MALL proof nets identify proofs modulo rule commutation

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We show that the proof nets introduced in [4, 5] for MALL (Multiplicative Additive Linear Logic, without units) identify cut-free proofs modulo rule commutation: two cut-free proofs translate to the same proof net *if and only if* one can be obtained from the other by a succession of rule commutations. This result holds with and without the mix rule, and we extend it with cut.

1 Introduction

The proof nets for MALL (Multiplicative Additive Linear Logic [2], without units) introduced in [4, 5] solved numerous issues with monomial proof nets [3], for example:

- There is a simple (deterministic) translation function from cut-free proofs to proof nets.
- Cut elimination is simply defined and strongly normalising.
- Proof nets form a semi (*i.e.*, unit-free) star-autonomous category with (co)products.

A proof net is a set of ***linkings*** on a sequent. Each linking is a set of ***links*** between complementary formula leaves (literal occurrences). Figure 1 illustrates the translation of a proof into a proof net.

In this paper we prove that the translation precisely captures proofs modulo rule commutation: two proofs translate to the same proof net *if and only if* one can be obtained from the other by a succession of rule commutations. A ***rule commutation*** is a transposition of adjacent rules that preserves subproofs immediately above, with possible duplication/identification, for example

$$\dfrac{\dfrac{}{P^{\perp}, P} \qquad \dfrac{\dfrac{}{Q, Q^{\perp}}}{Q,\ R\oplus Q^{\perp}}\oplus_2}{P^{\perp},\ P\otimes Q,\ R\oplus Q^{\perp}}\otimes \qquad\longrightarrow\qquad \dfrac{\dfrac{\dfrac{}{P^{\perp}, P} \qquad \dfrac{}{Q, Q^{\perp}}}{P^{\perp},\ P\otimes Q\ , Q^{\perp}}\otimes}{P^{\perp},\ P\otimes Q,\ R\oplus Q^{\perp}}\oplus_2$$

in which the lower $\otimes$-rule commutes over the $\oplus$-rule, or

$$\dfrac{\dfrac{\dfrac{}{P^{\perp}, P}}{P^{\perp}, P\oplus R}\oplus_1 \qquad \dfrac{\dfrac{}{Q^{\perp}, Q} \quad \dfrac{}{Q^{\perp}, Q}}{Q^{\perp},\ Q\&Q}\&}{P^{\perp},\ (P\oplus R)\otimes Q^{\perp},\ Q\&Q}\otimes \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{}{P^{\perp}, P}}{P^{\perp}, P\oplus R}\oplus_1 \quad \dfrac{}{Q^{\perp}, Q}}{P^{\perp}, (P\oplus R)\otimes Q^{\perp}, Q}\otimes \qquad \dfrac{\dfrac{\dfrac{}{P^{\perp}, P}}{P^{\perp}, P\oplus R}\oplus_1 \quad \dfrac{}{Q^{\perp}, Q}}{P^{\perp}, (P\oplus R)\otimes Q^{\perp}, Q}\otimes}{P^{\perp},\ P\otimes Q^{\perp},\ Q\&Q}\&$$

illustrating duplication (of the $\otimes$-rule and subproof $\dfrac{\dfrac{}{P^{\perp}, P}}{P^{\perp}, P\oplus R}\oplus_1$) as the $\otimes$-rule commutes over the $\&$-rule.

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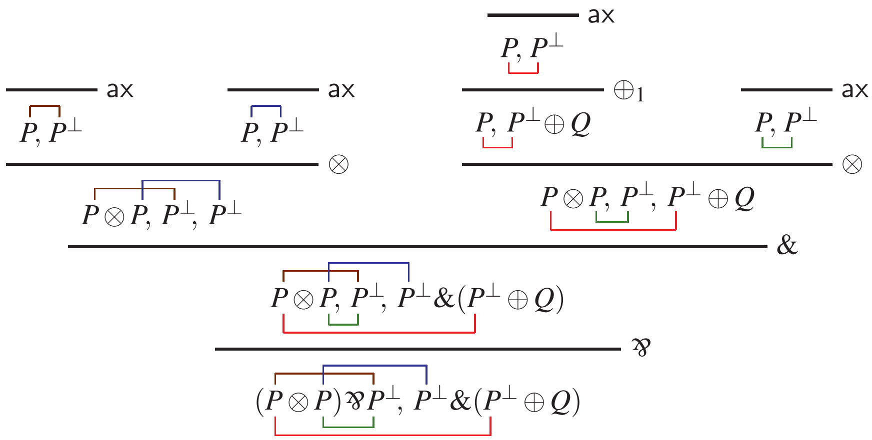


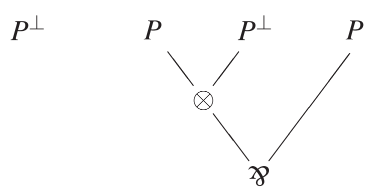
Figure 1: Example of the inductive translation of a MALL proof into a proof net. The concluding proof net has two linkings, one drawn above the sequent, the other below. Each has two links. The proof nets further up in the derivation have one or two linkings, correspondingly above/below the sequent.

2 Cut-free MALL

Let MALL$^-$ denote cut-free multiplicative-additive linear logic without units [2].[1] Formulas are built from literals (propositional variables $P, Q, \ldots$ and their negations $P^\perp$, $Q^\perp, \ldots$) by the binary connectives ***tensor*** $\otimes$, ***par*** $\parr$, ***with*** $\&$ and ***plus*** $\oplus$. Negation $(-)^\perp$ extends to arbitrary formulas with $P^{\perp\perp} = P$ on propositional variables and de Morgan duality: $(A \otimes B)^\perp = A^\perp \parr B^\perp$, $(A \parr B)^\perp = A^\perp \otimes B^\perp$, $(A \oplus B)^\perp = A^\perp \& B^\perp$, and $(A \& B)^\perp = A^\perp \oplus B^\perp$. We identify a formula with its parse tree, labelled with literals on leaves and connectives on internal vertices. A ***sequent*** is a disjoint union of formulas. Thus a sequent is a labelled forest. We write comma for disjoint union. For example,

$$P^\perp,\ (P \otimes P^\perp) \parr P$$

is the labelled forest



Sequents are proved using the following rules:

$$\frac{}{P, P^\perp}\,\mathsf{ax} \qquad \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta}\,\otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr$$

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B}\,\& \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2 \qquad \frac{\Gamma \quad \Delta}{\Gamma, \Delta}\,\mathsf{mix} \quad \text{(optional)}$$

[1] We treat cut in Section 5.

$$\dfrac{}{\{\overline{P,P^\perp}\}\ \rhd\ P,P^\perp}\,\mathsf{ax} \qquad \dfrac{\theta\ \rhd\ \Gamma,A \quad \theta'\ \rhd\ \Gamma,B}{\theta\cup\theta'\ \rhd\ \Gamma,A\&B}\,\& \qquad \dfrac{\theta\ \rhd\ \Gamma,A \qquad \theta'\ \rhd\ B,\Delta}{\{\lambda\cup\lambda' : \lambda\in\theta,\lambda'\in\theta'\}\ \rhd\ \Gamma,A\otimes B,\Delta}\,\otimes$$

$$\dfrac{\theta\ \rhd\ \Gamma,A,B}{\theta\ \rhd\ \Gamma,A⅋B}\,⅋ \qquad \dfrac{\theta\ \rhd\ \Gamma,A}{\theta\ \rhd\ \Gamma,A\oplus B}\,\oplus_1 \qquad \dfrac{\theta\ \rhd\ \Gamma,B}{\theta\ \rhd\ \Gamma,A\oplus B}\,\oplus_2 \qquad \dfrac{\theta\ \rhd\ \Gamma \qquad \theta'\ \rhd\ \Delta}{\{\lambda\cup\lambda' : \lambda\in\theta,\lambda'\in\theta'\}\ \rhd\ \Gamma,\Delta}\,\mathsf{mix}$$

Table 1: Alternative but equivalent definition of the function from MALL⁻ proofs to linking sets. Here $\theta\ \rhd\ \Gamma$ signifies that θ is a set of linkings on Γ. We use the implicit tracking of formula leaves downwards through rules. The base case is a singleton linking set whose only linking comprises a single link, between P and $P^\perp$.

The mix-rule is optional and absent by default. Our treatment is valid for MALL⁻ with and without mix.

Throughout this document P,Q,R range over propositional variables, $A,B,\ldots$ over formulas, and Γ,Δ,Σ over sequents. Each of the proof rules above yields an implicit tracking of subformula occurrences, mapping the vertices in the hypotheses to the ones in the conclusion. A formula occurrence in the conclusion of a rule ρ is ***generated*** by ρ if it is not in the image of this map.

3 Function from proofs to proof nets

A ***link*** on Γ is a pair (two-element set) of leaves in Γ. A ***linking*** on Γ is a set of links on Γ.[2] Every MALL⁻ proof Π of Γ defines a set θ_Π of linkings on Γ as follows. Define a ***&-resolution*** R of Π to be any result of deleting one branch above each &-rule of Π. By downwards tracking of formula leaves, the axiom rules of R determine a linking λ_R on Γ. Define $\theta_\Pi = \{\lambda_R : R \text{ is a \&-resolution of } \Pi\}$.

Table 1 defines the same function by induction. See Figure 1 for an example. The fact that this yields the same linking set as the resolution-based function follows from a simple structural induction on proofs. Note that $\otimes$ (resp. &) is ***multiplicative*** (resp. ***additive***): multiply (resp. add) the number of linkings in θ and θ' to obtain the number of linkings on the conclusion.[3]

A linking set is a ***proof net*** if it is the translation of a proof.[4]

4 Rule commutations

Tables 2, 3 and 4 exhaustively list the rule commutations of MALL⁻. Each commutation may be applied in context, *i.e.*, to any subproof. This collection of rule commutations is not ad hoc: they are generated systematically from a general definition of commutation, presented in the Appendix, which is more liberal than the one analysed by Kleene [7] and Curry [1] in the context of sequent calculus [8, Def. 5.2.1].

Our main result is that the kernel of the function from MALL⁻ proofs to proof nets coincides precisely with equivalence modulo rule commutations:

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[2]The paper [5] imposed additional conditions in the definition of a linking. We do not need these conditions here.

[3]This observation relies on θ and θ' having no common linking, which follows (by structural induction) from the fact that in any proof net on Γ, every linking touches every formula in Γ (*i.e.*, for every linking λ in the proof net, and every formula(-occurrence) A in Γ, some link of λ contains a leaf of A).

[4]In [4, 5] we defined a proof net via a geometric criterion on a linking set, and proved that a linking set meets this criterion if and only if it is the translation of a proof.

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma,A_1,A_2,B_1,B_2}}{\Gamma,A_1 \parr A_2,B_1,B_2}\,\parr}{\Gamma,A_1 \parr A_2,B_1 \parr B_2}\,\parr \qquad \overset{\mathsf{C}^{\parr}_{\parr}}{\longleftrightarrow} \qquad \dfrac{\dfrac{\dfrac{\Pi}{\Gamma,A_1,A_2,B_1,B_2}}{\Gamma,A_1,A_2,B_1 \parr B_2}\,\parr}{\Gamma,A_1 \parr A_2,B_1 \parr B_2}\,\parr$$

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma,A_i,B_j}}{\Gamma,A_1 \oplus A_2,B_j}\,\oplus_i}{\Gamma,A_1 \oplus A_2,B_1 \oplus B_2}\,\oplus_j \qquad \overset{\mathsf{C}^{\oplus}_{\oplus}}{\longleftrightarrow} \qquad \dfrac{\dfrac{\dfrac{\Pi}{\Gamma,A_i,B_j}}{\Gamma,A_i,B_1 \oplus B_2}\,\oplus_j}{\Gamma,A_1 \oplus A_2,B_1 \oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A_1} \quad \dfrac{\dfrac{\Pi_2}{A_2,\Delta,B_1} \quad \dfrac{\Pi_3}{B_2,\Sigma}}{A_2,\Delta,B_1 \otimes B_2,\Sigma}\,\otimes}{\Gamma,A_1 \otimes A_2,\Delta,B_1 \otimes B_2,\Sigma}\,\otimes \qquad \overset{\mathsf{C}^{\otimes}_{\otimes}}{\longleftrightarrow} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A_1} \quad \dfrac{\Pi_2}{A_2,\Delta,B_1}}{\Gamma,A_1 \otimes A_2,\Delta,B_1}\,\otimes \quad \dfrac{\Pi_3}{B_2,\Sigma}}{\Gamma,A_1 \otimes A_2,\Delta,B_1 \otimes B_2,\Sigma}\,\otimes$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A_1,B_1} \quad \dfrac{\Pi_2}{\Gamma,A_2,B_1}}{\Gamma,A_1 \& A_2,B_1}\,\& \quad \dfrac{\dfrac{\Pi_3}{\Gamma,A_1,B_2} \quad \dfrac{\Pi_4}{\Gamma,A_2,B_2}}{\Gamma,A_1 \& A_2,B_2}\,\&}{\Gamma,A_1 \& A_2,B_1 \& B_2}\,\& \qquad \overset{\mathsf{C}^{\&}_{\&}}{\longleftrightarrow} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A_1,B_1} \quad \dfrac{\Pi_3}{\Gamma,A_1,B_2}}{\Gamma,A_1,B_1 \& B_2}\,\& \quad \dfrac{\dfrac{\Pi_2}{\Gamma,A_2,B_1} \quad \dfrac{\Pi_4}{\Gamma,A_2,B_2}}{\Gamma,A_2,B_1 \& B_2}\,\&}{\Gamma,A_1 \& A_2,B_1 \& B_2}\,\&$$

Table 2: Homogeneous rule commutations. In the last conversion, note the reversal of Π_2 and Π_3.

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_i, B_1, B_2}}{\Gamma, A_1 \oplus A_2, B_1, B_2}\,\oplus_i}{\Gamma, A_1 \oplus A_2, B_1 \parr B_2}\,\parr \qquad \underset{\mathsf{C}^{\parr}_{\oplus}}{\overset{\mathsf{C}^{\oplus}_{\parr}}{\substack{\longrightarrow \\ \longleftarrow}}} \qquad \dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_i, B_1, B_2}}{\Gamma, A_i, B_1 \parr B_2}\,\parr}{\Gamma, A_1 \oplus A_2, B_1 \oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_i, B_1}}{\Gamma, A_1 \oplus A_2, B_1}\,\oplus_i \quad \dfrac{\dfrac{\Pi_2}{\Gamma, A_i, B_2}}{\Gamma, A_1 \oplus A_2, B_2}\,\oplus_i}{\Gamma, A_1 \oplus A_2, B_1 \& B_2}\,\& \qquad \underset{\mathsf{C}^{\&}_{\oplus}}{\overset{\mathsf{C}^{\oplus}_{\&}}{\substack{\longrightarrow \\ \longleftarrow}}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_i, B_1} \quad \dfrac{\Pi_2}{\Gamma, A_i, B_2}}{\Gamma, A_i, B_1 \& B_2}\,\&}{\Gamma, A_1 \oplus A_2, B_1 \& B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1, A_2, B_1}}{\Gamma, A_1 \parr A_2, B_1}\,\parr \quad \dfrac{\dfrac{\Pi_2}{\Gamma, A_1, A_2, B_2}}{\Gamma, A_1 \parr A_2, B_2}\,\parr}{\Gamma, A_1 \parr A_2, B_1 \& B_2}\,\& \qquad \underset{\mathsf{C}^{\&}_{\parr}}{\overset{\mathsf{C}^{\parr}_{\&}}{\substack{\longrightarrow \\ \longleftarrow}}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1, A_2, B_1} \quad \dfrac{\Pi_2}{\Gamma, A_1, A_2, B_2}}{\Gamma, A_1, A_2, B_1 \& B_2}\,\&}{\Gamma, A_1 \parr A_2, B_1 \& B_2}\,\parr$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_i}}{A_2, \Delta, B_1 \oplus B_2}\,\oplus_i}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \oplus B_2}\,\otimes \qquad \underset{\mathsf{C}^{\otimes}_{\oplus}}{\overset{\mathsf{C}^{\oplus}_{\otimes}}{\substack{\longrightarrow \\ \longleftarrow}}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_i}}{\Gamma, A_1 \otimes A_2, \Delta, B_i}\,\otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_1, B_2}}{A_2, \Delta, B_1 \parr B_2}\,\parr}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \parr B_2}\,\otimes \qquad \underset{\mathsf{C}^{\otimes}_{\parr}}{\overset{\mathsf{C}^{\parr}_{\otimes}}{\substack{\longrightarrow \\ \longleftarrow}}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_1, B_2}}{\Gamma, A_1 \otimes A_2, \Delta, B_1, B_2}\,\otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \parr B_2}\,\parr$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_1} \quad \dfrac{\Pi_3}{A_2, \Delta, B_2}}{A_2, \Delta, B_1 \& B_2}\,\&}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \& B_2}\,\otimes \qquad \underset{\mathsf{C}^{\otimes}_{\&}}{\overset{\mathsf{C}^{\&}_{\otimes}}{\substack{\longrightarrow \\ \longleftarrow}}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_1}}{\Gamma, A_1 \otimes A_2, \Delta, B_1}\,\otimes \quad \dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_3}{A_2, \Delta, B_2}}{\Gamma, A_1 \otimes A_2, \Delta, B_2}\,\otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \& B_2}\,\&$$

Table 3: Heterogeneous rule commutations. The last three have symmetric variants, obtained by switching $A_2 \otimes A_1$ for $A_1 \otimes A_2$ and exchanging hypotheses of rules from left to right, correspondingly. (The hypotheses are not ordered; however, we apply the convention that a hypothesis that contributes to one side of a $\otimes$ or & connective is drawn on that side.) Note that there are two copies of the subproof Π_1 on the right side of the final conversion.

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta} \quad \dfrac{\Pi_3}{\Sigma}}{\Delta,\Sigma}\,\mathsf{mix}}{\Gamma,\Delta,\Sigma}\,\mathsf{mix}
\qquad \overset{\mathsf{C}^{\mathsf{mix}}_{\mathsf{mix}}}{\longleftrightarrow} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta}}{\Gamma,\Delta}\,\mathsf{mix} \qquad \dfrac{\Pi_3}{\Sigma}}{\Gamma,\Delta,\Sigma}\,\mathsf{mix}
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_1} \quad \dfrac{\Pi_3}{B_2,\Sigma}}{\Delta,B_1 \otimes B_2,\Sigma}\,\otimes}{\Gamma,\Delta,B_1 \otimes B_2,\Sigma}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\otimes}}{\overset{\mathsf{C}^{\otimes}_{\mathsf{mix}}}{\rightleftarrows}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_1}}{\Gamma,\Delta,B_1}\,\mathsf{mix} \qquad \dfrac{\Pi_3}{B_2,\Sigma}}{\Gamma,\Delta,B_1 \otimes B_2,\Sigma}\,\otimes
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_i}}{\Delta,B_1 \oplus B_2}\,\oplus_i}{\Gamma,\Delta,B_1 \oplus B_2}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\oplus}}{\overset{\mathsf{C}^{\oplus}_{\mathsf{mix}}}{\rightleftarrows}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_i}}{\Gamma,\Delta,B_i}\,\mathsf{mix}}{\Gamma,\Delta,B_1 \oplus B_2}\,\oplus_i
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_1,B_2}}{\Delta,B_1 \parr B_2}\,\parr}{\Gamma,\Delta,B_1 \parr B_2}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\parr}}{\overset{\mathsf{C}^{\parr}_{\mathsf{mix}}}{\rightleftarrows}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_1,B_2}}{\Gamma,\Delta,B_1,B_2}\,\mathsf{mix}}{\Gamma,\Delta,B_1 \parr B_2}\,\parr
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_1} \quad \dfrac{\Pi_3}{\Delta,B_2}}{\Delta,B_1 \& B_2}\,\&}{\Gamma,\Delta,B_1 \& B_2}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\&}}{\overset{\mathsf{C}^{\&}_{\mathsf{mix}}}{\rightleftarrows}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_1}}{\Gamma,\Delta,B_1}\,\mathsf{mix} \qquad \dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_3}{\Delta,B_2}}{\Gamma,\Delta,B_2}\,\mathsf{mix}}{\Gamma,\Delta,B_1 \& B_2}\,\&
$$

Table 4: Mix rule commutations. The second conversion has a symmetric variant, in which, on the right-hand side, the mix rule applies to the hypothesis contributing to the right argument of the tensor. Since sequents are unordered, we do not need symmetric variants obtained by exchanging the hypotheses of the mix rule. Our general definition of rule commutation in the Appendix also allows a version of $\mathsf{C}^{\mathsf{mix}}_{\mathsf{mix}}$ with three applications of mix, two above and one below. However, this conversion can be generated from the top conversion listed above and is therefore not listed explicitly.

$$\frac{}{P, P^\perp}\,\mathsf{ax} \qquad \frac{\Gamma, A, B}{\Gamma, A \mathbin{⅋} B}\,⅋ \qquad \frac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B, \Delta}\,\otimes \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2$$

$$\frac{\Gamma \quad \Delta}{\Gamma, \Delta}\,\mathsf{mix} \quad \text{(optional)} \qquad \frac{\Gamma, A \qquad A^\perp, \Delta}{\Gamma, A * A^\perp, \Delta}\,* \qquad \frac{\Omega_1, \Gamma, A \qquad \Omega_2, \Gamma, B}{\Omega_1, \Omega_2, \Gamma, A \& B}\,\& \quad (\Omega_i \text{ is cut-only})$$

Table 5: Rules for deriving sequents in MALL*. Here P ranges over propositional variables, A, B range over MALL formulas, Γ, Δ range over MALL* sequents, and the Ω_i range over cut-only sequents (disjoint unions of cuts). Note that the &-rule may superimpose one or more cuts from its two hypotheses (the ones contained in Γ), or may leave all cut pairs separate (when putting all cuts in Ω_i).

THEOREM 1 *Two MALL$^-$ proofs translate to the same proof net if and only if they can be converted into each other by a series of rule commutations.*

We will obtain this result as a special case of Proposition 1.

5 Cut

Let MALL be MALL$^-$, as defined in Section 2, together with the rule

$$\frac{\Gamma, A \qquad A^\perp, \Delta}{\Gamma, \Delta}\,\mathsf{cut}$$

Table 6 lists the rule commutations for cut; the rule commutations for MALL not involving cut are exactly the same as in the cut-free case (Tables 2–4).

The translation of MALL proofs to proof nets [5] goes via a technically convenient variant MALL* of MALL in which cuts are retained in sequents. Extend MALL formulas to include ***cuts*** $A*A^\perp$ for any cut-free MALL formula A, where $*$ is the ***cut connective***. By definition $*$ is unordered, *i.e.*, $A*A^\perp = A^\perp*A$ (in contrast to MALL formulas, where connectives are ordered, *e.g.*, $A \otimes B \neq B \otimes A$ when $A \neq B$). Note that $*$ can only occur in outermost position.

As before, a ***sequent*** is a disjoint union of formulas (but now a formula may be a cut $A*A^\perp$). Sequents are derived in MALL* using the rules in Table 5. The system MALL* is an extension of MALL$^-$. The function taking a MALL$^-$ proof to a set of linkings on a MALL$^-$ sequent (defined in Section 3, page 3) extends in the obvious way to a function taking a MALL* proof Π to a set of linkings θ_Π on a MALL* sequent: a ***&-resolution*** R of Π is any result of deleting one branch above each &-rule of Π; by downwards tracking of formula leaves, the axiom rules of Π determine a linking λ_R; define $\theta_\Pi = \{\lambda_R : R \text{ is a \&-resolution of } \Pi\}$. Alternatively, the same function can be defined inductively, by means of a direct extension of the cut-free case in Table 1 [5].

A linking set on a sequent Γ is a ***proof net*** if it is the translation of a MALL* proof of Γ.

Every MALL* proof projects to a MALL proof by deleting all cuts, thereby turning each $*$ rule into a standard cut rule. Let θ be a set of linkings on a sequent. A MALL proof Π ***translates*** to θ, or is a ***sequentialisation*** of θ, denoted $\Pi \xrightarrow{s} \theta$, if Π is the projection of a MALL* proof translating to θ.

Restricted to the cut-free case, the sequentialisation relation $\xrightarrow{s}$ is a function taking a proof to a proof net, exactly the cut-free translation defined in Section 3. In the presence of cuts, more than one proof net may correspond to the same MALL proof. Examples can be found in [5].

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^{\perp},\Delta,B} \quad \dfrac{\Pi_3}{B^{\perp},\Sigma}}{A^{\perp},\Delta,\Sigma}\,\mathsf{cut}}{\Gamma,\Delta,\Sigma}\,\mathsf{cut} \qquad \overset{\mathsf{C}^{\mathsf{cut}}_{\mathsf{cut}}}{\longleftrightarrow} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^{\perp},\Delta,B}}{\Gamma,\Delta,B}\,\mathsf{cut} \qquad \dfrac{\Pi_3}{B^{\perp},\Sigma}}{\Gamma,\Delta,\Sigma}\,\mathsf{cut}$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A} \qquad \dfrac{\dfrac{\Pi_2}{A^{\perp},\Delta,B_1} \quad \dfrac{\Pi_3}{B_2,\Sigma}}{A^{\perp},\Delta,B_1 \otimes B_2,\Sigma}\,\otimes}{\Gamma, \Delta, B_1 \otimes B_2,\Sigma}\,\mathsf{cut} \qquad \underset{\mathsf{C}^{\mathsf{cut}}_{\otimes}}{\overset{\mathsf{C}^{\otimes}_{\mathsf{cut}}}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^{\perp},\Delta,B_1}}{\Gamma,\Delta,B_1}\,\mathsf{cut} \qquad \dfrac{\Pi_3}{B_2,\Sigma}}{\Gamma,\Delta,B_1 \otimes B_2,\Sigma}\,\otimes$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^{\perp},\Delta,B_i}}{A^{\perp},\Delta,B_1 \oplus B_2}\,\oplus_i}{\Gamma,\Delta,B_1 \oplus B_2}\,\mathsf{cut} \qquad \underset{\mathsf{C}^{\mathsf{cut}}_{\oplus}}{\overset{\mathsf{C}^{\oplus}_{\mathsf{cut}}}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^{\perp},\Delta,B_i}}{\Gamma,\Delta,B_i}\,\mathsf{cut}}{\Gamma,\Delta,B_1 \oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^{\perp},\Delta,B_1,B_2}}{A^{\perp},\Delta,B_1 \parr B_2}\,\parr}{\Gamma,\Delta,B_1 \parr B_2}\,\mathsf{cut} \qquad \underset{\mathsf{C}^{\mathsf{cut}}_{\parr}}{\overset{\mathsf{C}^{\parr}_{\mathsf{cut}}}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^{\perp},\Delta,B_1,B_2}}{\Gamma,\Delta,B_1,B_2}\,\mathsf{cut}}{\Gamma,\Delta,B_1 \parr B_2}\,\parr$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^{\perp},\Delta,B_1} \quad \dfrac{\Pi_3}{A^{\perp},\Delta,B_2}}{A^{\perp},\Delta,B_1 \& B_2}\,\&}{\Gamma,\Delta,B_1 \& B_2}\,\mathsf{cut} \qquad \underset{\mathsf{C}^{\mathsf{cut}}_{\&}}{\overset{\mathsf{C}^{\&}_{\mathsf{cut}}}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^{\perp},\Delta,B_1}}{\Gamma,\Delta,B_1}\,\mathsf{cut} \qquad \dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_3}{A^{\perp},\Delta,B_2}}{\Gamma,\Delta,B_2}\,\mathsf{cut}}{\Gamma,\Delta,B_1 \& B_2}\,\&$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B} \quad \dfrac{\Pi_3}{B^{\perp},\Sigma}}{\Delta,\Sigma}\,\mathsf{cut}}{\Gamma,\Delta,\Sigma}\,\mathsf{mix} \qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\mathsf{cut}}}{\overset{\mathsf{C}^{\mathsf{cut}}_{\mathsf{mix}}}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B}}{\Gamma,\Delta,B}\,\mathsf{mix} \qquad \dfrac{\Pi_3}{B^{\perp},\Sigma}}{\Gamma,\Delta,\Sigma}\,\mathsf{cut}$$

Table 6: MALL rule commutations involving cut. The second conversion has a symmetric variant in whose right-hand side the cut rule applies to the hypothesis contributing to the right argument of the tensor.

Let ***proof-net equivalence*** be the smallest equivalence relation on MALL proofs such that proofs that have a common translation are equivalent. Then our main result (Theorem 1) extends to MALL as follows.

THEOREM 2 *Two MALL proofs are proof-net equivalent if and only if they can be converted into each other by a series of rule commutations.*

The proof will be the subject of the following sections.

6 MALL* rule commutations

We will obtain Theorem 2 from a similar theorem for MALL*. To this end, we need to collect the rule commutations for MALL*. As in the cut-free case, these can be generated systematically from the general definition of rule commutation in the Appendix. Table 7 lists the rule commutations for $*$. The rule commutations for MALL* not involving $*$ or & are exactly the same as in the cut-free case (Tables 2–4), whereas the heterogeneous commutations involving & are obtained from the cut-free ones by the addition of Ω_1 and Ω_2, just as in the rule $\mathsf{C}^{\&}_{*}$. The commutation $\mathsf{C}^{\&}_{\&}$ is a bit more involved: it can be obtained from the one in Table 2 by the addition of $2^{2^2}-2=14$ variables Ω_{ijkl} with $i,j,k,l\in\{0,1\}$. Each cut occurring in the conclusion can be produced by one or more of the subproofs Π_1–Π_4. The variable Ω_{1001} captures those cuts that are produced by Π_1 as well as Π_4, but not by Π_2 or Π_3; in general the n^{th} index from the series i,j,k,l indicates whether or not the cuts in Ω_{ijkl} are produced by Π_n. A sequent occurring in the rule commutation is enriched with Ω_{ijkl} iff it occurs under Π_n for an n such that the n^{th} index from the series i,j,k,l is set to 1. The variable Ω_{0000} is not needed, as it would not occur in the conclusion, and the variable Ω_{1111} is superfluous, as it can be incorporated in Γ. Since the resulting rules do not fit on the page, below they are displayed using an abbreviation: Ω^n denotes the disjoint union of the sequents Ω_{ijkl} where the n^{th} index is set to 1. Likewise Ω^{mn} indicates the disjoint union of the sequents Ω_{ijkl} where either the m^{th} or the n^{th} index is set to one, i.e. the non-disjoint union of Ω^m and Ω^n.

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Omega^1,\Gamma,A_1,B_1}\quad\dfrac{\Pi_2}{\Omega^2,\Gamma,A_2,B_1}}{\Omega^{12},\Gamma,A_1\&A_2,B_1}\,\&\qquad\dfrac{\dfrac{\Pi_3}{\Omega^3,\Gamma,A_1,B_2}\quad\dfrac{\Pi_4}{\Omega^4,\Gamma,A_2,B_2}}{\Omega^{34},\Gamma,A_1\&A_2,B_2}\,\&}{\Omega^{1234},\Gamma,A_1\&A_2,B_1\&B_2}\,\&$$

$$\updownarrow\ \mathsf{C}^{\&}_{\&}$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Omega^1,\Gamma,A_1,B_1}\quad\dfrac{\Pi_3}{\Omega^3,\Gamma,A_1,B_2}}{\Omega^{13},\Gamma,A_1,B_1\&B_2}\,\&\qquad\dfrac{\dfrac{\Pi_2}{\Omega^2,\Gamma,A_2,B_1}\quad\dfrac{\Pi_4}{\Omega^4,\Gamma,A_2,B_2}}{\Omega^{24},\Gamma,A_2,B_1\&B_2}\,\&}{\Omega^{1234},\Gamma,A_1\&A_2,B_1\&B_2}\,\&$$

The rule commutations for MALL (cf. Section 5) are obtained from the ones of MALL* by omitting all cuts from sequents.

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^\perp,\Delta,B} \quad \dfrac{\Pi_3}{B^\perp,\Sigma}}{A^\perp,\Delta,B\ast B^\perp,\Sigma}\,\ast}{\Gamma,A\ast A^\perp,\Delta,B\ast B^\perp,\Sigma}\,\ast \qquad \overset{\mathsf{C}^\ast_\ast}{\longleftrightarrow} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^\perp,\Delta,B}}{\Gamma,A\ast A^\perp,\Delta,B}\,\ast \qquad \dfrac{\Pi_3}{B^\perp,\Sigma}}{\Gamma,A\ast A^\perp,\Delta,B\ast B^\perp,\Sigma}\,\ast$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A} \qquad \dfrac{\dfrac{\Pi_2}{A^\perp,\Delta,B_1} \quad \dfrac{\Pi_3}{B_2,\Sigma}}{A^\perp,\Delta,B_1\otimes B_2,\Sigma}\,\otimes}{\Gamma, A\ast A^\perp,\Delta,B_1\otimes B_2,\Sigma}\,\ast \qquad \underset{\mathsf{C}^\ast_\otimes}{\overset{\mathsf{C}^\otimes_\ast}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^\perp,\Delta,B_1}}{\Gamma,A\ast A^\perp,\Delta,B_1}\,\ast \qquad \dfrac{\Pi_3}{B_2,\Sigma}}{\Gamma,A\ast A^\perp,\Delta,B_1\otimes B_2,\Sigma}\,\otimes$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^\perp,\Delta,B_i}}{A^\perp,\Delta,B_1\oplus B_2}\,\oplus_i}{\Gamma,A\ast A^\perp,\Delta,B_1\oplus B_2}\,\ast \qquad \underset{\mathsf{C}^\ast_\oplus}{\overset{\mathsf{C}^\oplus_\ast}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^\perp,\Delta,B_i}}{\Gamma,A\ast A^\perp,\Delta,B_i}\,\ast}{\Gamma,A\ast A^\perp,\Delta,B_1\oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^\perp,\Delta,B_1,B_2}}{A^\perp,\Delta,B_1 \parr B_2}\,\parr}{\Gamma,A\ast A^\perp,\Delta,B_1\parr B_2}\,\ast \qquad \underset{\mathsf{C}^\ast_\parr}{\overset{\mathsf{C}^\parr_\ast}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^\perp,\Delta,B_1,B_2}}{\Gamma,A\ast A^\perp,\Delta,B_1,B_2}\,\ast}{\Gamma,A\ast A^\perp,\Delta,B_1\parr B_2}\,\parr$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma,A} \qquad \dfrac{\dfrac{\Pi_2}{A^\perp,\Omega_1,\Delta,B_1} \quad \dfrac{\Pi_3}{A^\perp,\Omega_2,\Delta,B_2}}{A^\perp,\Omega_1,\Omega_2,\Delta,B_1\&B_2}\,\&}{\Gamma,A\ast A^\perp,\Omega_1,\Omega_2,\Delta,B_1\&B_2}\,\ast \qquad \underset{\mathsf{C}^\ast_\&}{\overset{\mathsf{C}^\&_\ast}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_2}{A^\perp,\Omega_1,\Delta,B_1}}{\Gamma,A\ast A^\perp,\Omega_1,\Delta,B_1}\,\ast \qquad \dfrac{\dfrac{\Pi_1}{\Gamma,A} \quad \dfrac{\Pi_3}{A^\perp,\Omega_2,\Delta,B_2}}{\Gamma,A\ast A^\perp,\Omega_2,\Delta,B_2}\,\ast}{\Gamma,A\ast A^\perp,\Omega_1,\Omega_2,\Delta,B_1\&B_2}\,\&$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B} \quad \dfrac{\Pi_3}{B^\perp,\Sigma}}{\Delta,B\ast B^\perp,\Sigma}\,\ast}{\Gamma,\Delta,B\ast B^\perp,\Sigma}\,\mathsf{mix} \qquad \underset{\mathsf{C}^{\mathsf{mix}}_\ast}{\overset{\mathsf{C}^\ast_{\mathsf{mix}}}{\rightleftarrows}} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B}}{\Gamma,\Delta,B}\,\mathsf{mix} \qquad \dfrac{\Pi_3}{B^\perp,\Sigma}}{\Gamma,\Delta,B\ast B^\perp,\Sigma}\,\ast$$

Table 7: $\mathrm{MALL}^\ast$ rule commutations involving cut. The second conversion also has a symmetric variant in whose right-hand side the $\ast$ cut rule applies to the hypothesis contributing to the right argument of the tensor. Since the arguments of $\ast$ are unordered, we do not need symmetric variants obtained by exchanging the hypotheses of the $\ast$ cut rule.

$\beta\backslash\alpha$	$*$	mix	$\otimes$	$\oplus_1$	$\oplus_2$	$⅋$	$\&$
$*$	✓	✓	✓	✓	✓	✓	✓
mix	✓	✓	✓	✓	✓	✓	✓
$\otimes$	✓	✓	✓	✓	✓	✓	✓
$\oplus_1$	✓	✓	✓	✓	✓	✓	✓
$\oplus_2$	✓	✓	✓	✓	✓	✓	✓
$⅋$	○	○	○	✓	✓	✓	✓
$\&$	●	●	●	○	○	○	○

Table 8: Rule commutations. The check marks flag pairs $\frac{\alpha}{\beta}$ where a (lower) β-rule always commutes over an α-rule. The marks $\circ$ indicate situations where β-rules commute over α-rules only under certain syntactic restrictions, which can be found by studying the results of commuting α- over β-rules. The $\bullet$ denotes commutation under certain syntactic restrictions.

7 Proof of the MALL* rule commutation theorem

We say that a β-rule ***commutes over*** an α-rule if there is a valid MALL* rule commutation where a proof fragment in which the β-rule occurs immediately below one or more α-rules is replaced by a proof fragment in which this order is reversed. Using either the definition of rule commutation from the Appendix or the enumeration of Tables 2, 3, 4 and 7, enriched with Ωs as discussed above, it is not hard to check that this happens if and only if (i) no formula occurrence generated by one of the α-rules tracks to a subformula of a formula generated by the β-rule, and (ii) one of the following cases applies (*cf.* Table 8):

- $\beta \in \{\otimes, \oplus_1, \oplus_2, \mathsf{mix}, *\}$;
- $\beta = ⅋$ and $\alpha \neq \otimes, \mathsf{mix}, *$;
- $\beta = ⅋$, $\alpha = \otimes$, mix or $*$, and both arguments of the formula generated by the $⅋$-rule occur in the same hypothesis of the α-rule;
- $\beta = \&$, $\alpha \neq \otimes, \mathsf{mix}, *$, and the formula occurrences generated by the two α-rules track to the same same formula occurrence of the β-rule.
- $\beta = \&$, $\alpha = \otimes$, mix or $*$, the β-rule generates a formula $B_1 \& B_2$, and the hypotheses of the two α-rules that do not contain B_1 or B_2 are the same, and have identical subproofs.

This, in turn, yields exactly the rule commutations of Tables 2–7, enriched with Ωs as discussed in Section 6.

The following result, a generalisation of Theorem 1, is a crucial step towards proving Theorem 2.

PROPOSITION 1 *Two MALL* proofs translate to the same proof net if and only if they can be converted into each other by a series of rule commutations.*

Proof. If Π' can be obtained from Π by commuting rule occurrences, then Π and Π' translate to the same linking set: taking a &-resolution on either side of a commutation induces essentially the same &-resolutions (or deletions) of the subproofs Π_i. For example, in the last commutation in Table 3, if we choose *right* for the distinguished &-rule, we delete subproof Π_2 from both sides, and induce corresponding &-resolutions of Π_1 and Π_3. The converse is proved below. $\Box$

Given a set of linkings Λ on a MALL* sequent Γ, let $\Gamma{\restriction}\Lambda$ be obtained from the forest Γ by deleting all vertices that are not below a leaf of Γ that occurs in Λ (*i.e.*, in a link in a linking of Λ). A &-vertex w in Γ is ***toggled*** by Λ if both arguments of w occur in $\Gamma{\restriction}\Lambda$. A link a ***depends*** on w in Λ if there exist $\lambda, \lambda' \in \Lambda$ such that $a \in \lambda$, $a \notin \lambda'$, and w is the only & toggled by $\{\lambda, \lambda'\}$. Construct the ***graph*** $\mathscr{G}_\Lambda$ [5] from $\Gamma{\restriction}\Lambda$ by adding the edges of $\bigcup_{\lambda\in\Lambda}\lambda$, as well as all ***jump*** edges from leaves ℓ and ℓ' to any &-vertex on which the link $\{\ell, \ell'\}$ depends in Λ. Below we will need the following properties of a proof net θ on a MALL* sequent Γ, established in [5].

Any set of two linkings in θ toggles a &-vertex of Γ. (1)

Each root vertex (formula occurrence) in Γ occurs in $\mathscr{G}_\theta$. (2)

For every $\lambda \in \theta$ and each root &-vertex w in Γ there is a $\lambda' \in \theta$ such that w is the only & toggled by $\{\lambda, \lambda'\}$. (3)

A formula occurrence $A = A_1 \alpha A_2$ in a MALL* sequent Γ ***separates*** a proof net θ on Γ if (i) $\alpha \in \{⅋, \&\}$, (ii) $\alpha = \oplus$ and one of the A_i does not occur in $\mathscr{G}_\theta$, or (iii) $\alpha \in \{\otimes, *\}$ and $\mathscr{G}_\theta$ has no cycle through α.

LEMMA 1 *If the last rule of a MALL* proof generates A, then A separates the associated proof net.*

Proof. The only non-trivial cases are $A = A_1 \alpha A_2$ for $\alpha \in \{\otimes, *\}$. Let Γ_1 and Γ_2 be the hypotheses of the last rule ρ of the proof Π, let Π_i be the branch of Π above ρ proving Γ_i, let θ be the proof net of Π and θ_i that of Π_i. $\mathscr{G}_\theta$ could have a cycle through α only when in θ a link a in Γ_1 depends on a &-vertex w in Γ_2 (or vice versa). In that case there exist $\lambda, \lambda' \in \theta$ such that $a \in \lambda$, $a \notin \lambda'$, and w is the only & toggled by $\{\lambda, \lambda'\}$. Hence there must be λ_1, λ_1' in θ_1 and λ_2, λ_2' in θ_2 such that $a \in \lambda_1$, $a \notin \lambda_1'$ and w is the only & toggled by $\{\lambda_1 \cup \lambda_2, \lambda_1' \cup \lambda_2'\}$. However, by (1) there must be another &-vertex of Λ that is toggled by $\{\lambda_1 \cup \lambda_2, \lambda_1' \cup \lambda_2'\}$, namely one occurring in Γ_1 that is toggled by $\{\lambda_1, \lambda_1'\}$. □

LEMMA 2 *If a formula occurrence $A = A_1 \alpha A_2$ in a MALL* sequent Γ, A separates a proof net θ of Γ, A for which $\mathscr{G}_\theta$ is connected, then there is at most one instance σ of an α-rule that could generate A in the last step of a proof Π of Γ, A with proof net θ.*

Proof.

- Case $\alpha = ⅋$: the hypothesis of σ must be Γ, A_1, A_2.
- Case $\alpha = \&$: the hypotheses of σ must be Γ, A_1 and Γ, A_2.
- Case $\alpha = \oplus$: exactly one of the A_i, say A_d, is in $\mathscr{G}_\theta$ (2). Hence the hypothesis of σ must be Γ, A_d.
- Case $\alpha \in \{\otimes, *\}$: let Γ, A_1, A_2 be the sequent resulting from deleting the connective α in A from Γ, A. Since A separates θ and $\mathscr{G}_\theta$ is connected, the restriction of $\mathscr{G}_\theta$ to Γ, A_1, A_2 has two disconnected components, one on a sequent Γ_1, A_1 and the other on a sequent Γ_2, A_2, where $\Gamma_1 \cup \Gamma_2 = \Gamma$. Using (2), the hypotheses of σ must be Γ_1, A_1 and Γ_2, A_2. □

In each case the proof nets on the hypotheses of σ, induced by the branches of Π that prove these hypotheses, are completely determined by θ.

For Π a MALL* proof, let $\mathscr{G}_\Pi$ abbreviate $\mathscr{G}_{\theta_\Pi}$. We shall prove the following four lemmas by simultaneous structural induction.

LEMMA 3 *Let Π be a proof of a MALL* sequent $\Delta, A_1 \alpha A_2, \Sigma$ such that in $\mathscr{G}_\Pi$ any path between (vertices in) Δ, A_1 and A_2, Σ passes through the indicated occurrence of $\alpha \in \{\otimes, *\}$. Then Π can, by means of rule commutations, be converted into a proof Π'' whose last step is the α-rule with hypotheses Δ, A_1 and A_2, Σ.*

LEMMA 4 *Let Π be a proof of a MALL* sequent Γ whose proof net θ is separated by a formula occurrence A in Γ. Then, by means of a series of rule commutations, Π can be converted into a proof Π'' of Γ that generates A in its last step.*

LEMMA 5 *Let Π be a proof of a MALL* sequent Δ,Σ for nonempty sequents Δ and Σ, such that in $\mathscr{G}_\Pi$ there is no path between (vertices in) Δ and Σ. Then Π can, by means of rule commutations, be converted into a proof Π'' whose last step is the* mix-*rule with hypotheses Δ and Σ.*

LEMMA 6 *If two proofs Π and Π' of a MALL* sequent Γ translate to the same proof net on Γ, then Π can be converted into Π' by a series of rule commutations.*

Lemma 6 is the converse direction of Proposition 1 that must be established.

Proof. We prove Lemmas 3–6 by a simultaneous structural induction on Π (or equivalently, on Γ).

Induction base (applies to Lemma 6 only). The induction base is trivial, as a MALL sequent that can be proven in one step has at most one proof, a single application of ax.

Induction step for Lemma 3.

- First consider the case that the last step ρ of Π is an application of mix, say with hypotheses Γ_c and $\Gamma_d, A_1 \alpha A_2$.

  Let Π_d be the branch of Π above ρ proving $\Gamma_d, A_1 \alpha A_2$. Let $\Delta_d = \Delta \cap \Gamma_d$ and $\Sigma_d = \Sigma \cap \Gamma_d$. Since $\mathscr{G}_{\Pi_d}$ is a subgraph of $\mathscr{G}_\Pi$, any path in $\mathscr{G}_{\Pi_d}$ between (vertices in) Δ_d, A_1 and A_2, Σ_d passes through the indicated occurrence of α. Hence, by induction, Π_d can, by means of rule commutations, be converted into a proof Π'_d whose last step is the α-rule with hypotheses Δ_d, A_1 and A_2, Σ_d.

  Let Π_c be the branch of Π above ρ proving Γ_c. Let $\Delta_c = \Delta \cap \Gamma_c$ and $\Sigma_c = \Sigma \cap \Gamma_c$. Since $\mathscr{G}_{\Pi_c}$ is a subgraph of $\mathscr{G}_\Pi$, there is no path in $\mathscr{G}_{\Pi_c}$ between (vertices in) Δ_c and Σ_c. If Δ_c or Σ_c is empty, let $\Pi'_c = \Pi_c$. Otherwise, by induction, using Lemma 5, Π_c can, by means of rule commutations, be converted into a proof Π'_c whose last step is the mix-rule with hypotheses Δ_c and Σ_c.

  Let Π' be the proof obtained from Π by replacing Π_d with Π'_d and Π_c with Π'_c. Let Π'' be the proof with the same 3 or 4 subproofs yielding Δ_c, Σ_c, Δ_d, A_1 and A_2, Σ_d that first combines Δ_c with Δ_d, A_1 into Δ, A_1 using mix (provided Δ_c is nonempty), and likewise combines Σ_c with A_2, Σ_d into A_2, Σ using mix (provided Σ_c is nonempty), and then applies α to yield $\Delta, A_1 \alpha A_2, \Sigma$. By means of a few simple rule commutations, Π' can be converted into Π''.

- Next consider the case that the last step ρ of Π is an application of α generating the same formula $A_1 \alpha A_2$. Let the hypotheses of ρ be Γ_i, A_i for $i = 1,2$.

  Let Π_i be the branch of Π above ρ proving Γ_i, A_i. Let $\Delta_i = \Delta \cap \Gamma_i$ and $\Sigma_i = \Sigma \cap \Gamma_i$. Since $\mathscr{G}_{\Pi_1}$ is a subgraph of $\mathscr{G}_\Pi$, there is no path in $\mathscr{G}_{\Pi_1}$ between (vertices in) Δ_1, A_1 and Σ_1. In case Σ_1 is empty, let $\Pi'_1 = \Pi_1$. Otherwise, by induction, using Lemma 5, Π_1 can, by means of rule commutations, be converted into a proof Π'_1 whose last step is the mix-rule with hypotheses Δ_1, A_1 and Σ_1.

  In case Δ_2 is empty, let $\Pi'_2 = \Pi_2$. Otherwise, by means of rule commutations, Π_2 can be converted into a proof Π'_2 whose last step is the mix-rule with hypotheses Δ_2 and A_2, Σ_2.

  Let Π' be the proof obtained from Π by replacing Π_i with Π'_i for $i \in \{1,2\}$. Let Π'' be the proof with the same 2, 3 or 4 subproofs yielding Δ_1, A_1, Σ_1, Δ_2 and A_2, Σ_2 that first combines Δ_1, A_1 with Δ_2 into Δ, A_1 using mix (provided Δ_2 is nonempty), and likewise combines Σ_1 with A_2, Σ_2 into A_2, Σ using mix (provided Σ_1 is nonempty), and then applies α to yield $\Delta, A_1 \alpha A_2, \Sigma$. By means of a few simple rule commutations, Π' can be converted into Π''.

In the remaining cases let the last step of Π be a a β-rule ρ generating the formula $B = B_1\beta B_2 \neq A = A_1\alpha A_2$. We treat the case that B occurs in Σ; the other case follows by symmetry. Let $\Sigma = \Sigma', B_1\beta B_2$.

- Let $\beta = \oplus$. Let Π_d be the part of Π above ρ, proving the hypothesis Δ, A, Σ', B_d of ρ (where d is 1 or 2). Since $\mathscr{G}_{\Pi_d}$ is a subgraph of $\mathscr{G}_{\Pi}$, any path in $\mathscr{G}_{\Pi_d}$ between (vertices in) Δ, A_1 and A_2, Σ', B_d passes through the indicated occurrence of α. Thus, by induction, by a series of rule commutations Π_d can be be converted into a proof Π'_d of $\Delta, A_1\alpha A_2, \Sigma', B_d$ whose last step is the α-rule with hypotheses Δ, A_1 and A_2, Σ', B_d. Let Π' be the proof obtained from Π by replacing Π_d by Π'_d. In Π', ρ commutes over the α-rule generating A, thereby yielding the required proof Π''.
- Let $\beta \in \{\otimes, *\}$. Let Π_1 and Π_2 be the branches of Π above ρ proving the hypotheses $\Delta_1, A, \Sigma_1, B_1$ and Δ_2, Σ_2, B_2 of ρ, respectively. Here $\Delta = \Delta_1, \Delta_2$ and $\Sigma' = \Sigma_1, \Sigma_2$. We assume that A sides with B_1; the other case proceeds symmetrically. Since $\mathscr{G}_{\Pi_1}$ is a subgraph of $\mathscr{G}_{\Pi}$, any path in $\mathscr{G}_{\Pi_1}$ between (vertices in) Δ_1, A_1 and A_2, Σ_1, B_1 passes through the indicated occurrence of α. Thus, by induction, by a series of rule commutations Π_1 can be be converted into a proof Π'_1 of $\Delta_1, A_1\alpha A_2, \Sigma_1, B_1$ whose last step is the α-rule with hypotheses Δ_1, A_1 and A_2, Σ_1, B_1.

  In case Δ_2 is empty, let $\Pi'_2 = \Pi_2$. Otherwise, by induction, using Lemma 5, Π_2 can be be converted into a proof Π'_2 of Δ_2, Σ_2, B_2 whose last step is the mix-rule with hypotheses Δ_2 and Σ_2, B_2.

  Let Π' be the proof obtained from Π by replacing Π_i by Π'_i, for $i \in \{1,2\}$. Let Π'' be the proof with the same 3 or 4 subproofs yielding Δ_1, A_1, A_2, Σ_1, B_1, Δ_2 and Σ_2, B_2 that first combines Δ_2 with Δ_1, A_1 into Δ, A_1 using mix (provided Δ_2 is nonempty), and likewise combines Σ_2, B_2 with A_2, Σ_1, B_1 into A_2, Σ', B using β, and then applies α to yield Δ, A, Σ', B. By means of a few simple rule commutations, Π' can be converted into Π''.
- Let $\beta = \parr$. Let Π_ρ be the part of Π above ρ. Then Π_ρ proves the hypothesis $\Delta, A, \Sigma', B_1, B_2$ of ρ. Since $\mathscr{G}_{\Pi_\rho}$ is a subgraph of $\mathscr{G}_{\Pi}$, in $\mathscr{G}_{\Pi_\rho}$ any path between (vertices in) Δ, A_1 and A_2, Σ', B_1, B_2 passes through the indicated occurrence of α. Hence, by induction, using by Lemma 3, Π_ρ can, by means of rule commutations, be converted into a proof Π'_ρ whose last step is the α-rule with hypotheses Δ, A_1 and A_2, Σ', B_1, B_2. Let Π' be the proof obtained from Π by replacing Π_ρ by Π'_ρ. In Π' the $\parr$-rule ρ commutes over the α-rule generating A, thereby yielding the required proof Π''.
- Let $\beta = \&$. The rule ρ has hypotheses hypotheses $\Omega_1^\Delta, \Delta', A, \Omega_1^\Sigma, \Sigma'', B_1$ and $\Omega_2^\Delta, \Delta', A, \Omega_2^\Sigma, \Sigma'', B_2$

$$\dfrac{\begin{array}{c}\vdots\,\Pi_1\\ \Omega_1^\Delta, \Delta', A_1\alpha A_2, \Omega_1^\Sigma, \Sigma'', B_1\end{array} \qquad \begin{array}{c}\vdots\,\Pi_2\\ \Omega_2^\Delta, \Delta', A_1\alpha A_2, \Omega_2^\Sigma, \Sigma'', B_2\end{array}}{\Omega_1^\Delta, \Omega_2^\Delta, \Delta', A_1\alpha A_2, \Omega_1^\Sigma, \Omega_2^\Sigma, \Sigma'', B_1\& B_2}\,\&(\rho)$$

  with $\Delta = \Omega_1^\Delta, \Omega_2^\Delta, \Delta'$ and $\Sigma' = \Omega_1^\Sigma, \Omega_2^\Sigma, \Sigma''$. We claim that Ω_1^Δ, and by symmetry also Ω_2^Δ, is empty. For if not, let ℓ be a leaf in Ω_1^Δ that occurs in a link a in a linking v of $\mathscr{G}_{\Pi_1}$—such a leaf exists by (2). Then, using Table 1, v also occurs in $\mathscr{G}_{\Pi}$. Using (3), let $v' \in \theta_\Pi$ be such that β is the only $\&$ toggled by $\{v, v'\}$. Again using Table 1, v' must occur in $\mathscr{G}_{\Pi_2}$. Since ℓ does not occur in $\mathscr{G}_{\Pi_2}$, a cannot occur in v', and thus depends on β. Hence in $\mathscr{G}_\Pi$ there is a jump edge from ℓ to β. This contradicts the assumption that in $\mathscr{G}_\theta$ any path between (vertices in) Δ, A_1 and $A_2, \Sigma', B_1 \parr B_2$ passes through the indicated occurrence of α.

  Let Π_i be the branch of Π above ρ proving $\Delta', A, \Omega_i^\Sigma, \Sigma'', B_i$. Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_{\Pi}$, in $\mathscr{G}_{\Pi_i}$ any path between (vertices in) Δ', A_1 and $A_2, \Omega_i^\Sigma, \Sigma'', B_i$ passes through the indicated occurrence of α. Hence, by induction, using Lemma 3, Π_i can, by means of rule commutations, be converted into a proof Π'_i whose last step is the α-rule with hypotheses Δ', A_1 and $A_2, \Omega_i^\Sigma, \Sigma'', B_i$.

Thus the left hypotheses of Π_1' and Π_2' are both Δ',A_1, and we claim that the proof nets on them induced by the subproofs Π_{11}' and Π_{21}' of Π leading up to these hypotheses must be the same.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\Pi_{11}'\\ \Delta',A_1\end{array} \qquad A_2,\Omega_1^{\Sigma},\Sigma'',B_1}{\Delta',A_1\alpha A_2,\Omega_1^{\Sigma},\Sigma'',B_1}\,\alpha \qquad \dfrac{\begin{array}{c}\vdots\,\Pi_{21}'\\ \Delta',A_1\end{array} \qquad A_2,\Omega_2^{\Sigma},\Sigma'',B_2}{\Delta',A_1\alpha A_2,\Omega_2^{\Sigma},\Sigma'',B_2}\,\alpha}{\Delta',A_1\alpha A_2,\Omega_1^{\Sigma},\Omega_2^{\Sigma},\Sigma'',B_1\& B_2}\,\&(\rho)$$

For if not, let λ be a linking in the proof net of Π_{11}' but not in the proof net of Π_{21}'. (The symmetric case goes likewise.) Then, using Table 1, for some linking μ on $A_2,\Omega_1^{\Sigma},\Sigma'',B_1$, the linking $\nu := \lambda \cup \mu$ must be in the proof net θ of Π. Using (3), let $\nu' \in \theta$ be such that β is the only & toggled by $\{\nu,\nu'\}$. Again using Table 1, $\nu' = \lambda' \cup \mu'$ for some linking λ' in the proof net of Π_{21}'. Since there must be a link $a = \{\ell,\ell'\}$ such that $a \in \lambda$ but $a \notin \lambda'$ (or vice versa), in $\mathscr{G}_\theta$ there is a jump edge from ℓ to β. This contradicts the assumption that in $\mathscr{G}_\theta$ any path between (vertices in) Δ',A_1 and $A_2,\Sigma',B_1 ⅋ B_2$ passes through the indicated occurrence of α.

Therefore, by induction, using Lemma 6, Π_{11}' can be converted into Π_{21}' by a series of rule commutations. Let Π_2'' be obtained from Π_2' by replacing its subproof Π_{21}' by Π_{11}', and let Π' be the proof obtained from Π by replacing Π_1 by Π_1' and Π_2 by Π_2''. In Π', the α-rules generating A commute with the &-rule ρ, thereby yielding the required proof Π''. ■

Induction step for Lemma 4. Suppose that Π does not generate $A = A_1\alpha A_2$ in its last step. The case $\alpha \in \{\otimes,*\}$ is implied by Lemma 3. Therefore we assume here that $\alpha \in \{\oplus,⅋,\&\}$.

- First consider the case that the last step of Π is the application of a mix-rule ρ. Then $\Gamma = \Delta,A$ and A occurs in a hypothesis Δ_d,A of ρ (where $\Delta_d \subseteq \Delta$). Let Π_d be the branch of Π above ρ proving Δ_d,A. Its proof net is separated by A in Δ_d, for otherwise the proof net θ of Π would not be separated by A in Γ. Thus, by induction, by a series of rule commutations Π_d can be be converted into a proof Π_d' of Δ_d,A that generates A in its last step. Let Π' be the proof of Γ obtained by replacing Π_d by Π_d' in Π. In Π', ρ commutes over the α-rule generating A, thereby yielding the required proof Π''.

In the remaining cases let the last step of Π be the application of a β-rule ρ, generating the formula $B_1\beta B_2$. Thus $\Gamma = \Delta,A,B_1\beta B_2$.

- Let $\beta \in \{\otimes,\oplus,*\}$. Then A occurs in a hypothesis Δ_d,A,B_d of ρ (where d is 1 or 2, and $\Delta_d = \Delta$ in the case $\beta = \oplus$). Let Π_d be the branch of Π above ρ proving Δ_d,A,B_d. Its proof net is separated by A in Δ_d,A,B_d, for otherwise the proof net θ of Π would not be separated by A in Γ. Thus, by induction, by a series of rule commutations Π_d can be be converted into a proof Π_d' of Δ_d,A,B_d that generates A in its last step. Let Π' be the proof of Γ obtained by replacing Π_d by Π_d' in Π. In Π', ρ commutes over the α-rule generating A, thereby yielding the required proof Π''.
- Let $\beta = ⅋$. Let Π_ρ be the part of Π above ρ. Then Π_ρ proves the hypothesis Δ,A,B_1,B_2 of ρ, and its proof net is separated by A, for otherwise θ would not be separated by A. Thus, by induction, by a series of rule commutations Π_ρ can be be converted into a proof Π_ρ' of Δ,A,B_1,B_2 that generates A in its last step. As above, a rule commutation completes the argument.
- Let $\beta = \&$. Then ρ has hypotheses Ω_1,Δ',A,B_1 and Ω_2,Δ',A,B_2 with $\Delta = \Omega_1,\Omega_2,\Delta'$. Let Π_i be the branch of Π above ρ proving Ω_i,Δ',A,B_i. The proof nets of Π_1 and Π_2 are separated by A in Δ,A,B_i in exactly the same way, *i.e.*, in case $\alpha = \oplus$ choosing the same argument A_d, for otherwise

θ would not be separated by A. By induction, by a series of rule commutations the Π_i can be converted into proofs Π_i' of $\Omega_i, \Delta', A, B_i$ that generate A in their last steps. Let Π' be the proof of Γ obtained by replacing Π_i by Π_i' in Π, for $i = 1,2$. In Π', the &-rule ρ commutes over the α-rules generating A, thereby yielding the required proof Π''. ■

Induction step for Lemma 5.

- First consider the case that the last step ρ of Π is an application of mix, say with hypotheses Γ_1 and Γ_2. Let Π_i be the branch of Π above ρ, proving Γ_i (for $i = 1,2$). Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_i}$ there is no path between (vertices in) $\Delta_i := \Delta \cap \Gamma_i$ and $\Sigma_i := \Sigma \cap \Gamma_i$. In case Δ_i or Σ_i is empty, we let $\Pi_i' = \Pi_i$. Otherwise, by induction Π_i can, by means of rule commutations, be converted into a proof Π_i' whose last step is a mix-rule with hypotheses Δ_i and Σ_i. Let Π' be the proof obtained from Π by replacing Π_i with Π_i' for $i = 1,2$. In Π', ρ commutes over the 0, 1 or 2 mix-rules introduced immediately above it, thereby yielding the required proof Π''.

In the remaining cases let the last step of Π be the application of a β-rule ρ, generating the formula $B = B_1 \beta B_2$. We treat the case that B occurs in Σ; the other case follows by symmetry. Let $\Sigma = \Sigma', B_1 \beta B_2$.

- Let $\beta \in \{\otimes, *\}$. The hypotheses of this rule are Δ_i, Σ_i, B_i, for $i \in \{1,2\}$, where $\Delta = \Delta_1, \Delta_2$ and $\Sigma' = \Sigma_1, \Sigma_2$. Let Π_i be the branch of Π proving Δ_i, Σ_i, B_i. Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_i}$ there is no path between (vertices in) Δ_i and Σ_i, B_i. In case Δ_i is empty, we let $\Pi_i' = \Pi_i$. Otherwise, by induction Π_i can, by means of rule commutations, be converted into a proof Π_i' whose last step is a mix-rule with hypotheses Δ_i and Σ_i, B_i. Let Π' be the proof obtained from Π by replacing Π_i with Π_i' for $i = 1,2$. In Π', ρ commutes over the 1 or 2 mix-rules introduced immediately above it (possibly using $\mathsf{C}_\beta^{\mathsf{mix}}$ twice and $\mathsf{C}_{\mathsf{mix}}^{\mathsf{mix}}$ once), thereby yielding the required proof Π''.
- Let $\beta = \oplus$. The hypothesis of this rule is Δ, Σ', B_d, where d is 1 or 2. Let Π_d be the subproof of Π proving the latter sequent. Since $\mathscr{G}_{\Pi_d}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_d}$ there is no path between (vertices in) Δ and Σ', B_d. By induction Π_d can, by means of rule commutations, be converted into a proof Π_d' whose last step is an application of the mix-rule with hypotheses Δ and Σ', B_d. Let Π' be the proof obtained from Π by replacing Π_d with Π_d'. In Π', ρ commutes over the mix-rule introduced immediately above it, thereby yielding the required proof Π''.
- Let $\beta = \invamp$. The hypothesis of this rule is $\Delta, \Sigma', B_1, B_2$. Let Π_ρ be the subproof of Π proving the latter sequent. Since $\mathscr{G}_{\Pi_\rho}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_\rho}$ there is no path between Δ and Σ', B_1, B_2. By induction Π_ρ can, by means of rule commutations, be converted into a proof Π_ρ' whose last step is a mix-rule with hypotheses Δ and Σ', B_1, B_2. Let Π' be the proof obtained from Π by replacing Π_ρ with Π_ρ'. In Π', ρ commutes over the mix-rule introduced immediately above it, thereby yielding the required proof Π''.
- Let $\beta = \&$. The hypotheses of this rule are $\Omega_i^\Delta, \Delta', \Omega_i^\Sigma, \Sigma'', B_i$ for $i \in \{1,2\}$ with $\Delta = \Omega_1^\Delta, \Omega_2^\Delta, \Delta'$ and $\Sigma' = \Omega_1^\Sigma, \Omega_2^\Sigma, \Sigma''$. As in the induction step for Lemma 3, it follows that Ω_1 and Ω_2 are empty. Let Π_i be the branch of Π proving $\Delta', \Omega_i^\Sigma, \Sigma'', B_i$. Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_i}$ there is no path between (vertices in) Δ' and $\Omega_i^\Sigma, \Sigma'', B_i$. By induction Π_i can, by means of rule commutations, be converted into a proof Π_i' whose last step is a mix-rule with hypotheses Δ' and $\Omega_i^\Sigma, \Sigma'', B_i$. So the left hypotheses of Π_1' and Π_2' are both Δ', and we claim that the proof nets on them induced by the subproofs Π_{11}' and Π_{21}' of Π leading up to these hypotheses must be the same. The argument goes just as in the induction step for Lemma 3.

  Therefore, by induction, using Lemma 6, Π_{11}' can be converted into Π_{21}' by a series of rule commutations. Let Π_2'' be obtained from Π_2' by replacing its subproof Π_{21}' by Π_{11}', and let Π' be the proof obtained from Π by replacing Π_1 by Π'_1 and Π_2 by Π''_2. In Π', the mix-rules generating A commute with the &-rule ρ, thereby yielding the required proof Π''. ■

Induction step for Lemma 6. For the induction step, suppose Π and Π' are two proofs of a MALL* sequent Γ that have the same proof net θ.

First assume that $\mathscr{G}_\theta$ is connected. In that case the last steps of Π and Π' cannot be mix. Let A be the formula occurrence in Γ that is generated by the last step of Π'. By Lemma 1, A separates θ. Hence, using Lemma 4, by means of a series of rule commutations, Π can be converted into a proof Π'' of Γ that generates A in its last step. By Lemma 2, the last step σ of Π' is the same as the last step of Π''. Thus each hypothesis Γ_d of σ is proven by a subproof Π'_d of Π', and by a subproof Π''_d of Π''. As Π'_d and Π''_d have the same proof net, by induction they can be converted into each other by means of a series of rule commutations. It follows that also Π and Π' can be converted into each other by means of a series of rule commutations.

Next assume that $\mathscr{G}_\theta$ is disconnected; let $\Gamma = \Gamma_1, \Gamma_2$ with the Γ_i nonempty sequents, such that in $\mathscr{G}_\theta$ there is no path between (vertices in) Γ_1 and Γ_2. Using Lemma 5, Π can, by means of rule commutations, be converted into a proof Π_{mix} whose last step is the mix-rule with hypotheses Γ_i. Let Π_i be the branch of Π_{mix} proving Γ_i. Its proof net is simply the restriction of (the linkings in) θ to Γ_i. Likewise, Π' can, by means of rule commutations, be converted into a proof Π'_{mix} whose last step is the mix-rule with hypotheses Γ_i. Let Π'_i be the branch of Π_{mix} proving Γ_i. Since Π_i and Π'_i have the same proof net, by induction one can be converted into the other by a series of rule commutations. Consequently, Π can be converted into Π'. □

8 Proof of the MALL rule commutation theorem

We use Proposition 1 to derive Theorem 2. We shall need two lemmas connecting MALL* rule commutations with MALL rule commutations.

LEMMA 7 *If two MALL* proofs differ by a rule commutation, so do their projections to MALL proofs.*

Proof. This follows immediately from inspecting the rule commutations. □

In the other direction, one might expect that for each pair (Π_l, Π_r) of commuting MALL proofs, and for each MALL* proof Π^*_l that projects to Π_l, there exists a MALL* proof Π^*_r projecting to Π_r and commuting with Π^*_l. This is not the case, however. A counterexample is provided by taking Π^*_r to be

$$\dfrac{\dfrac{\dfrac{\Pi^*_{1a}}{\Omega_1,\Gamma,A_1} \quad \dfrac{\Pi^*_2}{A_2,\Delta,B_1}}{\Omega_1,\Gamma,A_1\otimes A_2,\Delta,B_1}\otimes \qquad \dfrac{\dfrac{\Pi^*_{1b}}{\Omega_2,\Gamma,A_1} \quad \dfrac{\Pi^*_3}{A_2,\Delta,B_2}}{\Omega_2,\Gamma,A_1\otimes A_2,\Delta,B_2}\otimes}{\Omega_1,\Omega_2,\Gamma,A_1\otimes A_2,\Delta,B_1\& B_2}\&$$

with Ω_1 or Ω_2 nonempty, and (Π_l, Π_r) the last rule commutation of Table 3. For (Π_l, Π_r) to be a valid rule commutation, the subproofs Π^*_{1a} and Π^*_{1b} must project to identical MALL proofs, even though they derive different MALL* sequents. This can be achieved by inserting a &-rule in each of these subproofs, where one superimposes two cuts, while the other keeps them disjoint. However, a weaker property *does* hold:

LEMMA 8 *For each MALL rule commutation* (Π_l, Π_r) *there is a MALL** *rule commutation* (Π_l^*, Π_r^*) *that projects to* (Π_l, Π_r).

Proof. Orient the pair (Π_l, Π_r) so that we avoid Π_l being an $\alpha\beta$-proof fragment (see the Appendix) with $\beta = \&$ and $\alpha \in \{\otimes, \mathsf{mix}, \mathsf{cut}\}$. Take Π_l^* to be an arbitrary MALL* proof projecting to Π_l. Going through the rule commutations of Tables 2–6, one can check that in each case it is straightforward to find the required proof Π_r^*. $\square$

COROLLARY 1

(a) *If two MALL proofs* Π_l *and* Π_r *translate to a common proof net then they can be converted into each other by rule commutations.*

(b) *If two MALL proofs* Π_l *and* Π_r *differ by a rule commutation then they have a common proof net.*

Proof. Suppose Π_l and Π_r translate to a common proof net θ. Then Π_l and Π_r must be projections of MALL* proofs Π_l^* and Π_r^* that translate to θ. By Proposition 1 Π_l^* and Π_r^* can be converted into each other by a series of rule commutations. By Lemma 7 the same holds for Π_l and Π_r.

Suppose Π_l and Π_r differ by a rule commutation. By Lemma 8 there are MALL* proofs Π_l^* and Π_r^* that differ by a rule commutation and project to Π_l and Π_r. By Proposition 1 Π_l^* and Π_r^* translate to the same proof net θ. Hence θ is a common proof net of Π_l and Π_r. $\square$

Finally, Theorem 2 is a direct consequence of Corollary 1. $\square$

9 Alternative treatments of cut

One of the innovations of the proof nets from [5] over the monomial ones from [2] is that the translation from cut-free proofs to proof nets is a function. This property does not extend to proofs with cut. In [5, Section 5.3.4] three alternative translations are discussed of which two *are* functions. One of these fails to identify proof nets modulo rule commutations. For the other, we conjecture that it does. However, for this notion "it is not immediately clear how to define a meaningful correctness criterion to characterise the image of the translation" [5].

Superimposing no cuts The first alternative is to restrict the rule for & in Table 5 by requiring that Γ may contain no cuts. This means the cuts appearing in the conclusion of the rule must be the disjoint union of the cuts appearing in the premises. Let $\mathrm{MALL}^*_{\mathsf{sep}}$ be the resulting alternative for MALL*. Now each MALL proof is the restriction of a unique $\mathrm{MALL}^*_{\mathsf{sep}}$ proof; hence the translation from MALL proofs to proof nets becomes a function.

Clearly, the resulting notion of proof-net equivalence on MALL proofs is included in the one from Section 5. In fact the conclusion is strict, for we loose the rule commutation $\mathsf{C}^*_\&$, as illustrated in [5, Section 5.3.4]. In general, the commutations $\mathsf{C}^\alpha_\&$ and $\mathsf{C}^\&_\alpha$ with $\alpha \in \{\otimes, \mathsf{mix}, *\}$ are no longer valid, because any cut included in Γ appears only once on the left, yet is duplicated on the right. (All other rule commutations remain valid.)

Superimposing as many cuts as possible The alternative of requiring Ω_1 and Ω_2 in the rule for & in Table 5 to be disjoint superimposes as many cuts as possible. As pointed out in [5, Section 5.3.4] it does not yield a function from MALL proofs to proof nets, for there may be a choice of how to identify cuts.

Local cuts A final variation considered in [5] is to depart from sets of linkings on a fixed cut sequent, and permit each linking its own set of cut pairs. Define a ***cut linking*** on a MALL sequent Γ as a linking on a sequent Ω,Γ with Ω a disjoint union of cuts. In order to abstract from the identity of the cut pairs we consider Ω (but not Γ) up to isomorphism. A MALL proof of Γ yields a set of cut linkings on Γ in the obvious way [5]. This yields a deterministic translation (function) from MALL proofs to sets of cut linkings.

Since the set of cut linkings of a MALL sequent Γ can be inferred from any MALL proof net of Γ, the kernel of this function (identifying MALL proofs that translate to the same set of cut linkings) includes proof-net equivalence as defined in Section 5. Thus, two MALL proofs that differ by rule commutations translate to the same set of cut linkings.

CONJECTURE 1 *Two MALL proofs translate to the same set of cut linkings if and only if they can be converted into each other by a series of rule commutations.*

10 Local rule commutations

The rule commutations $\mathsf{C}^{\&}_{\otimes}/\mathsf{C}^{\otimes}_{\&}$, $\mathsf{C}^{\&}_{\mathsf{mix}}/\mathsf{C}^{\mathsf{mix}}_{\&}$, $\mathsf{C}^{\&}_{\mathsf{cut}}/\mathsf{C}^{\mathsf{cut}}_{\&}$ and $\mathsf{C}^{\&}_{*}/\mathsf{C}^{*}_{\&}$ duplicate/identify premises, respectively; we refer to the other rule commutation as ***local***. The Appendix below concludes with a general definition of local rule commutation. In [6] a different notion of proof net, called a ***conflict net***, is proposed, such that two MALL proofs translate to the same conflict net if and only if they can be converted into each other by a series of local rule commutations.

Appendix: General concept of rule commutation

In order to properly define rule commutations in a sequent calculus, we consider rules—called ***abstract*** rules—that contain ***variables*** ranging over formulas and over sequents. The rules for MALL in Sections 2 and 5 are of this form. Thus, rather than seeing the rule for $\otimes$ as a template, of which there is an instance for each choice of A, B, Γ and Δ, we see it as a single rule containing four variables. When applying such a rule in a proof, formulas and sequents are substituted for the variables of the corresponding type.

Formally, a ***formula expression*** is built from formula variables, negated formula variables, literals and connectives; it is a ***formula*** if it contains only literals and connectives. Here a ***negated formula variable*** is a formula variable annotated with the subscript $^{\perp}$. A ***sequent expression*** is a multiset of sequent variables and formula expressions; it is a ***sequent*** if it does not contain any variables. Here a ***multiset*** of objects from a set S is a function $M : S \to \mathbb{N}$ indicating for each object in S how often it occurs in M. An object $x \in S$ with $M(x) > 0$ is called an ***element*** of M. Let $C(M) = \{x \in S \mid M(x) > 0\}$ denote the set of elements of M. In case $M(x) \in \{0,1\}$ for all $x \in S$, the multiset M is usually identified with the set $C(M)$.

An ***abstract rule*** is a pair $\frac{H}{\Gamma}$ of a set H of sequent expressions—the ***premises***—and a single sequent expression Γ—the conclusion. A ***concrete rule***—simply called ***rule*** outside of this appendix—is a pair $\frac{H}{\Gamma}$ of a *multi*set H of (variable-free) sequents and a single sequent Γ.

A ***substitution*** σ maps formula variables to formula expressions and sequent variables to sequent expressions;[5] it extends to negated formula variables $A^{\perp}$ by $\sigma(A^{\perp}) = \sigma(A)^{\perp}$, and further extends to a map from formula expressions to formula expressions and from (sets of) sequent expressions to (multisets of) sequent expressions. A substitution is ***closed*** if it maps formula variables to formulas and sequent

[5] In order to capture MALL*, we also allow sequent variables of special types—like "cut only" in Table 5—and for each type define the class of sequent expressions that may be substituted for it.

variables to sequents. If $\frac{H}{\Gamma}$ is an abstract rule and σ a (closed) substitution, then $\frac{\sigma(H)}{\sigma(\Gamma)}$ is a (closed) ***substitution instance*** of $\frac{H}{\Gamma}$; its ***collapse*** $\frac{C(\sigma(H))}{\sigma(\Gamma)}$ is again an abstract rule.

Given a collection of connectives to determine the valid formulas, a ***sequent calculus***—such as MALL—is given by a set of abstract rules.[6] It induces a set of concrete rules, namely the collapsed closed substitution instances of the abstract rules.

We now formalise proofs, extended to include the case where the conclusion is a sequent expression. When the conclusion is a standard sequent, the definition specialises to the familiar notion of sequent calculus proof. A ***proof*** Π in a sequent calculus is a well-founded, upwards branching tree whose nodes are labelled by sequent expressions and some of the leaves are marked "hypothesis", such that if Δ is the label of a node that is not a hypothesis and K is the multiset of labels of the children of this node then $\frac{K}{\Delta}$ is a substitution instance of one of the rules of that sequent calculus. Such a proof ***derives*** the abstract rule $\frac{H}{\Gamma}$, where H is the set of labels of the hypotheses, and Γ the label of the root of Π. A proof of a sequent expression Γ can be regarded as a proof of the abstract rule $\frac{H}{\Gamma}$ with $H = \emptyset$.

For α and β two abstract proof rules in a sequent calculus, an ***$\alpha\beta$-proof*** is a proof in which each non-hypothesis node is either the root and an application of β, or a child of the root and an application of α. A ***subproof*** Π' of a proof Π comprises all nodes in the tree Π above a given node, which is the root of Π'. A proof Π_f deriving a rule $\frac{H}{\Gamma}$, together with proofs Π_Δ of Δ for each $\Delta \in H$, composes into a proof Π' of Γ, such that the proofs Π_Δ are subproofs of Π'. If Π' itself is a subproof of a proof Π we say that Π_f is a ***proof fragment*** of Π; if Π_f is an $\alpha\beta$-proof, it is called an ***$\alpha\beta$-proof fragment*** of Π. For Π a proof and σ a substitution, $\sigma(\Pi)$ denotes the proof obtained from Π by applying σ to all its node labels.

An abstract rule is ***pure*** if (1) its premises are free of literals and connectives and thus are built from variables (sequent variables, formula variables and negated formula variables) only, and (2) each of these variables occurs exactly once in the conclusion. We define rule commutation for sequent calculi containing pure rules only. This includes MALL$^-$ and MALL*, but not MALL; however, the rule commutations of MALL can be derived as the projections of the ones for MALL*.

The implicit tracking of subformula occurrences described in Section 2 and utilised in Sections 3 and 5 can now be formalised as follows: a subformula occurrence within an occurrence of a formula or sequent substituted for a variable A, $A^\perp$ or Γ appearing in the premises of an abstract rule tracks to the corresponding subformula occurrence within the occurrence of the same formula or sequent substituted for A, $A^\perp$ or Γ in the conclusion of the rule.

It is not hard to show that any abstract rule derivable in a sequent calculus containing pure rules only can be obtained as a collapsed substitution instance of a pure rule derivable in that sequent calculus. Although we do not make use of this insight in our proofs, it helps to motivate the following definition.

A ***rule commutation*** is an (ordered) pair of an $\alpha\beta$-proof and a (different) $\beta\alpha$-proof deriving the same pure rule. An $\alpha\beta$-proof Π'_1 ***commutes*** with a $\beta\alpha$-proof Π'_2 if there exists a rule commutation (Π_1, Π_2) and a substitution σ such that $\sigma(\Pi_1) = \Pi'_1$ and $\sigma(\Pi_2) = \Pi'_2$. Two proofs ***differ by a rule commutation*** if one can be obtained from the other by the replacement of an $\alpha\beta$-proof fragment occurring in it by a commuting $\beta\alpha$-proof fragment. Thus a rule commutation is a transposition of adjacent rules that preserves subproofs immediately above, with possible duplication/identification.

We leave it to the reader to check that this definition, applied to MALL$^-$ and MALL*, generates exactly the rule commutations presented in Sections 4 and 6.

[6] By these definitions, the MALL axiom $\mathtt{ax}$, unlike the other rules, is still a template, of which an instance is obtained by filling in actual propositional variables for the metavariable P. If this is felt to be inelegant, one could rename "propositional variable" into "atom" and introduce "atom variables" and negated atom variables to formulate the axiom $\mathtt{ax}$. For simplicity, we abstain from doing this here.

In our definition of rule commutation it is essential that the rule derived by each of the two proofs Π_1 and Π_2 in a rule commutation is pure. Skipping this requirement would give rise to unwanted rule commutations. As an example, consider the rule commutation $\mathsf{C}^{\parr}_{\parr}$ of Figure 2 in which Γ, A_1 is substituted for Γ. The two sides of the resulting rule commutation define the same non-pure rule. Simply requiring—as we do—that the same proof Π, deriving the sequent $\Gamma, A_1, A_1, A_2, B_1, B_2$, is used at both sides of the commutation does not rule out that the roles of the two occurrences of A_1 are swapped at one side of the commutation, possibly leading to proofs inducing different proof nets.

Moreover, we cannot drop the requirement that Π_1 and Π_2 must be $\alpha\beta$- and $\beta\alpha$-rules, for that would give rise to the unwanted rule commutation

$$\dfrac{\dfrac{\dfrac{A,C \qquad D,\ E}{A, C\otimes D,\ E}\otimes}{A, C\otimes D, E\oplus F}\oplus_1 \qquad \dfrac{\dfrac{B,C \qquad D,\ F}{B, C\otimes D,\ F}\otimes}{B, C\otimes D, E\oplus F}\oplus_2}{A\&B, C\otimes D, E\oplus F}\& \quad\longleftrightarrow\quad \dfrac{\dfrac{\dfrac{A,C \qquad D,\ F}{A, C\otimes D,\ F}\otimes}{A, C\otimes D, E\oplus F}\oplus_2 \qquad \dfrac{\dfrac{B,C \qquad D,\ E}{B, C\otimes D,\ E}\otimes}{B, C\otimes D, E\oplus F}\oplus_1}{A\&B, C\otimes D, E\oplus F}\&$$

These two proofs derive the same pure rule, yet (when instantiated) induce different proof nets:

$$A\&B,\quad C\otimes D,\quad E\oplus F \qquad\qquad\qquad A\&B,\quad C\otimes D,\quad E\oplus F$$

Based on the above, we say that a concrete β-rule ***commutes over*** a concrete α-rule, if these rules occur in an $\alpha\beta$-proof fragment obtained as a substitution instance of an $\alpha\beta$-proof for which there exists a $\beta\alpha$-proof deriving the same pure rule. This definition of rule commutation is more liberal than the standard definition of rule commutation for a Gentzen sequent calculus [8, Def. 5.2.1], analysed by Kleene [7] and Curry [1]. That definition only covers the case where *each* β rule commutes over *each* α-rule, corresponding with the check marks in Table 8. Moreover, [8] requires—translated to our terminology—the source proof fragment to have two non-leaf nodes only (one for β and only one for α), thereby ruling out the commutation of & over any α.

Local rule commutations. Define a proof as ***non-repeating*** if all its hypothesis have a different label. A rule commutation (Π_1, Π_2) is ***local*** (*cf.* Section 10) if Π_1 and Π_2 are non-repeating.

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