E'[a^{n+1}/X] \longrightarrow Q''[\alpha^n] Q''$, $E'[a^{n+1}/X] \triangleleft_{\alpha^n} Q''$ and $E'[a^{n+1}/X] \triangleleft_{\alpha^n} Q''$. By Lemma 12 using that $a \neq \tau$, there exists a $X$-closed process expression $E''$ such that $E'' \longrightarrow E''$ and $E''[a^{n+1}/X]$. Moreover, there exists an $X$-closed process expression $F''$ such that $F'' \longrightarrow F''$ and $F''[a^{n+1}/X]$. In the latter case we would have $E''[a^{n+1}/X] \triangleleft_{\alpha^n} Q''[\alpha^n] Q''[\alpha^n]$, which is impossible by the choice of $n$. So the former case applies: we have $E'' \longrightarrow F''[\alpha^n] F''$, $E'' \in R F''$, and $E'' \not\longrightarrow F''$. The case that $F'' \not\longrightarrow F''$ proceeds by symmetry, so the symmetric closure of $R$ satisfies condition (4).

To show that $R$ (and its symmetric closure) satisfies $(\Box \mathcal{M})$, let $(E_k)_{k \in \omega}$ be an infinite sequence of $X$-closed process expressions such that $E_k \longrightarrow_{\alpha^n} E_{k+1}$ for all $k \in \omega$, and let $F_0$ be such that $E_0 \in R F_0$. Then $E_0[a^{n+1}/X] \triangleleft_{\alpha^n} F_0[a^{n+1}/X]$ and by Lemma 13 $E_0[a^{n+1}/X] \longrightarrow E[k+1][a^{n+1}/X]$ for all $k \in \omega$. Then, there exist a process expression $Q'$ such that $F_0 \longrightarrow Q'$ and $E_k[a^{n+1}/X] \triangleleft_{\alpha^n} Q'$ for some $k \in N$. By Lemma 12, using that $a \neq \tau$, there exists an $X$-closed process expression $E''$ such that $E'' \longrightarrow E''$ and $E''[a^{n+1}/X] \triangleleft_{\alpha^n} Q'$. Furthermore, $E_k \in R F''$. Application of the claim: Let $E[P/X] \triangleleft_{\alpha^n} F[P/X]$ for each closed process expression $P$. Then $E[a^{n+1}/X] \triangleleft_{\alpha^n} F[a^{n+1}/X]$. The claim yields $E \triangleleft_{\alpha^n} F$.

Now let $E[P/X] \triangleleft_{\alpha^n} F[P/X]$ for each closed $P$. Then $E[a^{n+1}/X] \triangleleft_{\alpha^n} F[a^{n+1}/X]$. Suppose that $E[a^{n+1}/X] \not\longrightarrow E'$ with $\alpha \in \mathcal{A}_r$. Then $E[a^{n+1}/X] \longrightarrow E'[a^{n+1}/X]$ by Lemma 13. So there exists a $Q'$ with $F[a^{n+1}/X] \longrightarrow Q'$ and $E'[a^{n+1}/X] \triangleleft_{\alpha^n} Q'$. By Lemma 12 there exists an $X$-closed process expression $F'$ such that $F \longrightarrow F'$ and $F' \longrightarrow F'[a^{n+1}/X]$. In the latter case we would have $E'[a^{n+1}/X] \triangleleft_{\alpha^n} Q' = a^n$, which is impossible by the choice of $n$. So the former case applies, and $E' \in R F'$. The claim yields $E \triangleleft_{\alpha^n} F'$. The other clause follows by symmetry, so $E \triangleleft_{\alpha^n} F$. 

The following is an immediate corollary of Propositions 11, 12, and 14.
Corollary 15. Let $E$ and $F$ be $X$-closed process expressions such that $E \leftrightarrow^A F$.

1. If $E \overset{\alpha}{\longrightarrow} E'$, then there exist $X$-closed process expressions $F_0, \ldots, F_n$ and $F'$ such that $F = F_0 \overset{\tau}{\longrightarrow} \cdots \overset{\tau}{\longrightarrow} F_n \overset{\alpha}{\longrightarrow} F'$ such that $E \leftrightarrow^A F_i$ ($0 \leq i \leq n$) and $E' \leftrightarrow^A F'$.

2. If $X$ is exposed in $E$, then there exist $k \geq 0$ and $X$-closed process expressions $F_0, \ldots, F_k$ such that $F = F_0 \overset{\tau}{\longrightarrow} \cdots \overset{\tau}{\longrightarrow} F_k$, $E \leftrightarrow^A F_i$ ($0 \leq i \leq k$), and $X$ is exposed in $F_k$.

3. If there is an infinite sequence of $X$-closed process expressions $(E_k)_{k \in \omega}$ such that $E = E_0$ and $E_k \overset{\tau}{\longrightarrow} E_{k+1}$, then there exists an $X$-closed process expression $F'$ such that $F \overset{\tau}{\longrightarrow} F'$ and $E_k \leftrightarrow^A F'$ for some $k \in \omega$.

Similarly, by combining Propositions 14 and Definition 5 we get the following corollary.

Corollary 16. Let $E$ and $F$ be $X$-closed process expressions such that $E \leftrightarrow^A F$. If $E \overset{\alpha}{\longrightarrow} E'$, then there exists an $X$-closed process expression $F'$ such that $F \overset{\alpha}{\longrightarrow} F'$ and $E' \leftrightarrow^A F'$.

3.2 Rooted divergence-preserving branching bisimulation up to $\leftrightarrow^A$

As was already illustrated by Milner [20], a suitable up-to relation is a crucial tool in the proof that a behavioural equivalence is compatible with the recursion construct. In [7], Milner’s notion of weak bisimulation up to weak bisimilarity is adapted to branching bisimulation up to branching bisimilarity. Here we make two further modifications. Not only do we add a divergence condition; we also incorporate rootedness into the relation.

Definition 17. Let $\mathcal{R}$ be a symmetric binary relation on $\mathcal{P}$, and denote by $\mathcal{R}^u$ the relation $\leftrightarrow^A : \mathcal{R} : \leftrightarrow^A$. We say that $\mathcal{R}$ is a rooted divergence-preserving branching bisimulation up to $\leftrightarrow^A$ if for all $P, Q \in \mathcal{P}$ such that $P \mathcal{R} Q$ the following three conditions are satisfied:

(U1) if $P \overset{\alpha}{\longrightarrow} P'$, then there exist $Q'$ such that $Q \overset{\alpha}{\longrightarrow} Q'$, and $P' \mathcal{R}^u Q'$.

(U2) if $P \overset{\tau}{\longrightarrow} P'$, then there exist $Q'$ and $Q''$ such that $Q \overset{\tau}{\longrightarrow} Q'' \overset{\alpha}{\longrightarrow} Q'$, $P' \mathcal{R}^u Q''$ and $P' \mathcal{R}^u Q'$.

(U3) if there exists an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $P = P_0$, and $P_k \overset{\tau}{\longrightarrow} P_{k+1}$ for all $k \in \omega$, then there also exists an infinite sequence of closed process expressions $(Q_\ell)_{\ell \in \omega}$ and a mapping $\sigma : \omega \rightarrow \omega$ such that $Q = Q_0$, and $Q_\ell \overset{\tau}{\longrightarrow} Q_{\sigma(\ell)+1}$ and $P_\sigma(\ell) \mathcal{R}^u Q_\ell$ for all $\ell \in \omega$.

Proposition 18. Let $P$ and $Q$ be closed process expressions and let $\mathcal{R}$ be a rooted divergence-preserving branching bisimulation up to $\leftrightarrow^A$. If $P \mathcal{R} Q$, then $P \leftrightarrow^A Q$.

Proof. If $P \mathcal{R} Q$ and $P \overset{\alpha}{\longrightarrow} P'$, then since $\mathcal{R}$ satisfies condition (U1) of Definition 17, there exists a $Q'$ such that $Q \overset{\alpha}{\longrightarrow} Q'$ and $P' \mathcal{R}^u Q'$. Furthermore, since $\mathcal{R}$ is symmetric, whenever $P \mathcal{R} Q$ also $Q \mathcal{R} P$, so if $Q \overset{\alpha}{\longrightarrow} Q'$, then by condition (U1) of Definition 17 there exists a $P'$ such that $P \overset{\alpha}{\longrightarrow} P'$ and $Q' \mathcal{R}^u P'$. It remains to establish that $P' \leftrightarrow^A Q'$, and for this, it suffices by Proposition 5 to prove that $\mathcal{R}^u$ is a branching bisimulation satisfying (D).

Note that, since $\leftrightarrow^A$ and $\mathcal{R}$ are both symmetric, also $\mathcal{R}^u$ is symmetric.

To prove that $\mathcal{R}^u$ satisfies (D), let $P_0$, $Q_0$ and $Q_1$ be closed process expressions such that $P_0 \leftrightarrow^A P_0$, $P_0 \mathcal{R} Q_0 \leftrightarrow^A Q_1$, and suppose that $P_1 \overset{\alpha}{\longrightarrow} P_1'$. Since $P_1 \leftrightarrow^A P_1'$ and $\mathcal{R}^u$ satisfies (D), there exist $P_0'$ and $P_0''$ such that $P_0 \overset{\tau}{\longrightarrow} P_0'' \overset{\alpha}{\longrightarrow} P_0'$, $P_1 \leftrightarrow^A P_0''$ and $P_1 \leftrightarrow^A P_0'$, and $P_1' \mathcal{R}^u P_0''$. So it follows by condition (U2) of Definition 17 that there exist $P_0$ and $Q_0'$ such that $Q_0' \mathcal{R}^u Q_0''$ and $P_0' \mathcal{R}^u Q_0''$. Hence, since $Q_0 \leftrightarrow^A Q_1$, by Lemma 2 there exist closed process expressions $Q_1$ and $Q_1'$ such that $Q_1 \overset{\alpha}{\longrightarrow} Q_1'' \overset{\alpha}{\longrightarrow} Q_1'$, $Q_0' \leftrightarrow^A Q_0''$ and $Q_0' \leftrightarrow^A Q_0''$. Note, moreover, that $P_1 \leftrightarrow^A P_0'' \mathcal{R}^u Q_0'' \leftrightarrow^A Q_1'$ whence $P_1 \mathcal{R}^u Q_1'$, and $P_1' \leftrightarrow^A P_0' \mathcal{R}^u Q_0' \leftrightarrow^A Q_1'$ whence $P_1' \mathcal{R}^u Q_1'$.
It remains to prove that $\mathcal{R}^n$ satisfies (17) of Proposition 3. To this end, let $P_0$, $P_1$, $Q_0$ and $Q_1$ be closed process expressions such that $P_1 \xrightarrow{\Delta} P_0 \mathcal{R} Q_0 \xrightarrow{\Delta} Q_1$, and suppose that there exists an infinite sequence of closed process expressions $(P_1, k)_{k \in \omega}$ such that $P_1 = P_{1,0}$ and $P_{1, k} \xrightarrow{\Delta} P_{1, k+1}$. Then, since $P_1 \xrightarrow{\Delta} P_0$, by Proposition 3 there exists an infinite sequence of closed process expressions $(P_0, k)_{k \in \omega}$ and a mapping $\sigma : \omega \rightarrow \omega$ such that $P_0 = P_{0,0}$, $P_{0, k} \xrightarrow{\Delta} P_{0, k+1}$ and $P_{1, \sigma(k)} \xrightarrow{\Delta} P_{0, k}$ for all $k \in \omega$. Hence, since $P_0 \mathcal{R} Q_0$ and $\mathcal{R}$ is a divergence-preserving branching bisimulation up to $\Delta$, there exists an infinite sequence of closed process expressions $(Q_0, m)_{m \in \omega}$ and a mapping $\sigma_p : \omega \rightarrow \omega$ such that $Q_0 = Q_{0,0}$, $Q_{0, m} \xrightarrow{\Delta} Q_{0, m+1}$ and $P_{0, \sigma_p(m)} \mathcal{R} Q_{0, m}$ for all $m \in \omega$. Hence, since $Q_0 \xrightarrow{\Delta} Q_1$, by Proposition 3 there exists an infinite sequence of closed process expressions $(Q_1, n)_{n \in \omega}$ and a mapping $\sigma_Q : \omega \rightarrow \omega$ such that $Q_1 = Q_{1,0}$, $Q_{1, n} \xrightarrow{\Delta} Q_{1, n+1}$ and $Q_{0, \sigma_Q(n)} \xrightarrow{\Delta} Q_{1, n}$ for all $n \in \omega$. We define
\[
\sigma = \sigma_p \circ \sigma_Q \circ \sigma_Q ,
\]
and then we have that $P_{1, \sigma(n)} \xrightarrow{\Delta} \mathcal{R}^n : \xrightarrow{\Delta} Q_{1, n}$, and hence $P_{1, \sigma(n)} \mathcal{R}^n Q_{1, n}$ for all $n \in \omega$. \[\square\]

To prove that $\xrightarrow{\Delta}$ is compatible with $\mu X.\_\_$ means to prove that if $E \xrightarrow{\Delta} F$, then $\mu X.E \xrightarrow{\Delta}$ $\mu X.F$. We first do this in the special case that $E$ and $F$ are $X$-closed. A crucial step in this proof will be to show that if $E \xrightarrow{\Delta} F$ for $X$-closed process expressions $E$ and $F$, then the symmetric closure $\mathcal{R}_{E,F}$ of the relation
\[
\{(G[\mu X.E/X], G[\mu X.F/X]) \mid G \in E \text{ and } G \text{ is } X\text{-closed}\}
\]
is a rooted branching bisimulation up to $\Delta$. The result then follows by taking $G := X$. Until Corollary 25 we fix $X$-closed process expressions $E$ and $F$ such that $E \xrightarrow{\Delta} F$.

For this application of the up-to technique from Definition 17, $\mathcal{R}^n$ could equally well have been defined as $\mathcal{R} : \xrightarrow{\Delta}$ $\mathcal{R}^n$. This less powerful technique is still valid by Proposition 15 yet is all we need in Lemma 19.\[\square\]

Lemma 19. For all $X$-closed process expressions $G$, if $G[\mu X.E/X] \xrightarrow{\alpha} P$, then there exists a $Q$ such that $G[\mu X.F/X] \xrightarrow{\alpha} Q$ and $P \mathcal{R}_{E,F} : \xrightarrow{\Delta} Q$.

Proof. Let $G$ be an $X$-closed process expression, and suppose that $G[\mu X.E/X] \xrightarrow{\alpha} P$; we proceed with induction on a derivation of this transition. By Lemma 22 there are two cases: either the transition under consideration stems directly from $G$, i.e., there exists a $G'$ such that $G \xrightarrow{\alpha} G'$ and $P = G'[\mu X.E/X]$, or $X$ is exposed in $G$, $\mu X.E \xrightarrow{\alpha} P$ and every derivation of $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation.

In the first case, Lemma 11 implies $G[\mu X.F/X] \xrightarrow{\alpha} G'[\mu X.F/X]$ and $P = G'[\mu X.E/X] \mathcal{R}_{E,F} G'[\mu X.F/X]$, so, since $\xrightarrow{\Delta}$ is reflexive, also $P \mathcal{R}_{E,F} : \xrightarrow{\Delta} G'[\mu X.F/X]$.

In the second case, since the considered derivation of the transition $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation, the last rule applied in this subderivation must be rule 2, it follows that the considered derivation of $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P$ as a proper subderivation. So by the induction hypothesis there exists a $Q$ such that $E[\mu X.F/X] \xrightarrow{\alpha} Q$ and $P \mathcal{R}_{E,F} : \xrightarrow{\Delta} Q$. Furthermore, since $E \xrightarrow{\Delta} F$, whence $E[\mu X.F/X] \xrightarrow{\Delta} F[\mu X.F/X]$, it follows that there exists an $R$ such that $F[\mu X.F/X] \xrightarrow{\alpha} R$ and $Q \xrightarrow{\Delta} R$. It follows from $F[\mu X.F/X] \xrightarrow{\alpha} R$ that $\mu X.F \xrightarrow{\alpha} R$. Since $X$ is exposed in $G$, Lemma 11 yields $G[\mu X.F/X] \xrightarrow{\alpha} R$. From $P \mathcal{R}_{E,F} : \xrightarrow{\Delta} Q$ and $Q \xrightarrow{\Delta} R$ it follows that $P \mathcal{R}_{E,F} : \xrightarrow{\Delta} R$. \[\square\]

As an immediate corollary to Lemma 11 we get that if $E \xrightarrow{\Delta} F$, then $\mathcal{R}_{E,F}$ satisfies the first condition of rooted divergence-preserving branching bisimulations up to $\xrightarrow{\Delta}$.

Corollary 20. $\mathcal{R}_{E,F}$ satisfies condition (17) of Definition 17.

With a little more work, Lemma 19 can also be used to derive that $\mathcal{R}_{E,F}$ satisfies the second condition of rooted divergence-preserving branching bisimulations up to $\xrightarrow{\Delta}$. To this end, we first prove the following lemma.
Lemma 21. Let $P$ and $Q$ be closed process expressions. If $P \rightarrow_{E,F} Q$ and $P \rightarrow_{R} P'$, then there exist $Q'$ and $Q''$ such that $Q \rightarrow Q'' \rightarrow Q'$, $P \rightarrow_{E,F} Q'$, and $P' \rightarrow_{E,F} Q''$. 

Proof. Suppose that $P \rightarrow_{E,F} Q$ and $P \rightarrow_{R} P'$. Then there exists an $R'$ such that $P \rightarrow_{R} P'$ and, according to the definition of $R_{E,F}$, there exists an $X$-closed process expression $G$ such that either $P = G[\mu X.E/X]$ or $P = G[\mu X.F/X]$ for $G = G[\mu X.E/X]$ and $R = G'[\mu X.E/X]$. Without loss of generality we assume that $P = G[\mu X.E/X]$ and $R = G'[\mu X.F/X]$. By Lemma 15, there exists an $R'$ such that $R \rightarrow_{R} R'$ and $P' \rightarrow_{E,F} Q''$. Hence, since $R \rightarrow_{E,F} Q$, there exist $Q'$ and $Q''$ such that $Q \rightarrow Q'' \rightarrow Q'$, $R \rightarrow_{E,F} Q''$ and $R' \rightarrow_{E,F} Q'$. It follows that $P \rightarrow_{E,F} Q''$ and $P' \rightarrow_{E,F} Q'$, so the proof of the lemma is complete.

Using that $P \rightarrow_{E,F} Q$ implies $P \rightarrow_{E,F} Q'$ by reflexivity of $\equiv_{E,F}$, and applying Lemma 21 with induction on the length of a transition that gives rise to $P \rightarrow_{R} P'$, it is straightforward to establish the following corollary.

Corollary 22. $R_{E,F}$ satisfies condition (2) of Definition 14.

It remains to establish that $R_{E,F}$ satisfies the third condition of rooted divergence-preserving branching bisimulations up to $\equiv_{E,F}$.

Lemma 23. Let $G$ and $H$ be $X$-closed process expressions such that $G \equiv_{E,F} H$. If there exists an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $G[\mu X.E/X] = P_0$ and $P_k \rightarrow_{r} P_{k+1}$ for all $k \in \omega$, then there also exists an infinite sequence of closed process expressions $(Q_k)_{k \in \omega}$ and a mapping $\sigma : \omega \rightarrow \omega$ such that $H[\mu X.F/X] = Q_0$, $Q_k \rightarrow_{r} Q_{k+1}$, and $P_{\sigma(k)} \rightarrow_{E,F} Q_k$ for all $k \in \omega$.

Proof. Suppose that there exists an infinite sequence of closed process expressions $(P_k)_{k \in \omega}$ such that $G[\mu X.E/X] = P_0$ and $P_k \rightarrow_{r} P_{k+1}$ for all $k \in \omega$. By Corollary 17 there is an infinite sequence of $\Delta$-closed process expressions $(G_k)_{k \in \omega}$ such that $P_k = G_k[\mu X.E/X]$ and either $G_k \rightarrow_{r} G_{k+1}$ or $G_k \rightarrow_{r} E \rightarrow_{r} G_{k+1}$ for all $k \in \omega$. We shall define simultaneously, with induction on $k$, an infinite sequence of $\Delta$-closed process expressions $(H_k)_{k \in \omega}$ with $H_0 = H$ and $H_k \rightarrow_{r} H_{k+1}[\mu X.F/X]$, and a mapping $\sigma : \omega \rightarrow \omega$, such that $G_{\sigma(k)} \equiv_{E,F} H_k$. This will suffice, because, for all $k \in \omega$, defining $Q_k$ as $H_k[\mu X.F/X]$ we obtain $Q_k \rightarrow_{r} Q_{k+1}$ and $P_{\sigma(k)} \rightarrow_{E,F} Q_k$ for all $k \in \omega$.

Suppose, by way of induction hypothesis, that $H_k$ and $\sigma(k)$ have been defined already, such that $G_{\sigma(k)} \equiv_{E,F} H_k$. By Corollary 17 there are two cases:

1. $G_{\sigma(k)+k} \rightarrow_{E,F} G_{\sigma(k)+k+1}$ for all $k \in \omega$. Then, since $G_{\sigma(k)} \equiv_{E,F} H_k$, by Corollary 19 there exists an $X$-closed process expression $H'$ such that $H_k \rightarrow_{r} H'$ and $G_{\sigma(k)+k} \equiv_{E,F} H'$ for some $k \in \omega$. We define $H_{k+1} = H'$ and $\sigma(k+1) = \sigma(k) + k$. Now $H_k[\mu X.F/X] \rightarrow_{r} H_{k+1}[\mu X.F/X]$ by Lemma 11 and $G_{\sigma(k+1)} \equiv_{E,F} H_{k+1}$.

2. There is a $k \in \omega$ such that $G_{\sigma(k)+i} \rightarrow_{E,F} H_{\sigma(k)+i+1}$ for all $i < k$, $X$ is exposed in $G_{\sigma(k)+k}$ and $E \rightarrow_{r} G_{\sigma(k)+k+1}$. Then, since $G_{\sigma(k)} \equiv_{E,F} H_k$, by Corollary 11 and with induction on $i$ there exists a sequence $F_0, \ldots, F_m$ and a mapping $\rho : \{0, \ldots, m\} \rightarrow \{0, \ldots, k\}$ with $\rho(m) = k$ such that $H_k = H_0 \rightarrow_{r} \cdots \rightarrow_{r} H_m$ and $G_{\sigma(k)+\rho(j)} \equiv_{E,F} H_j$. Using Corollary 16 we may furthermore assume that $X$ is exposed in $H_m$.

If $m > 0$, we define $H_{i+1} = H'_i$ and $\sigma(i+1) = \sigma(i) + \rho(i)$. Now $H_i[\mu X.F/X] \rightarrow H_{i+1}[\mu X.F/X]$ by Lemma 11, and $G_{\sigma(i+1)} \equiv_{E,F} H_{i+1}$.

So it remains to consider the case that $m = 0$. Since $E \equiv_{E,F} F$, there exists, by Corollary 16 an $X$-closed process expression $F'$ such that $F \rightarrow_{r} F'$ and $G_{\sigma(k)+k} \equiv_{E,F} F'$. We now define $H_{i+1} = E'$ and $\sigma(i+1) = \sigma(i) + 1$. We then have that $G_{\sigma(k)} \equiv_{E,F} H_{\sigma(k)+k}$, $H_k \rightarrow_{r} H_{\sigma(k)+k}[\mu X.F/X]$ by Lemma 11, so $\mu X.F \rightarrow_{r} H_{\sigma(k)+k}[\mu X.F/X]$ by rule 2 and Lemma 11 yields $H_k \rightarrow_{r} H_{\sigma(k)+k}[\mu X.F/X]$, using that $X$ is exposed in $H_k$. □
From Lemma 23 with \( G = H \) we immediately get the following corollary.

**Corollary 24.** \( \mathcal{R}_{E,F} \) satisfies condition (U2) of Definition 17.

The relation \( \mathcal{R}_{E,F} \) is symmetric by definition and we have now also proved that it satisfies conditions (U1), (U2) and (U3), so we have established the following result.

**Corollary 25.** \( \mathcal{R}_{E,F} \) is a rooted divergence-preserving branching bisimulation up to \( \equiv_{\Delta} \).

### 3.3 The main results

We can now establish that \( \equiv_{\Delta} \) is compatible with \( \alpha \), \( \mu X. \) and +.

**Proposition 26.** If \( E \equiv_{\Delta} F \), then \( \alpha.E \equiv_{\Delta} \alpha.F \) for all \( \alpha \in \mathcal{A} \), \( E + H \equiv_{\Delta} F + H \) and \( H + E \equiv_{\Delta} H + F \) for all process expressions \( H \), and \( \mu X.E \equiv_{\Delta} \mu X.F \).

**Proof.** To prove that \( \equiv_{\Delta} \) is compatible with \( \alpha \), and + is straightforward. (First, establish the property for closed terms, and then use that substitution distributes over \( \alpha \) and +.)

It remains to prove that \( \equiv_{\Delta} \) is compatible with \( \mu X. \) i.e., that \( E \equiv_{\Delta} F \) implies \( \mu X.E \equiv_{\Delta} \mu X.F \). Note that in the special case that \( E \) and \( F \) are X-closed this immediately follows from Corollary 25 and Proposition 18. Now, for the general case, let \( E \) and \( F \) be process expressions and suppose that \( E \equiv_{\Delta} F \). Let \( X, Y \) be a sequence of variables that at least includes the variables with a free occurrence in \( E \) or \( F \), and such that \( X \) does not occur in \( Y \). Then, according to the definition of \( \equiv_{\Delta} \) on process expressions with free variables (Definition 7), we have that, for every closed process expression \( P \) and for every sequence of closed process expressions \( \bar{P} \) of the same length as \( Y \), \( E[P, \bar{P}/X, \bar{Y}] \equiv_{\Delta} F[P, \bar{P}/X, \bar{Y}] \). So, clearly, also \( E[\bar{P}/\bar{Y}] \equiv_{\Delta} F[\bar{P}/\bar{Y}] \), and since \( E[\bar{P}/\bar{Y}] \) and \( F[\bar{P}/\bar{Y}] \) are X-closed, it follows that \( \mu X.E[\bar{P}/\bar{Y}] \equiv_{\Delta} \mu X.F[\bar{P}/\bar{Y}] \). Since \( X \) is not among the \( Y \), we may conclude that \( (\mu X.E)[\bar{P}/\bar{Y}] \equiv_{\Delta} (\mu X.F)[\bar{P}/\bar{Y}] \) for every sequence of closed process expressions \( \bar{P} \) of the same length as \( \bar{Y} \), and hence \( \mu X.E \equiv_{\Delta} \mu X.F \).

We have now obtained our main result that \( \equiv_{\Delta} \) is a congruence. In fact, it is the coarsest contained in \( \equiv_{\Delta} \).

**Theorem 27.** The relation \( \equiv_{\Delta} \) is the coarsest congruence contained in \( \equiv_{\Delta} \).

**Proof.** By Propositions 19 and 26 the relation \( \equiv_{\Delta} \) is a congruence. To prove that it is coarsest, it suffices to prove that for every relation \( \mathcal{R} \subseteq \equiv_{\Delta} \) that is compatible with + we have that \( \mathcal{R} \subseteq \equiv_{\Delta} \).

Let \( P \) and \( Q \) be closed process expressions, and suppose that \( P \mathcal{R} Q \).

Since by Proposition 12 the set of closed process expressions reachable from \( P \) and \( Q \) is finite and \( \mathcal{A} \) is non-empty, there exists a natural number \( n \in \omega \) such that for all \( R \) reachable from \( P \) or \( Q \) it holds that \( R \equiv_{\Delta} a^n \). This implies that for all \( R \) reachable from \( P \) or \( Q \) it holds that \( R \equiv_{\Delta} P + a^{n+1} \) and \( R \equiv_{\Delta} Q + a^{n+1} \).

Since \( \mathcal{R} \) is compatible with +, we have that \( P + a^{n+1} \mathcal{R} Q + a^{n+1} \), and hence \( P + a^{n+1} \equiv_{\Delta} Q + a^{n+1} \). To prove (R1), suppose that \( P \rightarrow^\alpha P' \). Then \( P + a^{n+1} \rightarrow^\alpha P' \), so by Lemma 2 there exist closed process expressions \( Q' \) and \( Q'' \) such that \( Q + a^{n+1} \rightarrow Q'' \rightarrow^\tau Q' \), \( P + a^{n+1} \equiv_{\Delta} Q'' \) and \( P' \equiv_{\Delta} Q' \). Since \( a \neq \tau \), we have that \( Q'' = Q + a^{n+1} \), for otherwise \( Q'' \) is reachable from \( Q \) and \( Q'' \rightarrow^\tau Q' \), moreover, \( Q'' \rightarrow^\tau Q' \), for otherwise \( P' \equiv_{\Delta} Q' = Q'' = Q + a^{n+1} \). Condition (R2) follows by symmetry.

### References


