

On Finite Bases for Weak Semantics: Failures versus Impossible Futures*

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Abstract. We provide a finite basis for the (in)equational theory of the process algebra BCCS modulo the weak failures preorder and equivalence. We also give positive and negative results regarding the axiomatizability of BCCS modulo weak impossible futures semantics.

1 Introduction

Labeled transition systems constitute a widely used model of concurrent computation. They model processes by explicitly describing their states and their transitions from state to state, together with the actions that produce these transitions. Several notions of behavioral semantics have been proposed, with the aim to identify those states that afford the same observations [14, 12]. For equational reasoning about processes, one needs to find an axiomatization that is sound and *ground-complete* modulo the semantics under consideration, meaning that all equivalent closed terms can be equated. Ideally, such an axiomatization is also *ω -complete*, meaning that all equivalent *open* terms can be equated. If such a finite axiomatization exists, it is said that there is a *finite basis* for the equational theory.

For concrete semantics, so in the absence of the silent action τ , the existence of finite bases is well-studied [16, 14, 7], in the context of the process algebra BCCSP, containing the basic process algebraic operators from CCS and CSP. However, for weak semantics, that take into account the τ , hardly anything is known on finite bases. In [12], Van Glabbeek presented a spectrum of weak semantics. For several of the semantics in this spectrum, a sound and ground-complete axiomatization has been given, in the setting of the process algebra BCCS (BCCSP extended by τ), see, e.g., [13]. But a finite basis has been given only for *weak*, *delay*, η - and *branching bisimulation* semantics [18, 11], and in case of an infinite alphabet of actions also for *weak impossible futures* semantics [22]. The reason for this lack of results on finite bases, apart from the inherent difficulties arising with weak semantics, may be that it is usually not so straightforward

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to define a notion of unique normal form for *open* terms in a *weak* semantics. Here we will employ a saturation technique, in which normal forms are saturated with subterms.

In this paper, we focus on two closely related weak semantics, based on failures and impossible futures. A *weak failure* consists of a trace $a_1 \cdots a_n$ and a set A , both of concrete actions. A state exhibits this weak failure pair if it can perform the trace $a_1 \cdots a_n$ (possibly intertwined with τ 's) to a state that cannot perform any action in A (even after performing τ 's). In a *weak impossible future*, A can be a set of traces. Weak failures semantics plays an essential role for the process algebra CSP [3]. For convergent processes, it coincides with testing semantics [8, 19], and thus is the coarsest congruence for the CCS parallel composition that respects deadlock behavior. Weak impossible futures semantics [21] is a natural variant of possible futures semantics [20]. In [15] it is shown that weak impossible futures semantics, with an additional root condition, is the coarsest congruence containing weak bisimilarity with explicit divergence that respects deadlock/livelock traces (or fair testing, or any liveness property under a global fairness assumption) and assigns unique solutions to recursive equations.

The heart of our paper is a finite basis for the inequational theory of BCCS modulo the weak failures *preorder*. The axiomatization consists of the standard axioms A1-4 for bisimulation, three extra axioms WF1-3 for failures semantics, and in case of a finite alphabet A , an extra axiom WF_A . The proof that A1-4 and WF1-3 are a finite basis in case of an infinite alphabet is a sub-proof of the proof that A1-4, WF1-3 and WF_A are a finite basis in case of a finite alphabet. Our proof has the same general structure as the beautiful proof for testing equivalences given in [8] and further developed in [17]. Pivotal to this is the construction of “saturated” sets of actions within a term [8]. Since here we want to obtain an ω -completeness result, we extend this notion to variables. Moreover, to deal with ω -completeness, we adopt the same general proof structure as in the strong case [9]. In this sense, our proof strategy can be viewed as a combination of the strategies proposed in [8] and [9]. Furthermore, we apply an algorithm from [2, 10, 6] to obtain a finite basis for BCCS modulo weak failures *equivalence* for free.

At the end, we investigate the equational theory of BCCS modulo weak impossible futures semantics. This shows a remarkable difference with weak failures semantics, in spite of the strong similarity between the definitions of these semantics (and between their ground-complete axiomatizations). As said, in case of an infinite alphabet, BCCS modulo the weak impossible futures preorder has a finite basis [22]. However, we show that in case of a finite alphabet, such a finite basis does not exist. Moreover, in case of weak impossible futures *equivalence*, there is no finite ground-complete axiomatization, regardless of the cardinality of the alphabet.

A finite basis for the equational theory of BCCSP modulo (concrete) failures semantics was given in [9]. The equational theory of BCCSP modulo (concrete) impossible futures semantics is studied in [4]. It is interesting to see that our results for weak semantics agree with their concrete counterparts, with very similar proofs. This raises a challenging open question: can one establish a general theorem to link the axiomatizability (or nonaxiomatizability) of concrete and weak semantics?

An extended abstract of this paper appears as [5].

2 Preliminaries

BCCS(A) is a basic process algebra for expressing finite process behavior. Its signature consists of the constant $\mathbf{0}$, the binary operator $_+ _$, and unary prefix operators $\tau_$ and $a_$, where a is taken from a nonempty set A of visible actions, called the *alphabet*, ranged over by a, b, c . We assume that $\tau \notin A$ and write A_τ for $A \cup \{\tau\}$, ranged over by α, β .

$$t ::= 0 \mid at \mid \tau t \mid t + t \mid x$$

Closed BCCS(A) terms, ranged over by p, q , represent finite process behaviors, where $\mathbf{0}$ does not exhibit any behavior, $p + q$ offers a choice between the behaviors of p and q , and αp executes action α to transform into p . This intuition is captured by the transition rules below. They give rise to A_τ -labeled transitions between closed BCCS terms.

$$\frac{}{\alpha x \xrightarrow{\alpha} x} \quad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \quad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

We assume a countably infinite set V of variables; x, y, z denote elements of V . Open BCCS terms, denoted by t, u, v, w , may contain variables from V . Write $\text{var}(t)$ for the set of variables occurring in t . The operational semantics is extended verbatim to open terms; variables generate no transition. We write $t \Rightarrow u$ if there is a sequence of τ -transitions $t \xrightarrow{\tau} \dots \xrightarrow{\tau} u$; furthermore $t \xrightarrow{\alpha}$ denotes that there is a term u with $t \xrightarrow{\alpha} u$, and likewise $t \Rightarrow \xrightarrow{\alpha}$ denotes that there are a terms u, v with $t \Rightarrow u \xrightarrow{\alpha} v$.

The *depth* of a term t , denoted by $|t|$, is the length of the *longest* trace of t , not counting τ -transitions. It is defined inductively as follows: $|\mathbf{0}| = |x| = 0$; $|at| = 1 + |t|$; $|\tau t| = |t|$; $|t + u| = \max\{|t|, |u|\}$.

A (closed) substitution, ranged over by σ, ρ , maps variables in V to (closed) terms. For open terms t and u , and a preorder \sqsubseteq (or equivalence \equiv) on closed terms, we define $t \sqsubseteq u$ (or $t \equiv u$) if $\sigma(t) \sqsubseteq \sigma(u)$ (resp. $\sigma(t) \equiv \sigma(u)$) for all closed substitutions σ . Clearly, $t \xrightarrow{\alpha} t'$ implies that $\sigma(t) \xrightarrow{\alpha} \sigma(t')$ for all substitutions σ .

An *axiomatization* is a collection of equations $t \approx u$ or of inequations $t \preceq u$. The (in)equations in an axiomatization E are referred to as *axioms*. If E is an equational axiomatization, we write $E \vdash t \approx u$ if the equation $t \approx u$ is derivable from the axioms in E using the rules of equational logic (reflexivity, symmetry, transitivity, substitution, and closure under BCCS contexts). For the derivation of an inequation $t \preceq u$ from an inequational axiomatization E , denoted by $E \vdash t \preceq u$, the rule for symmetry is omitted. We will also allow equations $t \approx u$ in inequational axiomatizations, as an abbreviation of $t \preceq u \wedge u \preceq t$.

An axiomatization E is *sound* modulo a preorder \sqsubseteq (or equivalence \equiv) if for all terms t, u , from $E \vdash t \preceq u$ (or $E \vdash t \approx u$) it follows that $t \sqsubseteq u$ (or $t \equiv u$). E is *ground-complete* for \sqsubseteq (or \equiv) if $p \sqsubseteq q$ (or $p \equiv q$) implies $E \vdash p \preceq q$ (or $E \vdash p \approx q$) for all closed terms p, q . Moreover, E is *ω -complete* if for all terms t, u with $E \vdash \sigma(t) \preceq \sigma(u)$ (or $E \vdash \sigma(t) \approx \sigma(u)$) for all closed substitutions σ , we have $E \vdash t \preceq u$ (or $E \vdash t \approx u$). When E is ω -complete as well as ground-complete, it is *complete* for \sqsubseteq (or \equiv) in the sense that $t \sqsubseteq u$ (or $t \equiv u$) implies $E \vdash t \preceq u$ (or $E \vdash t \approx u$) for all terms t, u . The equational theory of BCCS modulo a preorder \sqsubseteq (or equivalence \equiv) is said to

be *finitely based* if there exists a finite, ω -complete axiomatization that is sound and ground-complete for BCCS modulo \sqsubseteq (or \equiv).

A1-4 below are the core axioms for BCCS modulo bisimulation semantics. We write $t = u$ if $A1-4 \vdash t \approx u$.

$$\begin{array}{ll} A1 & x + y \approx y + x \\ A2 & (x + y) + z \approx x + (y + z) \\ A3 & x + x \approx x \\ A4 & x + \mathbf{0} \approx x \end{array}$$

Summation $\sum_{i \in \{1, \dots, n\}} t_i$ denotes $t_1 + \dots + t_n$, where summation over the empty set denotes $\mathbf{0}$. As binding convention, $+$ and summation bind weaker than α . For every term t there exists a finite set $\{\alpha_i t_i \mid i \in I\}$ of terms and a finite set Y of variables such that $t = \sum_{i \in I} \alpha_i t_i + \sum_{y \in Y} y$. The $\alpha_i t_i$ for $i \in I$ and the $y \in Y$ are called the *summands* of t . For a set of variables Y , we will often denote the term $\sum_{y \in Y} y$ by Y .

Definition 1 (Initial actions). For any term t , the set $\mathcal{I}(t)$ of initial actions is defined as $\mathcal{I}(t) = \{a \in A \mid t \Rightarrow^a\}$.

Definition 2 (Weak failures).

- A pair $(a_1 \dots a_k, B)$, with $k \geq 0$ and $B \subseteq A$, is a *weak failure pair* of a process p_0 if there is a path $p_0 \Rightarrow^{a_1} \dots \Rightarrow^{a_k} p_k$ with $\mathcal{I}(p_k) \cap B = \emptyset$.
- Write $p \leq_{\text{WF}} q$ if the weak failure pairs of p are also weak failure pairs of q .
- The *weak failures preorder* \sqsubseteq_{WF} is given by $p \sqsubseteq_{\text{WF}} q$ iff (1) $p \leq_{\text{WF}} q$ and (2) $p \xrightarrow{\tau}$ implies that $q \xrightarrow{\tau}$.
- *Weak failures equivalence* \equiv_{WF} is defined as $\sqsubseteq_{\text{WF}} \cap \sqsubseteq_{\text{WF}}^{-1}$.

It is well-known that $p \leq_{\text{WF}} q$ is *not* a *precongruence* for BCCS: e.g., $\tau \mathbf{0} \leq_{\text{WF}} \mathbf{0}$ but $\tau \mathbf{0} + a \mathbf{0} \not\leq_{\text{WF}} \mathbf{0} + a \mathbf{0}$. However, \sqsubseteq_{WF} is, meaning that $p_1 \sqsubseteq_{\text{WF}} q_1$ and $p_2 \sqsubseteq_{\text{WF}} q_2$ implies $p_1 + p_2 \sqsubseteq_{\text{WF}} q_1 + q_2$ and $\alpha p_1 \sqsubseteq_{\text{WF}} \alpha q_1$ for $\alpha \in A_\tau$. In fact, \sqsubseteq_{WF} is the coarsest precongruence contained in \leq_{WF} . Likewise, \equiv_{WF} is a *congruence* for BCCS.

3 A Finite Basis for Weak Failures Semantics

3.1 Axioms for the Weak Failures Preorder

On BCCS processes, the weak failures preorder as defined above coincides with the inverse of the must-testing preorder of [8]. A sound and ground-complete axiomatization of the must-testing preorder has been given in [8], in terms of a language richer than BCCS. After restriction to BCCS processes, and reversing the axioms, it consists of A1-4 together with the axioms:

$$\begin{array}{ll} N1 & \alpha x + \alpha y \approx \alpha(\tau x + \tau y) \\ N2 & \tau(x + y) \approx x + \tau y \\ N3 & \alpha x + \tau(\alpha y + z) \approx \tau(\alpha x + \alpha y + z) \\ E1 & x \approx \tau x + \tau y \end{array}$$

Here we simplify this axiomatization to A1-4 and WF1-3 from Tab. 1. In fact it is an easy exercise to derive WF1-3 from N1, N2 and E1, and N1, N2 and E1 from WF1-3. It is a little harder to check that N3 is derivable from the other three axioms (cf. Lem. 1).

WF1	$ax + ay \approx a(\tau x + \tau y)$
WF2	$\tau(x + y) \preceq \tau x + y$
WF3	$x \preceq \tau x + y$

Table 1. Axiomatization for the weak failures preorder

Theorem 1. $A1-4+WF1-3$ is sound and ground-complete for $BCCS(A)$ modulo \sqsubseteq_{WF} .

In this section, we extend this ground-completeness result with two ω -completeness results. The first one says, in combination with Theo. 1, that as long as our alphabet of actions is infinite, the axioms $A1-4+WF1-3$ constitute a finite basis for the inequational theory of $BCCS(A)$ modulo \sqsubseteq_{WF} .

Theorem 2. If $|A| = \infty$, then $A1-4+WF1-3$ is ω -complete for $BCCS(A)$ modulo \sqsubseteq_{WF} .

To get a finite basis for the inequational theory of $BCCS$ modulo \sqsubseteq_{WF} in case $|A| < \infty$, we need to add the following axiom:

$$WF_A \quad \sum_{a \in A} ax_a \preceq \sum_{a \in A} ax_a + y$$

where the x_a for $a \in A$ and y are distinct variables.

Theorem 3. If $|A| < \infty$, then $A1-4+WF1-3+WF_A$ is ω -complete for $BCCS(A)$ modulo \sqsubseteq_{WF} .

The rest of this section up to Sec. 3.4 is devoted to the proofs of Theorems 1–3. For a start, the inequations in Tab. 2 can be derived from $A1-4+WF1-3$:

D1	$\tau(x + y) + x \approx \tau(x + y)$
D2	$\tau(\tau x + y) \approx \tau x + y$
D3	$ax + \tau(ay + z) \approx \tau(ax + ay + z)$
D4	$\tau x \preceq \tau x + y$
D5	$\sum_{i \in I} ax_i \approx a(\sum_{i \in I} \tau x_i)$ for finite nonempty index sets I
D6	$\tau x + y \approx \tau x + \tau(x + y)$
D7	$\tau x + \tau y \approx \tau x + \tau(x + y) + \tau y$
D8	$\tau x + \tau(x + y + z) \approx \tau x + \tau(x + y) + \tau(x + y + z)$
D9	$\sum_{i \in I} \tau(at_i + y_i) \approx \sum_{i \in I} \tau(at + y_i)$ for finite I , where $t = \sum_{i \in I} \tau t_i$.

Table 2. Derived inequations

Lemma 1. D1-9 are derivable from $A1-4+WF1-3$.

Proof. We shorten “ $A1-4+WF1-3 \vdash$ ” to “ \vdash ”.

- By WF3, $\vdash x \preceq \tau x$, and thus $\vdash \tau x + x \preceq \tau x$. Moreover, by WF2, $\vdash \tau(x + x) \preceq \tau x + x$, hence $\vdash \tau x \preceq \tau x + x$. In summary, $\vdash \tau x \approx \tau x + x$. So $\vdash \tau(x + y) \approx \tau(x + y) + x + y + x \approx \tau(x + y) + x$.

2. By WF2, $\vdash \tau(x + \tau x) \preceq \tau x + \tau x = \tau x$, so by D1, $\vdash \tau \tau x \preceq \tau x$. Hence, by WF2, $\vdash \tau(\tau x + y) \preceq \tau \tau x + y \preceq \tau x + y$.
Moreover, by WF3, $\vdash \tau x + y \preceq \tau(\tau x + y)$.
3. By WF3, $\vdash y \preceq \tau y + \tau x$. So by WF1, $\vdash ay \preceq a(\tau x + \tau y) \approx ax + ay$. This implies $\vdash \tau(ay + z) \preceq \tau(ax + ay + z)$. Hence, by D1, $\vdash ax + \tau(ay + z) \preceq ax + \tau(ax + ay + z) \approx \tau(ax + ay + z)$.
Moreover, by WF2, $\vdash \tau(ax + ay + z) \preceq ax + \tau(ay + z)$.
4. By WF3 and D2, $\vdash \tau x \preceq \tau \tau x + y \approx \tau x + y$.
5. By induction on $|I|$, using WF1 and D2.
6. By D4 and D1, $\vdash \tau x + y \preceq \tau x + \tau(x + y) + y \approx \tau x + \tau(x + y)$.
Moreover, by WF2, $\vdash \tau x + \tau(x + y) \preceq \tau x + \tau x + y = \tau x + y$.
7. By D4 in one direction; by D6 and D1 in the other.
8. By D4 in one direction; by D6 and D1 in the other.
9. By D1, $\vdash \sum_{i \in I} \tau(at_i + y_i) \approx \sum_{i \in I} \tau(at_i + y_i) + u$, where $u = \sum_{i \in I} at_i$. Thus, by repeated application of D3, $\vdash \sum_{i \in I} \tau(at_i + y_i) \approx \sum_{i \in I} \tau(at_i + u + y_i) = \sum_{i \in I} \tau(u + y_i)$. By D5 we have $u = at$. \square

3.2 Normal Forms

The notion of a normal form, which is formulated in the following two definitions, will play a key role in the forthcoming proofs. For any set $L \subseteq A \cup V$ of actions and variables let $A_L = L \cap A$, the set of actions in L , and $V_L = L \cap V$, the set of variables in L .

Definition 3 (Saturated family). Suppose \mathcal{L} is a finite family of finite sets of actions and variables. We say \mathcal{L} is *saturated* if it is nonempty and

- $L_1, L_2 \in \mathcal{L}$ implies that $L_1 \cup L_2 \in \mathcal{L}$; and
- $L_1, L_2 \in \mathcal{L}$ and $L_1 \subseteq L_3 \subseteq L_2$ imply that $L_3 \in \mathcal{L}$.

Definition 4 (Normal form).

- (i) A term t is in τ normal form if

$$t = \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} at_a + V_L \right)$$

where the t_a are in normal form and \mathcal{L} is a saturated family of sets of actions and variables. We write $L(t)$ for $\bigcup_{L \in \mathcal{L}} L$; note that $L(t) \in \mathcal{L}$.

- (ii) t is in action normal form if

$$t = \sum_{a \in A_L} at_a + V_L$$

where the t_a are in normal form and $L \subseteq A \cup V$. We write $L(t)$ for L .

- (iii) t is in normal form if it is either in τ normal form or in action normal form.

Note that the definition of a normal form requires that for any $a \in A$, if $t \xrightarrow{a} t_1$ and $t \xrightarrow{a} t_2$, then t_1 and t_2 are syntactically identical.

We prove that every term can be equated to a normal form. We start with an example.

Example 1. Suppose $t = \tau(at_1 + \tau(bt_2 + ct_3) + x) + \tau(at_4 + \tau x + \tau y) + z$. Then t can be equated to a τ normal form with $\mathcal{L} = \{\{a, b, c, x\}, \{a, x, y\}, \{a, b, c, x, y, z\}, \{a, b, c, x, y\}, \{a, b, c, x, z\}, \{a, b, x, y\}, \{a, c, x, y\}, \{a, x, y, z\}, \{a, b, x, y, z\}, \{a, c, x, y, z\}\}$. We give a detailed derivation. By D2,

$$\vdash t \approx \tau(at_1 + bt_2 + ct_3 + x) + \tau(at_4 + x + y) + z$$

By D6,

$$\vdash t \approx \tau(at_1 + bt_2 + ct_3 + x) + \tau(at_4 + x + y) + \tau(at_4 + x + y + z)$$

Let $u_a = \tau t_1 + \tau t_4$, $u_b = t_2$ and $u_c = t_3$. By D9,

$$\vdash t \approx \tau(au_a + bu_b + cu_c + x) + \tau(au_a + x + y) + \tau(au_a + x + y + z)$$

By induction, u_a can be brought into a normal form t_a , and likewise for u_b and u_c . So

$$\vdash t \approx \tau(tu_a + bt_b + ct_c + x) + \tau(at_a + x + y) + \tau(at_a + x + y + z)$$

By D7,

$$\begin{aligned} \vdash t \approx & \tau(at_a + bt_b + ct_c + x) + \tau(at_a + x + y) \\ & + \tau(at_a + x + y + z) + \tau(at_a + bt_b + ct_c + x + y + z) \end{aligned}$$

Finally, by D8,

$$\begin{aligned} \vdash t \approx & \tau(at_a + bt_b + ct_c + x) + \tau(at_a + x + y) + \tau(at_a + x + y + z) \\ & + \tau(at_a + bt_b + ct_c + x + y + z) + \tau(at_a + bt_b + ct_c + x + y) \\ & + \tau(at_a + bt_b + ct_c + x + z) + \tau(at_a + bt_b + x + y) + \tau(at_a + ct_c + x + y) \\ & + \tau(at_a + bt_b + x + y + z) + \tau(at_a + ct_c + x + y + z) \\ = & \sum_{L \in \mathcal{L}} \tau(\sum_{a \in A_L} at_a + V_L) \end{aligned}$$

Lemma 2. For any term t , A1-4+WF1-3 $\vdash t \approx t'$ for some normal form t' .

Proof. By induction on $|t|$. We distinguish two cases.

- $t \xrightarrow{a}$. Let $t = \sum_{i \in I} a_i t_i + Y$. By D5,

$$\vdash t \approx \sum_{a \in \mathcal{I}(t)} a \left(\sum_{i \in I, a_i = a} \tau t_i \right) + Y.$$

By induction, for each $a \in \mathcal{I}(t)$,

$$\vdash \sum_{i \in I, a_i = a} \tau t_i \approx t_a$$

for some normal form t_a . So we are done.

- $t \xrightarrow{\tau}$. By D6, t can be brought in the form $\sum_{i \in I} \tau t_i$ with $I \neq \emptyset$, and using D2 one can even make sure that $t_i \not\xrightarrow{\tau}$ for $i \in I$. Using the first case in this proof, we obtain, for each $i \in I$,

$$\vdash t_i \approx \sum_{a \in A_{L(i)}} at_{a,i} + V_{L(i)}$$

for some $L(i) \subseteq A \cup V$. Thus

$$\vdash t \approx \sum_{i \in I} \tau \left(\sum_{a \in A_{L(i)}} at_{a,i} + V_{L(i)} \right).$$

For each $a \in \mathcal{I}(t)$, we define $u_a = \sum_{i \in I, a \in A_{L(i)}} \tau t_{a,i}$.

Then $|u_a| < |t|$. By induction, $\vdash u_a \approx t_a$ for some normal form t_a .

Define $\mathcal{L} = \{L(i) \mid i \in I\}$. By repeated application of D9 we obtain

$$\vdash t \approx \sum_{i \in I} \tau \left(\sum_{a \in A_{L(i)}} au_a + V_{L(i)} \right) \approx \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} at_a + V_L \right).$$

The latter term has the required form, except that the family \mathcal{L} need not be saturated. However, it is straightforward to saturate \mathcal{L} by application of D7 and D8. \square

Lemma 3. *Suppose t and u are both in normal forms and $t \sqsubseteq_{\text{WF}} u$. If $t \xrightarrow{\alpha} t_a$, then there exists a term u_a such that $u \xrightarrow{\alpha} u_a$ and $t_a \leq_{\text{WF}} u_a$.*

Proof. Suppose $t \sqsubseteq_{\text{WF}} u$ and $t \xrightarrow{\alpha} t_a$. Let σ be the closed substitution given by $\sigma(x) = \mathbf{0}$ for all $x \in V$. As (a, \emptyset) is a weak failure pair of $\sigma(t)$ and $\sigma(t) \sqsubseteq_{\text{WF}} \sigma(u)$, it is also a weak failure pair of u . Thus there exists a term u_a such that $u \xrightarrow{\alpha} u_a$. By the definition of a normal form, this term is unique. $(*)$

We now show that $t_a \leq_{\text{WF}} u_a$. Let ρ be a closed substitution. Consider a weak failure pair $(a_1 \cdots a_k, B)$ of $\rho(t_a)$. Then $(aa_1 \cdots a_k, B)$ is a weak failure pair of $\rho(t)$, and hence also of $\rho(u)$. It suffices to conclude that $(a_1 \cdots a_k, B)$ is a weak failure pair of $\rho(u_a)$. However, we can *not* conclude this directly, as possibly $u \Rightarrow x + u'$ where $(aa_1 \cdots a_k, B)$ is a weak failure pair of $\rho(x)$. To ascertain that nevertheless $(a_1 \cdots a_k, B)$ is a weak failure pair of $\rho(u_a)$, we define a modification ρ' of ρ such that for all $\ell \leq k$ and for all terms v , $\rho(v)$ and $\rho'(v)$ have the same weak failure pairs $(c_1 \cdots c_\ell, B)$, while for all $x \in V$, $(aa_1 \cdots a_k, B)$ is not a weak failure pair of $\rho'(x)$.

We obtain $\rho'(x)$ from $\rho(x)$ by replacing subterms bp at depth k by $\mathbf{0}$ if $b \notin B$ and by $bb\mathbf{0}$ if $b \in B$. That is,

$$\rho'(x) = \text{chop}_k(\rho(x))$$

with chop_m for all $m \geq 0$ inductively defined by

$$\begin{aligned} \text{chop}_m(\mathbf{0}) &= \mathbf{0} \\ \text{chop}_m(p + q) &= \text{chop}_m(p) + \text{chop}_m(q) \\ \text{chop}_m(\tau p) &= \tau \text{chop}_m(p) \\ \text{chop}_0(bp) &= \begin{cases} \mathbf{0} & \text{if } b \notin B \\ bb\mathbf{0} & \text{if } b \in B \end{cases} \\ \text{chop}_{m+1}(bp) &= b \text{chop}_m(p) \end{aligned}$$

We proceed to prove that ρ' has the desired properties mentioned above.

- A. For all $\ell \leq k$ and $c_1, \dots, c_\ell \in A$ and for all terms v , $\rho(v)$ and $\rho'(v)$ have the same weak failure pairs $(c_1 \cdots c_\ell, B)$,
 The difference between $\rho(v)$ and $\rho'(v)$ only appears within subterms of depth k , that is for terms p such that $\rho(v) \Rightarrow^{c_1} \cdots \Rightarrow^{c_k} p$ for certain $c_1, \dots, c_k \in A$. Such a subterm p of $\rho(v)$ corresponds to a subterm p' of $\rho'(v)$ —still satisfying $\rho'(v) \Rightarrow^{c_1} \cdots \Rightarrow^{c_k} p'$ —in which certain subterms bq are replaced by $\mathbf{0}$ if $b \notin B$ and by $bb\mathbf{0}$ if $b \in B$. For such corresponding subterms p and p' we have $\mathcal{I}(p) \cap B = \emptyset$ if and only if $\mathcal{I}(p') \cap B = \emptyset$. From this the claim follows immediately.
- B. For all $x \in V$, $(aa_1 \cdots a_k, B)$ is not a weak failure pair of $\rho'(x)$.

To this end we show that for all closed terms p , $\text{chop}_m(p)$ does not have any weak failure pair $(c_0 \cdots c_m, B)$ with $c_0, \dots, c_m \in A$. We apply induction on m .

Base case: Since the summands of $\text{chop}_0(p)$, when skipping over initial τ -steps, are $bb\mathbf{0}$ with $b \in \mathcal{I}(p) \cap B$, $\text{chop}_0(p)$ does not have a weak failure pair (c_0, B) .

Induction step: Let $m > 0$. By induction, for closed terms q , $\text{chop}_{m-1}(q)$ does not have weak failure pairs $(c_1 \cdots c_m, B)$. Since the transitions of $\text{chop}_m(p)$ are $\text{chop}_m(p) \xrightarrow{c} \text{chop}_{m-1}(q)$ for $p \xrightarrow{c} q$, it follows that $\text{chop}_m(p)$ does not have weak failure pairs $(c_0 \cdots c_m, B)$.

Now, since $(a_1 \cdots a_k, B)$ is a weak failure pair of $\rho(t_a)$, by property (A) it is also a weak failure pair of $\rho'(t_a)$, Therefore $(aa_1 \cdots a_k, B)$ is a weak failure pair of $\rho'(t)$, and hence also of $\rho'(u)$. Since according to property (B) it is *not* the case that $u \Rightarrow x + u'$ with $(aa_1 \cdots a_k, B)$ a weak failure pair of $\rho'(x)$, it must be the case that $u \Rightarrow^a u''$ such that $(a_1 \cdots a_k, B)$ is a weak failure pair of $\rho'(u'')$. By (*), $u'' = u_a$. Again by property (A), $(a_1 \cdots a_k, B)$ is a weak failure pair of $\rho(u_a)$. \square

3.3 ω -Completeness Proof

We are now in a position to prove Theo. 2 (ω -completeness in case of an infinite alphabet) and Theo. 3 (ω -completeness in case of a finite alphabet), along with Theo. 1 (ground completeness). We will prove these three theorems in one go. Namely, in the proof, two cases are distinguished; only in the second case ($\mathcal{I}(t) = A$), in which the A is guaranteed to be finite, will the axiom WF_A play a role.

Proof. Let $t \sqsubseteq_{\text{WF}} u$. We need to show that $\vdash t \preceq u$. We apply induction on $|t| + |u|$. By Lem. 2, we can write t and u in normal form.

We first prove that $L(t) \subseteq L(u)$. Suppose this is not the case. Then there exists some $a \in A_{L(t)} \setminus A_{L(u)}$ or some $x \in V_{L(t)} \setminus V_{L(u)}$. In the first case, let σ be the closed substitution with $\sigma(z) = \mathbf{0}$ for all $z \in V$; we find that (a, \emptyset) is a weak failure pair of $\sigma(t)$ but not of $\sigma(u)$, which contradicts the fact that $\sigma(t) \sqsubseteq_{\text{WF}} \sigma(u)$. In the second case, pick some $d > \max\{|t|, |u|\}$, and consider the closed substitution $\sigma(x) = a^d\mathbf{0}$ and $\sigma(z) = \mathbf{0}$ for $z \neq x$. Then (a^d, \emptyset) is weak failure pair of $\sigma(t)$. However, it can *not* be a weak failure pair of $\sigma(u)$, again contradicting $\sigma(t) \sqsubseteq_{\text{WF}} \sigma(u)$.

We distinguish two cases, depending on whether $\mathcal{I}(t) = A$ or not.

1. $\mathcal{I}(t) \neq A$. We distinguish three cases. Due to the condition that $t \xrightarrow{\tau}$ implies $u \xrightarrow{\tau}$, it cannot be the case that t is an action normal form and u a τ normal form.

(a) t and u are both action normal forms. So $t = \sum_{a \in A_L} at_a + V_L$ and $u = \sum_{a \in A_M} au_a + V_M$. We show that $L(t) = L(u)$. Namely, pick $b \in A \setminus A_L$, and let σ be the closed substitution with $\sigma(z) = \mathbf{0}$ for any $z \in V_L$, and $\sigma(z) = b\mathbf{0}$ for $z \notin V_L$. As $(\varepsilon, A \setminus \mathcal{I}(t))$ is a weak failure pair of t , and hence of u , it must be that $L(u) \subseteq L(t)$. Together with $L(t) \subseteq L(u)$ this gives $L(t) = L(u)$. By Lem. 3, for each $a \in \mathcal{I}(t)$, $t_a \leq_{\text{WF}} u_a$, and thus clearly $t_a \sqsubseteq_{\text{WF}} \tau u_a$. By induction, $\vdash t_a \preceq \tau u_a$ and hence $\vdash at_a \preceq au_a$. It follows that

$$\vdash t = \sum_{a \in A_L} at_a + V_L \preceq \sum_{a \in A_L} au_a + V_L = \sum_{a \in A_M} au_a + V_M = u$$

(b) Both t and u are τ normal forms:

$$t = \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} at_a + V_L \right)$$

and

$$u = \sum_{M \in \mathcal{M}} \tau \left(\sum_{a \in A_M} au_a + V_M \right)$$

By Lem. 3, for each $a \in \mathcal{I}(t)$, $t_a \leq_{\text{WF}} u_a$, and thus clearly $t_a \sqsubseteq_{\text{WF}} \tau u_a$. By induction, $\vdash t_a \preceq \tau u_a$. By these inequalities, together with D4,

$$\vdash t \preceq \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} au_a + V_L \right) + u \quad (1)$$

We now show that $\mathcal{L} \subseteq \mathcal{M}$. Take any $L \in \mathcal{L}$, pick $b \in A \setminus A_L$, and consider the closed substitution $\sigma(z) = \mathbf{0}$ for any $z \in V_L$, and $\sigma(z) = b\mathbf{0}$ for $z \notin V_L$. Since $\sigma(t) \xrightarrow{\tau} \sigma(\sum_{a \in A_L} at_a)$ and $\sigma(t) \sqsubseteq_{\text{WF}} \sigma(u)$, there exists an $M \in \mathcal{M}$ with $A_M \subseteq A_L$ and $V_M \subseteq V_L$. Since also $L \subseteq L(t) \subseteq L(u)$, and \mathcal{M} is saturated, it follows that $L \in \mathcal{M}$. Hence, $\mathcal{L} \subseteq \mathcal{M}$.

Since $\mathcal{L} \subseteq \mathcal{M}$,

$$\sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} au_a + V_L \right) + u = u \quad (2)$$

By (1) and (2), $\vdash t \preceq u$.

(c) t is an action normal form and u is a τ normal form. Then $\tau t \sqsubseteq_{\text{WF}} u$. Note that τt is a τ normal form, so according to the previous case,

$$\vdash \tau t \preceq u$$

By WF3,

$$\vdash t \preceq \tau t \preceq u$$

2. $\mathcal{I}(t) = A$. Note that in this case, $|A| < \infty$. So, according to Theo. 3, axiom WF_A is at our disposal. As before, we distinguish three cases.

(a) Both t and u are action normal forms. Since $L(t) \subseteq L(u)$ we have $t = \sum_{a \in A} at_a + W$ and $u = \sum_{a \in A} au_a + X$ with $W \subseteq X$. By WF_A ,

$$\vdash \sum_{a \in A} at_a \preceq \sum_{a \in A} at_a + u$$

By Lem. 3, for each $a \in A$, $t_a \leq_{\text{WF}} u_a$, and thus clearly $t_a \sqsubseteq_{\text{WF}} \tau u_a$. By induction, $\vdash t_a \preceq \tau u_a$. It follows, using $W \subseteq X$, that

$$\vdash t = \sum_{a \in A} at_a + W \preceq \sum_{a \in A} au_a + u + W = u$$

(b) Both t and u are τ normal forms.

$$t = \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} at_a + V_L \right)$$

and

$$u = \sum_{M \in \mathcal{M}} \tau \left(\sum_{a \in A_M} au_a + V_M \right)$$

By D1 and WF_A (clearly, in this case $A_{L(t)} = A$),

$$\vdash t \approx t + \sum_{a \in A} at_a \preceq t + \sum_{a \in A} at_a + u \quad (3)$$

By Lem. 3, for each $a \in A$, $t_a \leq_{\text{WF}} u_a$, and thus clearly $t_a \sqsubseteq_{\text{WF}} \tau u_a$. By induction, $\vdash t_a \preceq \tau u_a$. By these inequalities, together with (3),

$$\vdash t \preceq \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} au_a + V_L \right) + \sum_{a \in A} au_a + u$$

So by D1,

$$\vdash t \preceq \sum_{L \in \mathcal{L}} \tau \left(\sum_{a \in A_L} au_a + V_L \right) + u \quad (4)$$

Now for $L \in \mathcal{L}$ with $A_L \neq A$ we have $L \in \mathcal{M}$ using the same reasoning as in 1(b). For $L \in \mathcal{L}$ with $A_L = A$ we have $V_L \subseteq V_{L(t)} \subseteq V_{L(u)}$. By WF_A we have

$$\vdash \tau \left(\sum_{a \in A_L} au_a + V_L \right) \preceq \tau \left(\sum_{a \in A} au_a + V_{L(u)} \right) \quad (5)$$

As the latter is a summand of u we obtain $t \preceq u$.

(c) t is an action normal form and u is a τ normal form. This can be dealt with as in case 1(c).

This completes the proof. \square

3.4 Weak Failures Equivalence

In [2, 10] an algorithm is presented which takes as input a sound and ground-complete inequational axiomatization E for BCCSP modulo a preorder \sqsubseteq which *includes the ready simulation preorder* and is *initials preserving*,¹ and generates as output an equational axiomatization $\mathcal{A}(E)$ which is sound and ground-complete for BCCSP modulo the corresponding equivalence—its kernel: $\sqsubseteq \cap \sqsubseteq^{-1}$. Moreover, if the original axiomatization E is ω -complete, so is the resulting axiomatization. The axiomatization $\mathcal{A}(E)$ generated by the algorithm from E contains the axioms A1-4 for bisimulation equivalence and the axioms $\beta(\alpha x + z) + \beta(\alpha x + \alpha y + z) \approx \beta(\alpha x + \alpha y + z)$ for $\alpha, \beta \in A_\tau$ that are valid in ready simulation semantics, together with the following equations, for each inequational axiom $t \preceq u$ in E :

- $t + u \approx u$; and
- $\alpha(t + x) + \alpha(u + x) \approx \alpha(u + x)$ (for each $\alpha \in A_\tau$, and some variable x that does not occur in $t + u$).

Moreover, if E contains an equation (formally abbreviating two inequations), this equation is logically equivalent to the four equations in $\mathcal{A}(E)$ that are derived from it, and hence can be incorporated in the equational axiomatization unmodified.

Recently, we lifted this result to weak semantics [6], which makes the aforementioned algorithm applicable to all 87 preorders surveyed in [12] that are at least as coarse as the ready simulation preorder. Namely, among others, we show that

Theorem 4. *Let \sqsubseteq be a weak initials preserving precongruence² that contains the strong ready simulation preorder \sqsubseteq_{RS} and satisfies T2 (the second τ -law of CCS: $\tau x \approx \tau x + x$), and let E be a sound and ground-complete axiomatization of \sqsubseteq . Then $\mathcal{A}(E)$ is a sound and ground-complete axiomatization of the kernel of \sqsubseteq . Moreover, if E is ω -complete, then so is $\mathcal{A}(E)$.*

It is straightforward to check that weak failures meets the prerequisites of Theo. 4, and thus we can run the algorithm and obtain the axiomatization in Tab. 3 for weak failures equivalence. After simplification and omission of redundant axioms, we obtain the axiomatization in Tab. 4.

Lemma 4. *The axioms in Tab. 3 are derivable from the axioms in Tab. 4 together with A1-4.*

Proof. WF1 is unmodified. WF2^a and WF3^a can be trivially derived from WFE2. WF_A^a is derivable using A3.

To proceed, we have that WFE2 $\vdash \tau\tau x \approx \tau x$ (namely by substituting τx for y and invoking D1, which is also derivable from WFE2) and hence also WFE2 \vdash D2 (namely by substituting τx for x in WFE2 and invoking D1); using D2, the instances of WF2^b

¹ meaning that $p \sqsubseteq q$ implies that $I(p) \subseteq I(q)$, where the set $I(p)$ of *strongly* initial actions is $I(p) = \{\alpha \in A_\tau \mid p \xrightarrow{\alpha}\}$

² meaning that $p \sqsubseteq q$ implies that $\mathcal{I}_\tau(p) \subseteq \mathcal{I}_\tau(q)$, where the set $\mathcal{I}_\tau(p)$ of *weak* initial actions is $\mathcal{I}_\tau(p) = \{\alpha \in A_\tau \mid p \Rightarrow^\alpha\}$

WF1	$ax + ay \approx a(\tau x + \tau y)$
WF2 ^a	$\tau(x + y) + \tau x + y \approx \tau x + y$
WF2 ^b	$\alpha(\tau(x + y) + z) + \alpha(\tau x + y + z) \approx \alpha(\tau x + y + z)$
WF3 ^a	$x + \tau x + y \approx \tau x + y$
WF3 ^b	$\alpha(x + z) + \alpha(\tau x + y + z) \approx \alpha(\tau x + y + z)$
RS	$\beta(\alpha x + z) + \beta(\alpha x + \alpha y + z) \approx \beta(\alpha x + \alpha y + z)$
WF _A ^a	$\sum_{a \in A} ax_a + \sum_{a \in A} ax_a + y \approx \sum_{a \in A} ax_a + y$
WF _A ^b	$\beta(\sum_{a \in A} ax_a + z) + \beta(\sum_{a \in A} ax_a + y + z) \approx \beta(\sum_{a \in A} ax_a + y + z)$

Table 3. Axiomatization generated by the algorithm

WF1	$ax + ay \approx a(\tau x + \tau y)$
WFE2	$\tau(x + y) + \tau x \approx \tau x + y$
WFE3	$ax + \tau(ay + z) \approx \tau(ax + ay + z)$
WFE _A	$\tau(\sum_{a \in A} ax_a + z) + \tau(\sum_{a \in A} ax_a + y + z) \approx \tau(\sum_{a \in A} ax_a + y + z)$

Table 4. Axiomatization for weak failures equivalence

and WF3^b with $\alpha = \tau$, as well as the instance of RS with $\beta = \alpha = \tau$, are derivable from WFE2.

The instances of WF2^b and WF3^b with $\alpha \neq \tau$, are derivable from WF1 and the instances with $\alpha = \tau$; the same holds for the instances of RS and WF_A^b with $\beta \neq \tau$.

Finally in the remaining instances of RS (with $\beta = \tau$ and $\alpha = a \in A$), we have WFE2 $\vdash \tau(ax + z) + \tau(ax + ay + z) \approx \tau(ax + z) + ay$, and thus they can be derived from WFE3. The instance of WF_A^b with $\beta = \tau$ is exactly WFE_A. \square

The axioms WF1, WFE2-3 already appeared in [13]. A1-4+WF1+WFE2-3 is sound and ground-complete for BCCS modulo \equiv_{WF} (see also [13, 6]). By Theo. 2 and Theo. 3 (together with Lem. 4), we have:

Corollary 1. *If $|A| = \infty$, then the axiomatization A1-4+WF1+WFE2-3 is ω -complete for BCCS(A) modulo \equiv_{WF} .*

Corollary 2. *If $|A| < \infty$, then the axiomatization A1-4+WF1+WFE2-3+WFE_A is ω -complete for BCCS(A) modulo \equiv_{WF} .*

4 Weak Impossible Futures Semantics

Weak impossible futures semantics is closely related to weak failures semantics. Only, instead of the set of actions in the second argument of a weak failure pair (see Def. 2), an impossible future pair contains a set of *traces*.

Definition 5 (Weak impossible futures).

- A sequence $a_1 \cdots a_k \in A^*$, with $k \geq 0$, is a *trace* of a process p_0 if there is a path $p_0 \xrightarrow{a_1} \cdots \xrightarrow{a_k} p_k$; it is a *completed trace* of p_0 if moreover $\mathcal{I}(p_k) = \emptyset$. Let $\mathcal{T}(p)$ denote the set of traces of process p , and $\mathcal{CT}(p)$ its set of completed traces.

- A pair $(a_1 \cdots a_k, B)$, with $k \geq 0$ and $B \subseteq A^*$, is a *weak impossible future* of a process p_0 if there is a path $p_0 \Rightarrow^{a_1} \Rightarrow \cdots \Rightarrow^{a_k} \Rightarrow p_k$ with $\mathcal{T}(p_k) \cap B = \emptyset$.
- The *weak impossible futures preorder* \sqsubseteq_{WIF} is given by $p \sqsubseteq_{\text{WIF}} q$ iff (1) the weak impossible futures of p are also weak impossible futures of q , (2) $\mathcal{T}(p) = \mathcal{T}(q)$ and (3) $p \xrightarrow{\tau}$ implies that $q \xrightarrow{\tau}$.
- *Weak impossible futures equivalence* \equiv_{WIF} is defined as $\sqsubseteq_{\text{WIF}} \cap \sqsubseteq_{\text{WIF}}^{-1}$.

\sqsubseteq_{WIF} is a precongruence, and \equiv_{WF} a congruence, for BCCS [22]. The requirement (2) $\mathcal{T}(p) = \mathcal{T}(q)$ is necessary for this precongruence property. Without it we would have $\tau a \mathbf{0} \sqsubseteq \tau a \mathbf{0} + b \mathbf{0}$ but $c(\tau a \mathbf{0}) \not\sqsubseteq c(\tau a \mathbf{0} + b \mathbf{0})$.

A sound and ground-complete axiomatization for \sqsubseteq_{WIF} is obtained by replacing axiom WF3 in Tab. 1 by the following axiom (cf. [22], where a slightly more complicated, but equivalent, axiomatization is given):

$$\text{WIF3} \quad x \preceq \tau x$$

However, surprisingly, there is no finite sound and ground-complete axiomatization for \equiv_{WIF} . We will show this in Sec. 4.1. A similar difference between the impossible futures preorder and equivalence in the concrete case (so in the absence of τ) was found earlier in [4]. We note that, since weak impossible futures semantics is not coarser than ready simulation semantics, the algorithm from [2, 10, 6] to generate an axiomatization for the equivalence from the one for the preorder, does not work in this case.

In Sec. 4.2 we establish that the sound and ground-complete axiomatization for BCCS modulo \sqsubseteq_{WIF} is ω -complete in case $|A| = \infty$, and in Sec. 4.3 that there is no such finite basis for the inequational theory of BCCS modulo \sqsubseteq_{WIF} in case $|A| < \infty$. Again, these results correspond to (in)axiomatizability results for the impossible futures preorder in the concrete case [4], with very similar proofs.

4.1 Nonexistence of an Axiomatization for Equivalence

We now prove that for any (nonempty) A there does *not* exist any finite, sound, ground-complete axiomatization for BCCS(A) modulo \equiv_{WIF} . The cornerstone for this negative result is the following infinite family of closed equations, for $m \geq 0$:

$$\tau a^{2m} \mathbf{0} + \tau(a^m \mathbf{0} + a^{2m} \mathbf{0}) \approx \tau(a^m \mathbf{0} + a^{2m} \mathbf{0})$$

It is not hard to see that they are sound modulo \equiv_{WIF} . We start with a few lemmas.

Lemma 5. *If $p \sqsubseteq_{\text{WIF}} q$ then $\mathcal{CT}(p) \subseteq \mathcal{CT}(q)$.*

Proof. A process p has a completed trace $a_1 \cdots a_k$ iff it has a weak impossible future $(a_1 \cdots a_k, A)$. □

Lemma 6. *Suppose $t \sqsubseteq_{\text{WIF}} u$. Then for any t' with $t \Rightarrow^{\tau} t'$ there is some u' with $u \Rightarrow^{\tau} u'$ such that $\text{var}(u') \subseteq \text{var}(t')$.*

Proof. Let $t \Rightarrow^{\tau} t'$. Fix some $m > |t|$, and consider the closed substitution ρ defined by $\rho(x) = \mathbf{0}$ if $x \in \text{var}(t')$ and $\rho(x) = a^m \mathbf{0}$ if $x \notin \text{var}(t')$. Since $\rho(t) \Rightarrow \rho(t')$ with $|\rho(t')| = |t'| < m$, and $\rho(t) \sqsubseteq_{\text{WIF}} \rho(u)$, clearly $\rho(u) \Rightarrow q$ for some q with $|q| < m$. From the definition of ρ it then follows that there must exist $u \Rightarrow u'$ with $\text{var}(u') \subseteq \text{var}(t')$. In case $u \Rightarrow^{\tau} u'$ we are done, so assume $u' = u$. Let σ be the substitution with $\sigma(x) = \mathbf{0}$ for all $x \in V$. Since $\sigma(t) \xrightarrow{\tau}$ and $t \sqsubseteq_{\text{WIF}} u$ we have $\sigma(u) \xrightarrow{\tau}$, so $u \xrightarrow{\tau} u''$ for some u'' . Now $\text{var}(u'') \subseteq \text{var}(u) = \text{var}(u') \subseteq \text{var}(t')$. \square

Lemma 7. Assume that, for terms t, u , closed substitution σ , action a and integer m :

1. $t \equiv_{\text{WIF}} u$;
2. $m > |u|$;
3. $\mathcal{CT}(\sigma(u)) \subseteq \{a^m, a^{2m}\}$; and
4. there is a closed term p' such that $\sigma(t) \Rightarrow^{\tau} p'$ and $\mathcal{CT}(p') = \{a^{2m}\}$.

Then there is a closed term q' such that $\sigma(u) \Rightarrow^{\tau} q'$ and $\mathcal{CT}(q') = \{a^{2m}\}$.

Proof. According to proviso (4) of the lemma, we can distinguish two cases.

- There exists some $x \in V$ such that $t \Rightarrow t'$ with $t' = t'' + x$ and $\sigma(x) \Rightarrow^{\tau} p'$ where $\mathcal{CT}(p') = \{a^{2m}\}$. Consider the closed substitution ρ defined by $\rho(x) = a^m \mathbf{0}$ and $\rho(y) = \mathbf{0}$ for any $y \neq x$. Then $a^m \in \mathcal{CT}(\rho(t)) = \mathcal{CT}(\rho(u))$, using Lem. 5, and this is only possible if $u \Rightarrow u'$ for some $u' = u'' + x$. Hence $\sigma(u) \Rightarrow^{\tau} p'$.
- $t \Rightarrow^{\tau} t'$ with $\mathcal{CT}(\sigma(t')) = \{a^{2m}\}$. Since $|t'| \leq |t| = |u| < m$, clearly, for any $x \in \text{var}(t')$, either $|\sigma(x)| = 0$ or $\text{norm}(\sigma(x)) > m$, where $\text{norm}(p)$ denotes the length of the shortest completed trace of p . Since $t \equiv_{\text{WIF}} u$, by Lem. 6, $u \Rightarrow^u u'$ with $\text{var}(u') \subseteq \text{var}(t')$. Hence, for any $x \in \text{var}(u')$, either $|\sigma(x)| = 0$ or $\text{norm}(\sigma(x)) > m$. Since $|u'| < m$, $a^m \notin \mathcal{CT}(\sigma(u'))$. It follows from $\mathcal{CT}(\sigma(u)) \subseteq \{a^m, a^{2m}\}$ that $\mathcal{CT}(\sigma(u')) = \{a^{2m}\}$. And $u \Rightarrow^{\tau} u'$ implies $\sigma(u) \Rightarrow^{\tau} \sigma(u')$. \square

Lemma 8. Assume that, for E an axiomatization sound for \sqsubseteq_{WIF} , closed terms p, q , closed substitution σ , action a and integer m :

1. $E \vdash p \approx q$;
2. $m > \max\{|u| \mid t \approx u \in E\}$;
3. $\mathcal{CT}(q) \subseteq \{a^m, a^{2m}\}$; and
4. there is a closed term p' such that $p \Rightarrow^{\tau} p'$ and $\mathcal{CT}(p') = \{a^{2m}\}$.

Then there is a closed term q' such that $q \Rightarrow^{\tau} q'$ and $\mathcal{CT}(q') = \{a^{2m}\}$.

Proof. By induction on the derivation of $E \vdash p \approx q$.

- Suppose $E \vdash p \approx q$ because $\sigma(t) = p$ and $\sigma(u) = q$ for some $t \approx u \in E$ or $u \approx t \in E$ and closed substitution σ . The claim then follows by Lem. 7.
- Suppose $E \vdash p \approx q$ because $E \vdash p \approx r$ and $E \vdash r \approx q$ for some r . Since $r \equiv_{\text{WIF}} q$, by proviso (3) of the lemma and Lem. 5, $\mathcal{CT}(r) \subseteq \{a^m, a^{2m}\}$. Since there is a p' such that $p \Rightarrow^{\tau} p'$ with $\mathcal{CT}(p') = \{a^{2m}\}$, by induction, there is an r' such that $r \Rightarrow^{\tau} r'$ and $\mathcal{CT}(r') = \{a^{2m}\}$. Hence, again by induction, there is a q' such that $q \Rightarrow^{\tau} q'$ and $\mathcal{CT}(q') = \{a^{2m}\}$.

- Suppose $E \vdash p \approx q$ because $p = p_1 + p_2$ and $q = q_1 + q_2$ with $E \vdash p_1 \approx q_1$ and $E \vdash p_2 \approx q_2$. Since there is a p' such that $p \Rightarrow^{\tau} p'$ and $\mathcal{CT}(p') = \{a^{2m}\}$, either $p_1 \Rightarrow^{\tau} p'$ or $p_2 \Rightarrow^{\tau} p'$. Assume, without loss of generality, that $p_1 \Rightarrow^{\tau} p'$. By induction, there is a q' such that $q_1 \Rightarrow^{\tau} q'$ and $\mathcal{CT}(q') = \{a^{2m}\}$. Now $q \Rightarrow^{\tau} q'$.
- Suppose $E \vdash p \approx q$ because $p = cp_1$ and $q = cq_1$ with $c \in A$ and $E \vdash p_1 \approx q_1$. In this case, proviso (4) of the lemma can not be met.
- Suppose $E \vdash p \approx q$ because $p = \tau p_1$ and $q = \tau q_1$ with $E \vdash p_1 \approx q_1$. By proviso (4) of the lemma, either $\mathcal{CT}(p_1) = \{a^{2m}\}$ or there is a p' such that $p_1 \Rightarrow^{\tau} p'$ and $\mathcal{CT}(p') = \{a^{2m}\}$. In the first case, $q \Rightarrow^{\tau} q_1$ and $\mathcal{CT}(q_1) = \{a^{2m}\}$ by Lem. 5. In the second, by induction, there is a q' such that $q_1 \Rightarrow^{\tau} q'$ and $\mathcal{CT}(q') = \{a^{2m}\}$. Again $q \Rightarrow^{\tau} q'$. \square

Theorem 5. *There is no finite, sound, ground-complete axiomatization for $\text{BCCS}(A)$ modulo \equiv_{WIF} .*

Proof. Let E be a finite axiomatization over $\text{BCCS}(A)$ that is sound modulo \equiv_{WIF} . Let m be greater than the depth of any term in E . Clearly, there is no term r such that $\tau(a^m \mathbf{0} + a^{2m} \mathbf{0}) \Rightarrow^{\tau} r$ and $\mathcal{CT}(r) = \{a^{2m}\}$. So according to Lem. 8, the closed equation $\tau a^{2m} \mathbf{0} + \tau(a^m \mathbf{0} + a^{2m} \mathbf{0}) \approx \tau(a^m \mathbf{0} + a^{2m} \mathbf{0})$ cannot be derived from E . Nevertheless, it is valid modulo \equiv_{WIF} . \square

In the same way as above, one can establish the nonderivability of the equations $a^{2m+1} \mathbf{0} + a(a^m \mathbf{0} + a^{2m} \mathbf{0}) \approx a(a^m \mathbf{0} + a^{2m} \mathbf{0})$ from any given finite equational axiomatization sound for \equiv_{WIF} . As these equations are valid modulo (strong) 2-nested simulation equivalence, this negative result applies to all BCCS-congruences that are at least as fine as weak impossible futures equivalence and at least as coarse as strong 2-nested simulation equivalence. Note that the corresponding result of [1] can be inferred.

4.2 A Finite Basis for Preorder if $|A| = \infty$

In this section, we show that A1-4+WF1-2+WIF3 is ω -complete in case $|A| = \infty$. Note that this result was originally obtained in [22]. However, our proof is much simpler. First, let us note that A1-4+WF1-2+WIF3 \vdash D1, D2, D5.

Lemma 9. *For any closed terms p, q , if $p \sqsubseteq_{\text{WIF}} q$, then A1-4+WF1-2+WIF3 $\vdash p \preceq q$.*

Proof. Let $p \sqsubseteq_{\text{WIF}} q$. We prove $\vdash p \preceq q$ by induction on $|p| + |q|$. We distinguish two cases:

- $q \not\neq$. Then $p \not\neq$ since $p \sqsubseteq_{\text{WIF}} q$. Suppose $p = \sum_{i \in I} a_i p_i$ and $q = \sum_{j \in J} b_j q_j$. Clearly, we have $\mathcal{I}(p) = \mathcal{I}(q)$. By D5, we have

$$\vdash p \approx \sum_{a \in \mathcal{I}(p)} a \left(\sum_{a_i = a, i \in I} \tau p_i \right)$$

and

$$\vdash q \approx \sum_{a \in \mathcal{I}(p)} a \left(\sum_{b_j = a, j \in J} \tau q_j \right)$$

Since $p \sqsubseteq_{\text{WIF}} q$, for each $a \in \mathcal{I}(p)$, the following relation holds:

$$\sum_{a_i=a, i \in I} \tau p_i \sqsubseteq \sum_{b_j=a, j \in J} \tau q_j$$

By induction,

$$\vdash \sum_{a_i=a, i \in I} \tau p_i \preceq \sum_{b_j=a, j \in J} \tau q_j$$

and thus

$$\vdash a \left(\sum_{a_i=a, i \in I} \tau p_i \right) \preceq a \left(\sum_{b_j=a, j \in J} \tau q_j \right)$$

Summing these up for $a \in \mathcal{I}(p)$, we obtain that

$$\vdash p \preceq q$$

- $q \xrightarrow{\tau}$. By D2, we can write $p \approx \sum_{i \in I} \alpha_i p_i$ and $q \approx \sum_{j \in J} \beta_j q_j$ such that for each $\alpha_i = \tau$ (resp. $\beta_j = \tau$), $p_i \not\xrightarrow{\tau}$ (resp. $q_j \not\xrightarrow{\tau}$). Applying D1, for each $i \in I$ with $\alpha_i = \tau$, the summands of p_i are also made summands of p , and likewise for q . (*)

For each $i \in I$ with $\alpha_i = \tau$ we have $p \xrightarrow{\tau} p_i$. Since $p \sqsubseteq_{\text{WIF}} q$ and no q_j with $\beta_j = \tau$ contains a τ -summand, either $\mathcal{T}(q) \subseteq \mathcal{T}(p_i)$ or there exists $q \xrightarrow{\tau} q_j$ such that $\mathcal{T}(q_j) \subseteq \mathcal{T}(p_i)$. Since $q \xrightarrow{\tau}$, in either case there exists some $j_i \in J$ such that $b_{j_i} = \tau$ and $\mathcal{T}(q_{j_i}) \subseteq \mathcal{T}(p_i)$. It follows that

$$p_i \sqsubseteq_{\text{WIF}} p_i + q_{j_i}$$

Since $p_i \not\xrightarrow{\tau}$ and $q_{j_i} \not\xrightarrow{\tau}$, by the previous case,

$$\vdash p_i \preceq p_i + q_{j_i}$$

Hence by WF2,

$$\vdash \tau p_i \preceq \tau(p_i + q_{j_i}) \preceq p_i + \tau q_{j_i}$$

and thus

$$\vdash p = \sum_{\alpha_i=\tau} \tau p_i + \sum_{a \in \mathcal{I}(p)} \sum_{\alpha_i=a, i \in I} a p_i \preceq \sum_{\alpha_i=\tau} (p_i + \tau q_{j_i}) + \sum_{a \in \mathcal{I}(p)} \sum_{\alpha_i=a, i \in I} a p_i$$

By (*),

$$\vdash \sum_{\alpha_i=\tau} p_i + \sum_{a \in \mathcal{I}(p)} \sum_{\alpha_i=a, i \in I} a p_i \approx \sum_{a \in \mathcal{I}(p)} \sum_{\alpha_i=a, i \in I} a p_i$$

Since $p \sqsubseteq_{\text{WIF}} q$, $\mathcal{I}(p) = \mathcal{I}(q)$. Using (*), it is easy to see that for each $a \in \mathcal{I}(p)$,

$$\sum_{\alpha_i=a, i \in I} a p_i \sqsubseteq_{\text{WIF}} \sum_{\beta_j=a, j \in J} a q_j$$

So by the previous case,

$$\vdash \sum_{\alpha_i=a, i \in I} a p_i \preceq \sum_{\beta_j=a, j \in J} a q_j$$

It follows that

$$\vdash p \preceq \sum_{\alpha_i=\tau} \tau q_{j_i} + \sum_{a \in \mathcal{I}(p)} \sum_{\alpha_i=a, i \in I} a p_i \preceq \sum_{\alpha_i=\tau} \tau q_j + \sum_{a \in \mathcal{I}(p)} \sum_{\beta_j=a, j \in J} a q_j$$

By WIF3,

$$\vdash \sum_{\alpha_i=\tau} \tau q_j + \sum_{a \in \mathcal{I}(p)} \sum_{\beta_j=a, j \in J} a q_j \preceq q$$

Hence $\vdash p \preceq q$ □

With this ground-completeness result at hand, it is straightforward to apply the *inverted substitution* technique of Groote [16] to derive:

Theorem 6. *If $|A| = \infty$, then A1-4+WF1-2+WIF3 is ω -complete for $\text{BCCS}(A)$ modulo \sqsubseteq_{WIF} .*

Proof. Given an inequational axiomatization E and open terms t, u such that $E \vdash \sigma(t) \preceq \sigma(u)$ for all closed substitutions σ , the technique of inverted substitutions is a method to prove $E \vdash t \preceq u$. It does so by means of a closed substitution ρ encoding open terms into closed terms, and an decoding operation R that turns closed terms back into open terms. By assumption we have $E \vdash \rho(t) \preceq \rho(u)$. The pair (ρ, R) should be chosen in such a way that, in essence, applying R to all terms occurring in a proof of $\rho(t) \preceq \rho(u)$ yields a proof of $t \preceq u$. As observed in [16], this technique is applicable when three conditions are met, one of which being that $R(\rho(t)) = t$ and $R(\rho(u)) = u$. In fact, [16] dealt with equational logic only, but the very same reasoning applies to inequational logic [4].

Here we use the same pair (ρ, R) that was used by Groote to obtain most of the applications of the technique in [16]—it could be called the *default* (inverted) substitution. It is obtained by selecting for each variable $x \in V$ an action $a_x \in A$, not occurring in t or u . This is possible because $|A| = \infty$. Now the default substitution ρ is given by $\rho(x) = a_x \mathbf{0}$ and the default inverted substitution R replaces any maximal subterm of the form $a_x p$ into the variable x . Groote showed that with this particular (inverted) substitution, 2 out of his 3 conditions are always met, and the third one simply says that for each axiom $t \preceq u$ in E we should have that $E \vdash R(t) \preceq R(u)$. This condition is clearly met for the axioms A1-4+WF1-2+WIF3, and hence this axiomatization is ω -complete. □

Note that we could have used the same method to obtain Theo. 2, but not Theo. 3.

4.3 Nonexistence of a Finite Basis for Preorder if $|A| < \infty$

$1 < |A| < \infty$. We prove that the inequational theory of $\text{BCCS}(A)$ modulo \sqsubseteq_{WIF} does *not* have a finite basis in case of a finite alphabet with at least two elements. The cornerstone for this negative result is the following infinite family of inequations, for $m \geq 0$:

$$\tau(a^m x) + \Phi_m \preceq \Phi_m$$

with

$$\Phi_m = \tau(a^m x + x) + \sum_{b \in A} \tau(a^m x + a^m b \mathbf{0})$$

It is not hard to see that these inequations are sound modulo \sqsubseteq_{WIF} . Namely, given a closed substitution ρ , we have $\mathcal{T}(\rho(\tau(a^m x))) \subseteq \mathcal{T}(\rho(\Phi_m))$ and $\rho(\Phi_m) \xrightarrow{\tau}$. To argue that $\rho(\tau(a^m x) + \Phi_m)$ and $\rho(\Phi_m)$ have the same impossible futures, we only need to consider the transition $\rho(\tau(a^m x) + \Phi_m) \xrightarrow{\tau} a^m \rho(x)$ (all other cases being trivial). If $\rho(x) = \mathbf{0}$, then $\rho(\Phi_m) \xrightarrow{\tau} a^m \mathbf{0} + \mathbf{0}$ generates the same impossible futures (ε, B) . If, on the other hand, $b \in \mathcal{I}(\rho(x))$ for some $b \in A$, then $\mathcal{T}(a^m \rho(x) + a^m b \mathbf{0}) = \mathcal{T}(a^m \rho(x))$, so $\rho(\Phi_m) \xrightarrow{\tau} a^m \rho(x) + a^m b \mathbf{0}$ generates the same impossible futures (ε, B) .

We have already defined the traces and completed traces of closed terms. Now we extend these definitions to open terms by allowing (completed) traces of the form $a_1 \cdots a_k x \in A^*V$. We do this by treating each variable occurrence x in a term as if it were a subterm $x\mathbf{0}$ with x a visible action, and then apply Def. 5. Under this convention, $\mathcal{CT}(\Phi_m) = \{a^m x, x, a^m b \mid b \in A\}$. We write $\mathcal{T}_V(t)$ for the set of traces of t that end in a variable, and $\mathcal{T}_A(t)$ for ones that end in an action.

Observation 1. *Let $m > |t|$ or $a_m \in V$. Then $a_1 \cdots a_m \in \mathcal{T}(\sigma(t))$ iff there is a $k < m$ and $y \in V$ such that $a_1 \cdots a_k y \in \mathcal{T}_V(t)$ and $a_{k+1} \cdots a_m \in \mathcal{T}(\sigma(y))$.*

Lemma 10. *If $|A| > 1$ and $t \sqsubseteq_{\text{WIF}} u$ then $\mathcal{T}_A(t) = \mathcal{T}_A(u)$ and $\mathcal{T}_V(t) = \mathcal{T}_V(u)$.*

Proof. Let σ be the closed substitution defined by $\sigma(x) = \mathbf{0}$ for all $x \in V$. Then $t \sqsubseteq_{\text{WIF}} u$ implies $\sigma(t) \sqsubseteq_{\text{WIF}} \sigma(u)$ and hence $\mathcal{T}_A(t) = \mathcal{T}(\sigma(t)) = \mathcal{T}(\sigma(u)) = \mathcal{T}_A(u)$ by Def. 5.

For the second statement fix distinct actions $a, b \in A$ and an injection $\lceil \cdot \rceil : V \rightarrow \mathbb{Z}_{>0}$ (which exists because V is countable). Let $m = |u| + 1 = |t| + 1$. Define the closed substitution ρ by $\rho(z) = a^{\lceil z \rceil, m} b \mathbf{0}$ for all $z \in V$. Again, by Def. 5, $t \sqsubseteq_{\text{WIF}} u$ implies $\mathcal{T}(\rho(t)) = \mathcal{T}(\rho(u))$. By Obs. 1, for all terms v we have $a_1 \cdots a_k y \in \mathcal{T}_V(v)$ iff $a_1 \cdots a_k a^{\lceil y \rceil, m} b \in \mathcal{T}(\rho(v))$ with $k < m$. Hence $\mathcal{T}_V(v)$ is completely determined by $\mathcal{T}(\rho(v))$ and thus $\mathcal{T}_V(t) = \mathcal{T}_V(u)$. \square

Lemma 11. *Let $|A| > 1$. Suppose $t \sqsubseteq_{\text{WIF}} u$ and $t \Rightarrow^{\tau} t'$. Then there is a term u' such that $u \Rightarrow^{\tau} u'$ and $\mathcal{T}_V(u') \subseteq \mathcal{T}_V(t')$.*

Proof. Define ρ exactly as in the previous proof. Since $\rho(t) \Rightarrow \rho(t')$ and $t \sqsubseteq_{\text{WIF}} u$ there must be a u' with $\rho(u) \Rightarrow q$ and $\mathcal{T}(q) \subseteq \mathcal{T}(\rho(t'))$. Since $\rho(x)$ is τ -free for $x \in V$ it must be that $q = \rho(u')$ for some term u' with $u \Rightarrow u'$. Given the relationship between $\mathcal{T}_V(v)$ and $\mathcal{T}(\rho(v))$ for terms v observed in the previous proof, it follows that $\mathcal{T}_V(u') \subseteq \mathcal{T}_V(t')$. In case $u \Rightarrow^{\tau} u'$ we are done, so assume $u' = u$. Let σ be the substitution with $\sigma(x) = \mathbf{0}$ for all $x \in V$. Since $\sigma(t) \xrightarrow{\tau}$ and $t \sqsubseteq_{\text{WIF}} u$ we have $\sigma(u) \xrightarrow{\tau}$, so $u \xrightarrow{\tau} u''$ for some u'' . Now $\mathcal{T}_V(u'') \subseteq \mathcal{T}_V(u) = \mathcal{T}_V(u') \subseteq \mathcal{T}_V(t')$. \square

Lemma 12. *Let $|A| > 1$. Assume that, for some terms t, u , substitution σ , action a and integer m :*

1. $t \sqsubseteq_{\text{WIF}} u$;
2. $m \geq |u|$; and

3. $\sigma(t) \Rightarrow^{\tau} \hat{t}$ for a term \hat{t} without traces ax for $x \in V$ or $a^m b$ for $b \in A$.

Then $\sigma(u) \Rightarrow^{\tau} \hat{u}$ for a term \hat{u} without traces ax for $x \in V$ or $a^m b$ for $b \in A$.

Proof. Based on proviso (3) there are two cases to consider.

- $y \in \mathcal{T}_V(t)$ for some $y \in V$ and $\sigma(y) \Rightarrow^{\tau} \hat{t}$. In that case $y \in \mathcal{T}_V(u)$ by Lem. 10, so $\sigma(u) \Rightarrow^{\tau} \hat{t}$.
- $t \Rightarrow^{\tau} t'$ for some term t' such that $\hat{t} = \sigma(t)$. By Lem. 11 there is a term u' with $u \Rightarrow^{\tau} u'$ and $\mathcal{T}_V(u') \subseteq \mathcal{T}_V(t')$. Take $\hat{u} = \sigma(u')$. Clearly $\sigma(u) \Rightarrow^{\tau} \sigma(u')$. Suppose $\sigma(u')$ would have a trace $a^m b$. Then, by Obs. 1, there is a $k \leq m$ and $y \in V$ such that $a^k y \in \mathcal{T}_V(u')$ and $a^{m-k} b \in \mathcal{T}(\sigma(y))$. Since $\mathcal{T}_V(u') \subseteq \mathcal{T}_V(t')$ we have $a^m b \in \mathcal{T}(\sigma(t'))$, which is a contradiction. The case $ax \in \mathcal{T}(\sigma(u))$ is dealt with in the same way. \square

Lemma 13. Let $|A| > 1$ and let E be an axiomatization sound for \sqsubseteq_{WIF} . Assume that, for some terms v, w , action a and integer m :

1. $E \vdash v \preceq w$;
2. $m \geq \max\{|u| \mid t \preceq u \in E\}$; and
3. $v \Rightarrow^{\tau} \hat{v}$ for a term \hat{v} without traces ax for $x \in V$ or $a^m b$ for $b \in A$.

Then $w \Rightarrow^{\tau} \hat{w}$ for a term \hat{w} without traces ax for $x \in V$ or $a^m b$ for $b \in A$.

Proof. By induction on the derivation of $E \vdash v \preceq w$.

- Suppose $E \vdash v \preceq w$ because $\sigma(t) = v$ and $\sigma(u) = w$ for some $t \preceq u \in E$ and substitution σ . The claim then follows by Lem. 12.
- Suppose $E \vdash v \preceq w$ because $E \vdash v \preceq u$ and $E \vdash u \preceq w$ for some u . By induction, $u \Rightarrow^{\tau} \hat{u}$ for a term \hat{u} without traces ax or $a^m b$. Hence, again by induction, $w \Rightarrow^{\tau} \hat{w}$ for a term \hat{w} without traces ax or $a^m b$.
- Suppose $E \vdash v \preceq w$ because $v = v_1 + v_2$ and $w = w_1 + w_2$ with $E \vdash v_1 \preceq w_1$ and $E \vdash v_2 \preceq w_2$. Since $v \Rightarrow^{\tau} \hat{v}$, either $v_1 \Rightarrow^{\tau} \hat{v}$ or $v_2 \Rightarrow^{\tau} \hat{v}$. Assume, without loss of generality, that $v_1 \Rightarrow^{\tau} \hat{v}$. By induction, $w_1 \Rightarrow^{\tau} \hat{w}$ for a term \hat{w} without traces ax or $a^m b$. Now $w \Rightarrow^{\tau} \hat{w}$.
- Suppose $E \vdash v \preceq w$ because $v = cv_1$ and $w = cw_1$ with $c \in A$ and $E \vdash v_1 \preceq w_1$. In this case, proviso (3) of the lemma can not be met.
- Suppose $E \vdash v \preceq w$ because $v = \tau v_1$ and $w = \tau w_1$ with $E \vdash v_1 \preceq w_1$. Then either $v_1 = \hat{v}$ or $v_1 \Rightarrow^{\tau} \hat{v}$. In the first case, w_1 has no traces ax or $a^m b$ by Lem. 10 and proviso (3) of the lemma; hence w has no such traces either. In the second case, by induction, $w_1 \Rightarrow^{\tau} \hat{w}$ for a term \hat{w} without traces ax or $a^m b$. Again $w \Rightarrow^{\tau} \hat{w}$. \square

Theorem 7. If $1 < |A| < \infty$, then the inequational theory of $\text{BCCS}(A)$ modulo \sqsubseteq_{WIF} does not have a finite basis.

Proof. Let E be a finite axiomatization over $\text{BCCS}(A)$ that is sound modulo \sqsubseteq_{WIF} . Let m be greater than the depth of any term in E . According to Lem. 13, the inequation $\tau(a^m x) + \Phi_m \preceq \Phi_m$ cannot be derived from E . Yet it is sound modulo \sqsubseteq_{WIF} . \square

$|A| = 1$. We prove that the inequational theory of $\text{BCCS}(A)$ modulo \sqsubseteq_{WIF} does *not* have a finite basis in case of a singleton alphabet. The cornerstone for this negative result is the following infinite family of inequations, for $m \geq 0$:

$$a^m x \not\approx a^m x + x$$

If $|A| = 1$, then these inequations are clearly sound modulo \sqsubseteq_{WIF} . Note that given a closed substitution ρ , $\mathcal{T}(\rho(x)) \subseteq \mathcal{T}(\rho(a^m x))$.

Lemma 14. *If $t \sqsubseteq_{\text{WIF}} u$ then $\mathcal{T}_V(t) \subseteq \mathcal{T}_V(u)$.*

Proof. Fix $a \in A$ and an injection $\ulcorner \cdot \urcorner : V \rightarrow \mathbb{Z}_{>0}$. Let $m = |u| + 1$. Define the closed substitution ρ by $\rho(z) = a^{\ulcorner z \urcorner \cdot m} \mathbf{0}$ for all $z \in V$. By Lem. 5, $\mathcal{CT}(\rho(t)) \subseteq \mathcal{CT}(\rho(u))$. Now suppose $a_1 \cdots a_k y \in \mathcal{T}_V(t)$. Then $a_1 \cdots a_k a^{\ulcorner y \urcorner \cdot m} \in \mathcal{CT}(\rho(t)) \subseteq \mathcal{CT}(\rho(u))$ and $k < m$. This is only possible if $a_1 \cdots a_k y \in \mathcal{T}_V(u)$. \square

Lemma 15. *Assume that, for terms t, u , substitution σ , action a , variable x , integer m :*

1. $t \sqsubseteq_{\text{WIF}} u$;
2. $m > |u|$; and
3. $x \in \mathcal{T}_V(\sigma(u))$ and $a^k x \notin \mathcal{T}_V(\sigma(u))$ for $1 \leq k < m$.

Then $x \in \mathcal{T}_V(\sigma(t))$ and $a^k x \notin \mathcal{T}_V(\sigma(t))$ for $1 \leq k < m$.

Proof. Since $x \in \mathcal{T}_V(\sigma(u))$, by Obs. 1 there is a variable y with $y \in \mathcal{T}_V(u)$ and $x \in \mathcal{T}_V(\sigma(y))$. Consider the closed substitution ρ given by $\rho(y) = a^m \mathbf{0}$ and $\rho(z) = \mathbf{0}$ for $z \neq y$. Then $m > |u| = |t|$, and $y \in \mathcal{T}_V(u)$ implies $a^m \in \mathcal{T}(\rho(u)) = \mathcal{T}(\rho(t))$, so by Obs. 1 there is some $k < m$ and $z \in V$ such that $a^k z \in \mathcal{T}_V(t)$ and $a^{m-k} \in \mathcal{T}(\rho(z))$. As $k < m$ it must be that $z = y$. Since $a^k y \in \mathcal{T}_V(t)$ and $x \in \mathcal{T}_V(\sigma(y))$, Obs. 1 implies that $a^k x \in \mathcal{T}_V(\sigma(t))$. By Lem. 14, $a^k x \notin \mathcal{T}_V(\sigma(t))$ for $1 \leq k < m$. Hence we obtain $k = 0$. \square

Lemma 16. *Assume that, for E an axiomatization sound for \sqsubseteq_{WIF} and for terms v, w , action a , variable x and integer m :*

1. $E \vdash v \approx w$;
2. $m > \max\{|u| \mid t \approx u \in E\}$; and
3. $x \in \mathcal{T}_V(w)$ and $a^k x \notin \mathcal{T}_V(w)$ for $1 \leq k < m$.

Then $x \in \mathcal{T}_V(v)$ and $a^k x \notin \mathcal{T}_V(v)$ for $1 \leq k < m$.

Proof. By induction on the derivation of $E \vdash v \approx w$.

- Suppose $E \vdash v \approx w$ because $\sigma(t) = v$ and $\sigma(u) = w$ for some $t \approx u \in E$ and substitution σ . The claim then follows by Lem. 15.
- Suppose $E \vdash v \approx w$ because $E \vdash v \approx u$ and $E \vdash u \approx w$ for some u . By induction, $x \in \mathcal{T}_V(u)$ and $a^k x \notin \mathcal{T}_V(u)$ for $1 \leq k < m$. Hence, again by induction, $x \in \mathcal{T}_V(v)$ and $a^k x \notin \mathcal{T}_V(v)$ for $1 \leq k < m$.

- Suppose $E \vdash v \preceq w$ because $v = v_1 + v_2$ and $w = w_1 + w_2$ with $E \vdash v_1 \preceq w_1$ and $E \vdash v_2 \preceq w_2$. Since $x \in \mathcal{T}_V(w)$, either $x \in \mathcal{T}_V(w_1)$ or $x \in \mathcal{T}_V(w_2)$. Assume, without loss of generality, that $x \in \mathcal{T}_V(w_1)$. Since $a^k x \notin \mathcal{T}_V(w)$ for $1 \leq k < m$, surely $a^k x \notin \mathcal{T}_V(w_1)$ for $1 \leq k < m$. By induction, $x \in \mathcal{T}_V(v_1)$, and hence $x \in \mathcal{T}_V(v)$. For $1 \leq k < m$ we have $a^k x \notin \mathcal{T}_V(w)$ and hence $a^k x \notin \mathcal{T}_V(v)$, by Lem. 14.
- Suppose $E \vdash v \preceq w$ because $v = cv_1$ and $w = cw_1$ with $c \in A$ and $E \vdash v_1 \approx w_1$. In this case, proviso (3) of the lemma can not be met.
- Suppose $E \vdash v \preceq w$ because $v = \tau v_1$ and $w = \tau w_1$ with $E \vdash v_1 \approx w_1$. Then, by proviso (3) of the lemma, $x \in \mathcal{T}_V(w_1)$ and $a^k x \notin \mathcal{T}_V(w_1)$ for $1 \leq k < m$. By induction, $x \in \mathcal{T}_V(v_1)$ and $a^k x \notin \mathcal{T}_V(v_1)$ for $1 \leq k < m$. Hence $x \in \mathcal{T}_V(v)$ and $a^k x \notin \mathcal{T}_V(v)$ for $1 \leq k < m$. \square

Theorem 8. *If $|A| = 1$, then the inequational theory of $\text{BCCS}(A)$ modulo \sqsubseteq_{WIF} does not have a finite basis.*

Proof. Let E be a finite axiomatization over $\text{BCCS}(A)$ that is sound modulo \sqsubseteq_{WIF} . Let m be greater than the depth of any term in E . According to Lem. 16, the inequation $a^m x \preceq a^m x + x$ cannot be derived from E . Yet, since $|A| = 1$, it is sound modulo \sqsubseteq_{WIF} . \square

To conclude this subsection, we have

Theorem 9. *If $|A| < \infty$, then the inequational theory of $\text{BCCS}(A)$ modulo \sqsubseteq_{WIF} does not have a finite basis.*

Concluding, in spite of the close resemblance between weak failures and weak impossible futures semantics, there is a striking difference between their axiomatizability properties.

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