Tree Transducers and their Applications to XML

Tarragona Lectures
22-26.3.2004

Sebastian Maneth
Swiss Inst. of Technology, Lausanne
Sebastian.Maneth@epfl.ch

Overview

1st Day: History of Tree Transducers, Finite-State Models
2nd Day: Context-Free Models
TODAY: **Properties of Macro Tree Tranducers**
4th Day: XML Type Checking using MTTs
5th Day: Complexity of MTTs

Today

1. Expressive Power of Macro Tree Transducers
2. Closure under regular look ahead
3. Inverses of MTTs preserve REGT

Macro Tree Transducer

Notation For ranked alphabet Σ and a set A

$$\langle \Sigma, S \rangle := \{ \langle \sigma, a \rangle^{(k)} \mid \sigma \in \Sigma^{(k)},\ a \in A \}$$

MTT $M = (Q, \Sigma, \Delta, q_0, R)$

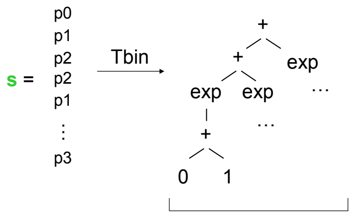
R: $\langle \underset{\uparrow \atop \in Q^{(m)}}{q}, \underset{\uparrow \atop \in \Sigma^{(k)}}{\sigma}(x_1,..,x_k) \rangle (y_1, \ldots, y_m) \rightarrow t \in T_{\langle Q, X_k \rangle \cup \Delta}(Y_m)$

Sentential forms: $\xi, \xi' \in T_{\langle Q, T_\Sigma \rangle \cup \Delta}$

$\xi \Rightarrow_M \xi'$ iff $\exists$ node u: $\xi/u = \langle q, \sigma(s_1,..,s_k) \rangle (\xi_1,..,\xi_m)$ and

$\xi' = \xi [u \leftarrow t [x_i \leftarrow s_i] [y_j \leftarrow \xi_j]]$.

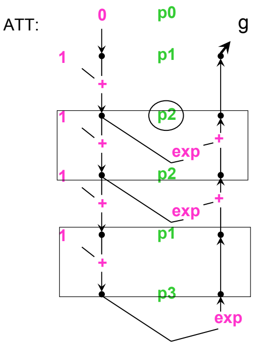
Macro Tree Transducer



repr. of decimal value of s

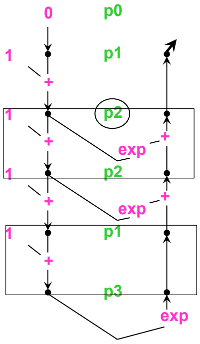
Can NOT be done by T^R, but by ATT. Can it be done by an MTT?

Macro Tree Transducer



Macro Tree Transducer

ATT:



MTT:

$\langle q_0, p0(x) \rangle \rightarrow \langle q, x \rangle(0)$

$\langle q, p1(x) \rangle(y) \rightarrow \langle q, x \rangle(+(1, y))$

Macro Tree Transducer

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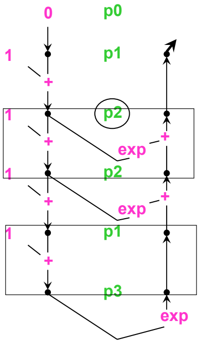

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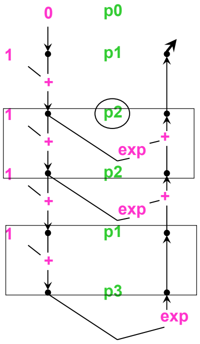

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NO! ↗

Macro Tree Transducer

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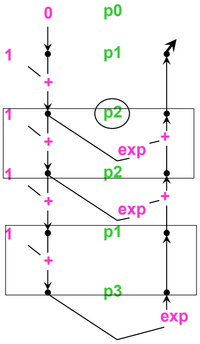

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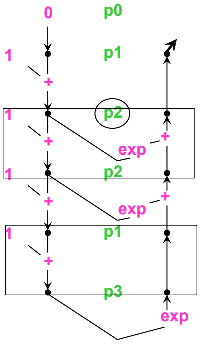

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Macro Tree Transducer

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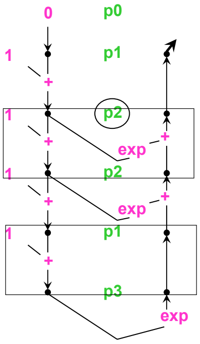

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↗

DETERMINISTIC, can be made
TOTAL.

OI vs. IO

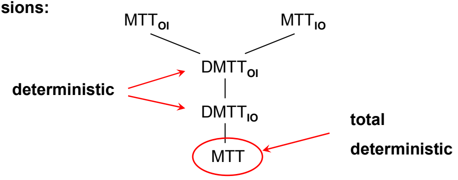
For the partial nondet case (like for CFT grammar):

→ unrestricted = OI, and

→ MTT_{OI} and MTT_{IO} are uncomparable

(to show this, fix an input tree and consider the MTT as a CFT grammar)

Inclusions:



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1. Expressive Power of MTTs

Theorem: ATT ⊆ MTT.

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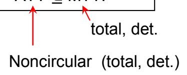

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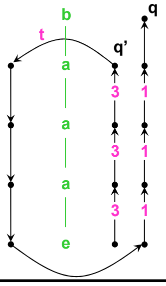
Theorem: ATT ⊆ MTT.

Proof idea: “Don’t go back, but JUMP back.”



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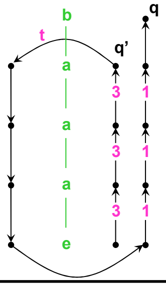

$\langle q, ba^3e \rangle$

$\Rightarrow_M \langle q, a^3e \rangle(t\langle q', a^3e \rangle)$

$\Rightarrow_M 1\langle q, a^2e \rangle(t\langle q', a^3e \rangle)$

$\Rightarrow^2_M 1^3\langle q, e \rangle(t\langle q', a^3e \rangle)$

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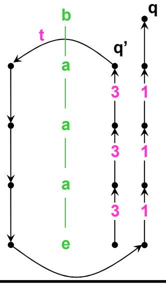

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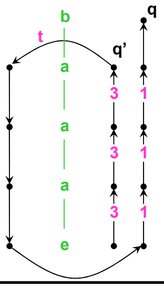
“push” into parameter what we need later!



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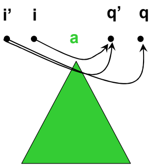
WHAT do we need later??



→ **q’** needs **i, i’**

→ **q** needs **i’**

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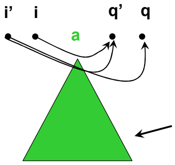

→ **q’** needs **i, i’**

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Depends on input tree!!

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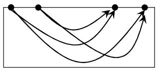
Solution 1: Use **regular look-ahead** to compute “needs”

Solution 2: Assume worst case (all) “needs” and stop at circularities.



All is needed.

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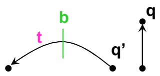

$\langle \mathbf{q}, b(x) \rangle \rightarrow \langle \mathbf{q}, x \rangle($

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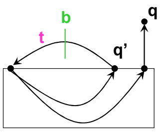

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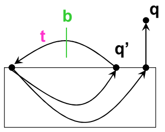

$$\langle \mathbf{q}, b(x) \rangle \rightarrow \langle \mathbf{q}, x \rangle(\; t \langle \mathbf{q'},x \rangle(\; t \langle \mathbf{q'},x \rangle($$

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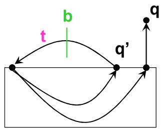

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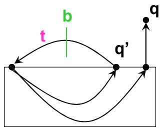

$$\langle \mathbf{q}, b(x) \rangle \rightarrow \langle \mathbf{q}, x \rangle(\; t \langle \mathbf{q'},x \rangle(\; \mathbf{dummy}))$$

---

1. Expressive Power of MTTs

Do ATT and MTT have the **same power??**

---

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$$\langle q_0, a(x_1) \rangle \rightarrow \langle q,x_1 \rangle(\langle q,x_1 \rangle(e))$$
$$\langle q, a(x_1) \rangle(y_1) \rightarrow \langle q,x_1 \rangle(\langle q,x_1 \rangle(y_1))$$
$$\langle q, e \rangle(y_1) \rightarrow a(y_1)$$

$$\langle q_0, a^n(e) \rangle \Rightarrow_M \langle q, a^{n-1}(e) \rangle(\langle q, a^{n-1}(e) \rangle(e))$$

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$$\langle q_0, a^n(e) \rangle \Rightarrow_M \langle q, a^{n-1}(e) \rangle(\langle q, a^{n-1}(e) \rangle(e))$$

$$\Rightarrow_M^2 \langle q, a^{n-2}(e) \rangle(\langle q, a^{n-2}(e) \rangle(\langle q, a^{n-2}(e) \rangle(\langle q, a^{n-2}(e) \rangle(e))))$$

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Do ATT and MTT have the **same power??**

$\langle q_0, a(x_1)\rangle \rightarrow \langle q,x_1\rangle(\langle q,x_1\rangle(e))$
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$\Rightarrow_M^* \underbrace{\langle q, e\rangle(\langle q, e\rangle(\dots \langle q, e\rangle}_{2^n \text{ times}}(e)\dots))$

---

1. Expressive Power of MTTs

Size-to-Height increase of MTTs can be EXPONENTIAL!!

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$\langle q, a(x_1)\rangle(y_1) \rightarrow \langle q,x_1\rangle(\langle q,x_1\rangle(y_1))$
$\langle q, e\rangle(y_1) \rightarrow a(y_1)$

$\langle q_0, \boxed{a^n(e)}\rangle \Rightarrow_M \langle q, a^{n-1}(e)\rangle(\langle q, a^{n-1}(e)\rangle(e))$

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---

1. Expressive Power of MTTs

Size-to-Height increase of MTTs can be EXPONENTIAL!!

Theorem: ATT $\subsetneq$ MTT.

(size-to-height inc. of ATTs is LINEAR)

$\langle q_0, \boxed{a^n(e)}\rangle \Rightarrow_M \langle q, a^{n-1}(e)\rangle(\langle q, a^{n-1}(e)\rangle(e))$

$\Rightarrow_M^2 \langle q, a^{n-2}(e)\rangle(\langle q, a^{n-2}(e)\rangle(\langle q, a^{n-2}(e)\rangle(\langle q, a^{n-2}(e)\rangle(e))))$

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---

1. Expressive Power of MTTs

Size increase of MTTs?

→ At most DOUBLE EXP.

EXERCISE:

Can you give an example of an MTT with **DEXP size** inc.?

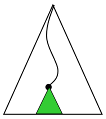
(**hint:** try to generalize the previous example
to output a full k-ary tree, instead of a monadic one)

---

2. Closure under Reg. Look-Ah.

Let A = (P, Σ, δ) be a (deterministic) BU finite tree automaton.

Regular look-ahead = ability to determine A's state for any node of the input tree.



input tree

What is δ(▲) ??

---

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MTT^R = MTT w. r. LA = B-REL ; MTT
↓
Rules:

$\langle q, \sigma(x_1,\dots, x_k)\rangle(y_1,\dots,y_m) \rightarrow t \quad \langle p_1,\dots, p_k\rangle$

A's state for x_1 A's state for x_k

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A's state for x_1 A's state for x_k

$rhs(q, \sigma, \langle p_1,..,p_k\rangle) := t$

---

2. Closure under Reg. Look-Ah.

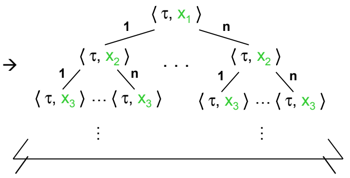
Theorem: $MTT^R \subseteq MTT$.

Proof. Let $M = (Q, \Sigma, \Delta, q_0, R, A)$ an MTT^R with $P = \{p_1,.., p_n\}$.

MTT $M' := (Q \cup \{\tau^{(n)}\}, \Sigma, \Delta, q_0, K)$

K:

$\langle q, \sigma(x_1,.., x_k)\rangle(y_1,..,y_m) \rightarrow$



$rhs(q, \sigma, \langle p_1,..,p_1\rangle)$ . . . $rhs(q, \sigma, \langle p_n,..,p_n\rangle)$

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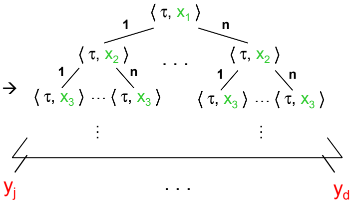
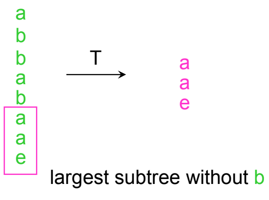


$\delta(\sigma, p_1,.., p_1) = p_j$

$\delta(\sigma, p_n,.., p_n) = p_d$

Example:

$\Sigma = \Delta = \{a^{(1)}, b^{(1)}, e^{(0)}\}$



largest subtree without b

Nondeterminsitic TOP:

q0(e) → e

q0(a/b(x)) → q0(x)

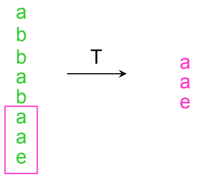
q0(a(x)) → a(q(x))

q(a(x)) → a(q(x))

q(e) → e

Example:

$\Sigma = \Delta = \{a^{(1)}, b^{(1)}, e^{(0)}\}$



largest subtree without b

TOP^R M for T:

Look-ahead Automaton A:

δ(e) = p1

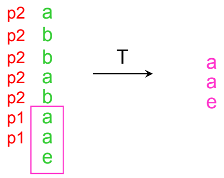
δ(a, p1) = p1

δ(b, p1) = p2

δ(a/b, p2) = p2

Example:

$\Sigma = \Delta = \{a^{(1)}, b^{(1)}, e^{(0)}\}$



largest subtree without b

TOP^R M for T:

Look-ahead Automaton A:

δ(e) = p1

δ(a, p1) = p1

δ(b, p1) = p2

δ(a/b, p2) = p2

Example:

$\Sigma = \Delta = \{ a^{(1)}, b^{(1)}, e^{(0)} \}$

p2 a
p2 b
p2 b
p2 a
p2 b
p1 $\boxed{a}$
p1 $\boxed{a}$
$\boxed{e}$

$\xrightarrow{T}$ a a e

largest subtree without b

TOP^R M for T:

Look-ahead Automaton A:

$\delta(e) = p1$
$\delta(a, p1) = p1$
$\delta(b, p1) = p2$
$\delta(a/b, p2) = p2$

$\langle q0, a/b\,(x_1)\rangle \rightarrow \langle id, x_1\rangle \quad \langle p1\rangle$

$\langle q0, a/b\,(x_1)\rangle \rightarrow \langle q0, x_1\rangle \quad \langle p2\rangle$

$\langle q0, e\rangle \rightarrow e \quad \langle p1/p2\rangle$

$\langle id, \underset{e}{a/b}\,(x_1)\rangle \rightarrow \underset{e}{a/b}(\langle id, x_1\rangle) \quad \langle p1/p2\rangle$

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$\langle q0, a/b\,(x_1)\rangle \rightarrow \langle \tau, x_1\rangle($
$\langle id, x_1\rangle, \langle q0, x_1\rangle)$

$\langle id, a/b\,(x_1)\rangle \rightarrow a/b(\langle id, x_1\rangle)$

$\langle q0/id, e\rangle \rightarrow e$

$\langle \tau, a(x_1)\rangle(y_1, y_2) \rightarrow \langle \tau, x_1\rangle($

TOP^R M for T:

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TOP^R M for T:

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$\langle \tau, e\rangle(y_1, y_2) \rightarrow y_1$

MTT N for T.

---

a
● b
b
a
b
a
a
e

$\xrightarrow{T}$ a a e

$\langle q0, a\,(\bullet)\rangle \Rightarrow_N \langle \tau, \bullet\rangle($
$\langle id, \bullet\rangle, \langle q0, \bullet\rangle)$

$\langle q0, a/b\,(x_1)\rangle \rightarrow \langle \tau, x_1\rangle($
$\langle id, x_1\rangle, \langle q0, x_1\rangle)$

$\langle id, a/b\,(x_1)\rangle \rightarrow a/b(\langle id, x_1\rangle)$

$\langle q0/id, e\rangle \rightarrow e$

$\langle \tau, a(x_1)\rangle(y_1, y_2) \rightarrow \langle \tau, x_1\rangle(y_1, y_2)$

$\langle \tau, b(x_1)\rangle(y_1, y_2) \rightarrow \langle \tau, x_1\rangle(y_2, y_2)$

$\langle \tau, e\rangle(y_1, y_2) \rightarrow y_1$

MTT N for T.

a
● b
● b
a
b
a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, a(\bullet)\rangle \Rightarrow^*_N \langle \tau, \bullet\rangle(\langle q0, \bullet\rangle, \langle q0, \bullet\rangle)$

$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

$\langle id, a/b\,(x_1)\rangle \to a/b(\langle id, x_1\rangle)$

$\langle q0/id, e\rangle \to e$

$\langle \tau, a(x_1)\rangle(y_1, y_2) \to \langle \tau, x_1\rangle(y_1, y_2)$

$\langle \tau, b(x_1)\rangle(y_1, y_2) \to \langle \tau, x_1\rangle(y_2, y_2)$

$\langle \tau, e\rangle(y_1, y_2) \to y_1$

MTT N for T.

---

a
● b
b
● a
b
a
a
e

$\xrightarrow{T}$

a
a
e

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MTT N for T.

---

a
● b
b
a
b
a
a
● e

$\xrightarrow{T}$

a
a
e

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$\langle \tau, b(x_1)\rangle(y_1, y_2) \to \langle \tau, x_1\rangle(y_2, y_2)$

$\langle \tau, e\rangle(y_1, y_2) \to y_1$

MTT N for T.

---

a
● b
b
a
b
a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, a(\bullet)\rangle \Rightarrow^*_N \langle q0, \bullet\rangle$

$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

$\langle id, a/b\,(x_1)\rangle \to a/b(\langle id, x_1\rangle)$

$\langle q0/id, e\rangle \to e$

$\langle \tau, a(x_1)\rangle(y_1, y_2) \to \langle \tau, x_1\rangle(y_1, y_2)$

$\langle \tau, b(x_1)\rangle(y_1, y_2) \to \langle \tau, x_1\rangle(y_2, y_2)$

$\langle \tau, e\rangle(y_1, y_2) \to y_1$

MTT N for T.

---

a
b
● b
a
b
a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, ab..\rangle \Rightarrow^*_N \langle \tau, \bullet\rangle(\langle id, \bullet\rangle, \langle q0, \bullet\rangle)$

$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

$\langle id, a/b\,(x_1)\rangle \to a/b(\langle id, x_1\rangle)$

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$\langle \tau, a(x_1)\rangle(y_1, y_2) \to \langle \tau, x_1\rangle(y_1, y_2)$

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MTT N for T.

---

a
b
● b
● a
b
a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, ab..\rangle \Rightarrow^*_N \langle \tau, \bullet\rangle(\langle q0, \bullet\rangle, \langle q0, \bullet\rangle)$

$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

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MTT N for T.

a
b
● b
a
b
a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, ab..\rangle \Rightarrow^*_N \langle q0, \bullet\rangle$

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MTT N for T.

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a
b
b
● a
b
a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, ab..\rangle \Rightarrow^*_N \langle q0, \bullet\rangle$

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MTT N for T.

---

a
b
b
a
● b
a
a
e

$\xrightarrow{T}$

a
a
e

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$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

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MTT N for T.

---

a
b
b
a
b
● a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, ab..\rangle \Rightarrow^*_N \langle \tau, \bullet\rangle(\langle id, \bullet\rangle, \langle q0, \bullet\rangle)$

$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

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$\langle \tau, e\rangle(y_1, y_2) \to y_1$

MTT N for T.

---

a
b
b
a
b
● a
a
e

$\xrightarrow{T}$

a
a
e

$\langle q0, ab..\rangle \Rightarrow^*_N \langle id, \bullet\rangle$

$\Rightarrow^*_N$ a(a(e))

$\langle q0, a/b\,(x_1)\rangle \to \langle \tau, x_1\rangle(\langle id, x_1\rangle, \langle q0, x_1\rangle)$

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MTT N for T.

---

2. Closure under Reg. Look-Ah.

Theorem: $MTT^R \subseteq MTT$.

How many times is each node visited??

- → for B-REL ; MTT
- → for the corresponding MTT (without rLA)

EXERCISE

Why is this construction useful??

- → (top-down) READ ONLY access to s

E.g. in Functional Languages:

```
match s => {
  case a -> ..
  case b -> ..
.. }
```

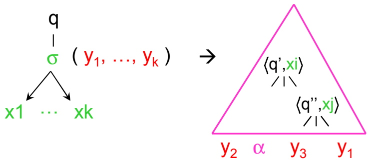
3. Inverses of MTTs preserve REGT

Theorem: If $M \in$ MTT and $K \in$ REGT, then $M^{-1}(K) \in$ REGT.

Proof. Let $M = (Q, \Sigma, \Delta, q_0, R)$ and

$A = (P, \Delta, \delta, F)$ be a det. bu TA with $L(A) = K$.

Idea: Run A on the rhs's of the rules of M.



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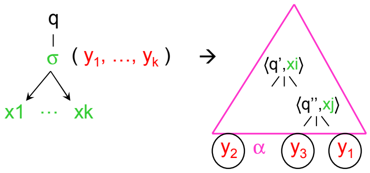

Which states of A?

---

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Proof. Let $M = (Q, \Sigma, \Delta, q_0, R)$ and

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Define $B = (D, \Sigma, \beta, G)$

D = all mappings d s.t. $\forall q \in Q^{(m)}$, $d(q): P^m \to P$

$\forall \sigma \in \Sigma^{(k)}$ and $d_1,..,d_k \in D$ let $\beta(\sigma,d_1,..,d_k) := d$

where $\forall q \in Q^{(m)}$, $p_1,..,p_m \in P$:

$$d(q)\,(p_1,..,p_m) = \delta'(\, \mathrm{rhs}(q, \sigma)\,[\, y_j \leftarrow p_j\,]\,)$$

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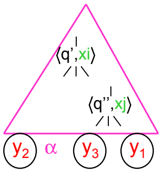

E.g. $d(q)\,(p_1, p_1, .., p_1) = ?$

---

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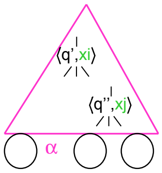

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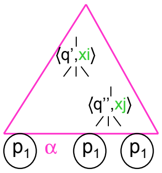

E.g. $d(q)\,(p_1, p_1, .., p_1) = ?$

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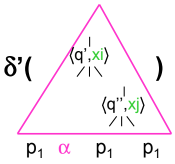
$d(q)\,(p_1,..,p_m) \;=\; \delta'(\, rhs(\, q, \sigma\,)\,[\, y_j \leftarrow p_j\,]\,)$



E.g. $d(q)\,(p_1, p_1, .., p_1\,) \;=\;$?

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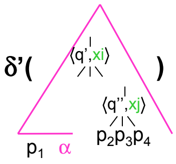


E.g. $d(q)\,(p_1, p_1, .., p_1\,) \;=\;$?

$\delta'(\langle\, q'',xj\,\rangle,\, p_2,\, p_3,\, p_4) \;=\; d_j\,(q'')\,(p_2,\, p_3,\, p_4)$

---

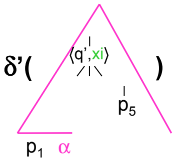
3. Inverses of MTTs preserve REGT

Theorem: If $M \in$ MTT and $K \in$ REGT, then $M^{-1}(K) \in$ REGT.

Proof. $\forall\, \sigma \in \Sigma^{(k)}$ and $d_1,..,d_k \in D$ let $\beta(\sigma,d_1,..,d_k) := d$

where $\forall\, q \in Q^{(m)}$, $p_1,..,p_m \in P$:

$d(q)\,(p_1,..,p_m) \;=\; \delta'(\, rhs(\, q, \sigma\,)\,[\, y_j \leftarrow p_j\,]\,)$



E.g. $d(q)\,(p_1, p_1, .., p_1\,) \;=\;$?

$\delta'(\langle\, q'',xj\,\rangle,\, p_2,\, p_3,\, p_4) \;=\; d_j\,(q'')\,(p_2,\, p_3,\, p_4)$

$= p_5$

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EXERCISE:

→ What is the complexity of this construction?

(hint: take as measure the number of states of the new automaton.)