From edge-disjoint paths to independent paths

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Abstract

Let \( f(k) \) denote the maximum such that every simple undirected graph containing two vertices \( s, t \) and \( k \) edge-disjoint \( s-t \) paths, also contains two vertices \( u, v \) and \( f(k) \) independent \( u-v \) paths. Here, a set of paths is independent if none of them contains an interior vertex of another. We prove that

\[
    f(k) = \begin{cases} 
        k & \text{if } k \leq 2, \\
        3 & \text{otherwise.}
    \end{cases}
\]

Since independent paths are edge-disjoint, it is clear that \( f(k) \leq k \) for every positive integer \( k \).

Let \( \mathcal{P} \) be a set of edge-disjoint \( s-t \) paths in a graph \( G \). Clearly, if \( |\mathcal{P}| \leq 1 \), then the paths in \( \mathcal{P} \) are independent. If \( \mathcal{P} = \{P_1, P_2\} \), a set of two independent \( u-v \) paths can easily be obtained as follows. Set \( u := s \) and let \( v \) be the vertex that belongs to both \( P_1 \) and \( P_2 \) and is closest to \( s \) on \( P_1 \). Then, the \( u-v \) subpaths of \( P_1 \) and \( P_2 \) are independent. This proves that \( f(k) = k \) if \( k \leq 2 \).

The lower bound for \( f(k), k \geq 3 \), is provided by the following lemma.

**Lemma 1.** Let \( G = (V, E) \) be a graph. If there are two vertices \( s, t \in V \) with 3 edge-disjoint \( s-t \) paths in \( G \), then there are two vertices \( u, v \in V \) with 3 independent \( u-v \) paths in \( G \).

**Proof.** Let \( P_1, P_2, P_3 \) denote 3 edge-disjoint \( s-t \) paths, and let \( S = \{s_1, s_2, s_3\} \), where \( s_i \) neighbors \( s \) on \( P_i \), \( 1 \leq i \leq 3 \). Consider the connected component \( G' \) of \( G \setminus \{s\} \) containing \( t \). Then, \( G' \) contains all vertices from \( S \). Let \( T \) be a spanning tree of \( G' \). Select \( v \) such that the \( s_i-v \) subpaths of \( T \), \( 1 \leq i \leq 3 \), are independent. This vertex \( v \) belongs to every subpath of \( T \) that has two vertices from \( S \) as endpoints.

To see that this vertex exists, consider the \( s_1-s_3 \) subpath \( P_{1,3} \) of \( T \) and the \( s_2-s_3 \) subpath \( P_{2,3} \) of \( T \). Set \( v \) to be the vertex that belongs to both \( P_{1,3} \) and \( P_{2,3} \) and is closest to \( s_2 \) on \( P_{2,3} \) (if \( P_{1,3} \) contains \( s_2 \), then \( v = s_2 \)). Set \( u := s \), and obtain 3 independent \( u-v \) paths in \( G \) by moving from \( u \) to \( s_i \), and then along the \( s_i-v \) subpath of \( T \) to \( v \), \( 1 \leq i \leq 3 \).

For the upper bound, consider the following family of graphs, the recursive diamond graphs \( \square \). The recursive diamond graph of order 0 is \( G_0 = (\{s, t\}, \{s\}) \), and the diamond graph \( G_p \) of order \( p \geq 1 \) is obtained from \( G_{p-1} \) by replacing each edge \( e = xy \) by the set of edges \( \{xp, pe, qe, qy\} \), where \( pe \) and \( qe \) are new vertices. See Figure 1 for an illustration.

The following lemma entails the upper bound for \( f(k), k \geq 3 \).

**Lemma 2.** For every \( k \geq 3 \), there is a graph \( G = (V, E) \) containing two vertices \( s, t \in V \) with \( k \) edge-disjoint \( s-t \) paths, but no two vertices \( u, v \in V \) with \( 4 \) independent \( u-v \) paths.

**Proof.** Consider the diamond graph \( G = G_p \) of order \( p = \lceil \log k \rceil \). \( G \) has \( 2^p \geq k \) edge-disjoint \( s-t \) paths.

Let \( u, v \) be any two vertices in \( G \). We will show that there are at most 3 independent \( u-v \) paths.

Observe that each recursive diamond graph \( G_r \) contains 4 edge-disjoint copies of \( G_{r-1} \). The extremities of \( G_r \) are the vertices \( s \) and \( t \), and the extremities of a subgraph \( H \) of \( G_r \) that is isomorphic to \( G_r', r' < r \), are the two vertices from \( H \) whose neighborhoods in \( G_r \) are not a subset of \( V(H) \).

Let \( Q \) be the smallest vertex set containing \( u \) and \( v \) such that \( G[Q] \) is a recursive diamond graph. Let \( q \) be the order of the recursive diamond graph \( G[Q] \).

If \( q = 0 \), then \( uv \) is an edge in \( G \), and either \( u \) or \( v \) has degree 2. But then, the number of independent \( u-v \) paths in \( G \) is at most 2 since independent paths pass through distinct neighbors of \( u \) and \( v \).

If \( q > 0 \), then \( uv \) is not an edge in \( G \). Decompose \( G[Q] \) into 4 edge-disjoint graphs \( H_1, \ldots, H_4 \) isomorphic to \( G_{q-1} \) such that \( u \in V(H_1) \) and the \( H_i \) are ordered cyclically by their index (i.e., \( V(H_1) \cap \)}

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Figure 1: The recursive diamond graphs of order 0, 1, 2, and 3.

Since we chose $Q$ to be minimum, $u$ and $v$ do not belong to the same $H_i$, $1 \leq i \leq 4$. If $u \notin V(H_2) \cup V(H_4)$, then the extremities of $H_1$ are a $u$–$v$-vertex cut of size 2 in $G[Q]$ and in $G$. Otherwise, suppose, without loss of generality, that $u \in V(H_1) \cap V(H_3)$. Since $v \notin V(H_1) \cup V(H_3)$, the other two extremities of $H_1$ and $H_3$ form a $u$–$v$-vertex cut $C$ of size 2 in $G[Q]$. The set $C$ is also a $u$–$v$-vertex cut in $G$, unless $q < p$ and $u$ is an extremity of another subgraph $J$ of $G$ isomorphic to $G_q$ that is edge-disjoint from $G[Q]$. In the latter case, add the other extremity of $J$ to $C$ to obtain a $u$–$v$-vertex cut in $G$ of size 3.

Since $G$ has a $u$–$v$-vertex cut of size at most 3, by Menger’s theorem \[6\], there are at most 3 independent $u$–$v$ paths in $G$. \qed

**An application** Lemma \[1\] has been used in an algorithm \[2\] for the detection of backdoor sets to ease Satisfiability solving. A backdoor set of a propositional formula is a set of variables such that assigning truth values to the variables in the backdoor set moves the formula into a polynomial-time decidable class; see \[3\] for a survey. The class of nested formulas was introduced by Knuth \[5\] and their satisfiability can be decided in polynomial time. To find a backdoor set to the class of nested formulas, the algorithm from \[2\] considers the clause-variable incidence graph of the formula. If the formula is nested, this graph does not contain a $K_{2,3}$-minor with the additional property that the independent set of size 3 is obtained by contracting 3 connected subgraphs containing a variable each. In the correctness proof of the algorithm it is shown that in certain cases the formula does not have a small backdoor set. This is shown by exhibiting two vertices $u, v$ and 3 independent $u$–$v$ paths in an auxiliary graph using Lemma \[4\]. Expanding these edges to the paths they represent in the formula’s incident graph gives rise to a $K_{2,3}$-minor with the desired property.

On the other hand, Lemma \[2\] shows the limitations of this approach if we would like to enlarge the target class to more general formulas.

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**References**


