

Justified Representation in Approval-Based Committee Voting

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Abstract We consider approval-based committee voting, i.e. the setting where each voter approves a subset of candidates, and these votes are then used to select a fixed-size set of winners (committee). We propose a natural axiom for this setting, which we call *justified representation (JR)*. This axiom requires that if a large enough group of voters exhibits agreement by supporting the same candidate, then at least one voter in this group has an approved candidate in the winning committee. We show that for every list of ballots it is possible to select a committee that provides *JR*. However, it turns out that several prominent approval-based voting rules may fail to output such a committee. In particular, while Proportional Approval Voting (*PAV*) always outputs a committee that provides *JR*, Sequential Proportional Approval Voting (*SeqPAV*), which is a tractable approximation to *PAV*, does not have this property. We then introduce a stronger version of the *JR* axiom, which we call *extended justified representation (EJR)*, and show that *PAV* satisfies *EJR*, while other rules we consider do not; indeed, *EJR* can be used to characterize *PAV* within the class of weighted *PAV* rules. We also consider several other questions related to *JR* and *EJR*, including the relationship between *JR/EJR* and core stability, and the complexity of the associated computational problems.

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1 Introduction

Aggregation of preferences is a central problem in the field of social choice. While the most-studied scenario is that of selecting a single candidate out of many, it is often the case that one needs to select a fixed-size *set of winners* (a *committee*): this includes domains such as parliamentary elections, the hiring of faculty members, or (automated) agents deciding on a set of plans (LeGrand et al. 2007; Davis et al. 2014; Elkind et al. 2015; Skowron et al. 2016; Elkind et al. 2017; Aziz et al. 2016). The study of the computational complexity of voting rules that output committees is an active research direction (Procaccia et al. 2008; Meir et al. 2008; Caragiannis et al. 2010; Lu and Boutilier 2011; Cornaz et al. 2012; Betzler et al. 2013; Skowron et al. 2015a,b).

In this paper we consider approval-based rules, where each voter lists the subset of candidates that she approves of. There is a growing literature on voting rules that are based on approval ballots: the Handbook on Approval Voting (Laslier and Sanver 2010) provides a very useful survey of pre-2010 research on this topic, and after this seminal book was published, various aspects of approval voting continued to attract a considerable amount of attention (see, e.g., the papers of Caragiannis et al. 2010; Endriss 2013; Duddy 2014). One of the advantages of approval ballots is their simplicity: compared to ranked ballots, approval ballots reduce the cognitive burden on voters (rather than providing a full ranking of the candidates, a voter only needs to decide which candidates to approve) and are also easier to communicate to the election authority. The most straightforward way to aggregate approvals is to have every approval for a candidate contribute one point to that candidate’s score and select the candidates with the highest score. This rule is called *Approval Voting (AV)*. AV has many desirable properties in the single-winner case (Brams and Fishburn 2007; Brams et al. 2006; Endriss 2013), including its “simplicity, propensity to elect Condorcet winners (when they exist), its robustness to manipulation and its monotonicity” (Brams 2010, p. viii). However, for the case of multiple winners, the merits of AV are “less clear” (Brams 2010, p. viii). For example, AV may fail proportional representation: if the goal is to select $k > 1$ winners, 51% of the voters approve the same k candidates, and the remaining voters approve a disjoint set of k candidates, then the voters in minority do not get any of their approved candidates selected.

As a consequence, over the years, several multi-winner rules based on approval ballots have been proposed (see, e.g., the survey by Kilgour 2010); we will now briefly describe the rules that will be considered in this paper (see Section 2 for formal definitions). Under *Proportional Approval Voting (PAV)*, each voter’s contribution to the committee’s total score depends on how many candidates from the voter’s approval set have been elected. In the canonical variant of this rule the marginal utility of the ℓ -th approved candidate is $\frac{1}{\ell}$, i.e. this rule is associated with the weight vector $(1, \frac{1}{2}, \frac{1}{3}, \dots)$; other weight vectors can be used as well, resulting in the family

of *weighted PAV rules*. A sequential variant of *PAV* is known as *Sequential Proportional Approval Voting (SeqPAV)*; again, by varying the weight vector, we obtain the family of *weighted SeqPAV rules*. Another way to modulate the approvals is through computing a satisfaction score for each voter based on the ratio of the number of their approved candidates appearing in the committee and their total number of approved candidates; this idea leads to *Satisfaction Approval Voting (SAV)*. One could also use a distance-based approach: *Minimax Approval Voting (MAV)* selects a set of k candidates that minimizes the maximum Hamming distance from the submitted ballots. Finally, one could adapt classic rules that provide fully proportional representation, such as the Chamberlin–Courant rule (Chamberlin and Courant 1983) or the Monroe rule (Monroe 1995), to work with approval ballots, by using each voter’s ballot as a scoring vector. All the rules informally described above have a more egalitarian objective than *AV*. For example, Steven Brams, a proponent of *AV* in single-winner elections, has argued that *SAV* is more suitable for equitable representation in multi-winner elections (Brams and Kilgour 2014).

The relative merits of approval-based multi-winner rules and the complexity of winner determination under these rules have been examined in great detail in both economics and computer science in recent years (Brams and Fishburn 2007; LeGrand et al. 2007; Meir et al. 2008; Caragiannis et al. 2010; Aziz et al. 2015; Byrka and Sornat 2014; Misra et al. 2015). On the other hand, there has been limited axiomatic analysis of these rules from the perspective of representation (see, however, Section 7).

In this paper, we introduce the notion of *justified representation (JR)* in approval-based voting. Briefly, a committee is said to provide justified representation for a given set of ballots if every large enough group of voters with shared preferences is allocated at least one representative. A rule is said to satisfy justified representation if it always outputs a committee that provides justified representation. This concept is related to the *Droop proportionality criterion* (Droop 1881) and Dummett’s *solid coalition property* (Dummett 1984; Tideman and Richardson 2000; Elkind et al. 2017), but is specific to approval-based elections.

We show that every set of ballots admits a committee that provides justified representation; moreover, such a committee can be computed efficiently, and checking whether a given committee provides *JR* can be done in polynomial time as well. This shows that justified representation is a reasonable requirement. However, it turns out that many popular multi-winner approval-based rules fail *JR*; in particular, this is the case for *AV*, *SAV*, *MAV* and the canonical variant of *SeqPAV*. On the positive side, *JR* is satisfied by some of the weighted *PAV* rules, including the canonical *PAV* rule, as well as by the weighted *SeqPAV* rule associated with the weight vector $(1, 0, \dots)$ and by the Monroe rule. Also, *MAV* satisfies *JR* for a restricted domain of voters’ preferences. We then consider a strengthening of the *JR* axiom, which we call *extended justified representation (EJR)*. This axiom captures the intuition that a large group of voters with similar preferences may deserve not just one, but several representatives. *EJR* turns out to be a more demanding property than *JR*: of all voting rules considered in this paper, only the canonical *PAV* rule satisfies *EJR*. Thus, in particular, *EJR* characterizes the canonical *PAV* rule within the class of weighted *PAV* rules. However, we show that it is computationally hard to check whether a given committee provides *EJR*.

We also consider other strengthenings of *JR*, which we call *semi-strong justified representation* and *strong justified representation*; however, it turns out that for some inputs the requirements imposed by these axioms are impossible to satisfy. Finally, we explore the relationship between *JR/EJR* and core stability in a non-transferable utility game that can be associated with a multiwinner approval voting scenario. We show that, even though *EJR* may appear to be similar to core stability, it is, in fact, a strictly weaker condition. Indeed, the core stability condition appears to be too demanding, as none of the voting rules considered in our work is guaranteed to produce a core stable outcome, even when the core is known to be non-empty. We conclude the paper by discussing related work and identifying several directions for future work.

2 Preliminaries

We consider a social choice setting with a set $N = \{1, \dots, n\}$ of voters and a set C of candidates. Each voter $i \in N$ submits an approval ballot $A_i \subseteq C$, which represents the subset of candidates that she approves of. We refer to the list $\mathbf{A} = (A_1, \dots, A_n)$ of approval ballots as the *ballot profile*. We will consider *approval-based multi-winner voting rules* that take as input a tuple (N, C, \mathbf{A}, k) , where k is a positive integer that satisfies $k \leq |C|$, and return a subset $W \subseteq C$ of size k , which we call the *winning set*, or *committee* (Kilgour and Marshall 2012). We omit N and C from the notation when they are clear from the context. Several approval-based multi-winner rules are defined below. Whenever the description of the rule does not uniquely specify a winning set, we assume that ties are broken according to some deterministic procedure; however, most of our results do not depend on the tie-breaking rule.

2.1 Approval-Based Multi-Winner Rules

Approval Voting (AV) Under *AV*, the winners are the k candidates that receive the largest number of approvals. Formally, the *approval score* of a candidate $c \in C$ is defined as $|\{i \in N \mid c \in A_i\}|$, and *AV* outputs a set W of size k that maximizes $\sum_{c \in W} |\{i \in N \mid c \in A_i\}|$. *AV* has been adopted by several academic and professional societies such as the Institute of Electrical and Electronics Engineers (IEEE), the International Joint Conference on Artificial Intelligence (IJCAI), and the Society for Social Choice and Welfare (SSCW).

Satisfaction Approval Voting (SAV) A voter’s *satisfaction score* is the fraction of her approved candidates that are elected. *SAV* maximizes the sum of voters’ satisfaction scores. Formally, *SAV* outputs a set $W \subseteq C$ of size k that maximizes $\sum_{i \in N} \frac{|W \cap A_i|}{|A_i|}$. This rule was proposed with the aim of “representing more diverse interests” than *AV* (Brams and Kilgour 2014).¹

Proportional Approval Voting (PAV) Under *PAV*, a voter is assumed to derive a utility of $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}$ from a committee that contains exactly j of her

¹ *SAV* is equivalent to *equal and even cumulative voting* (e.g., see Alcalde-Unzu and Vorsatz 2009), where for each voter, a score of 1 is divided evenly among all candidates in the voter’s approval set, and the k candidates with the highest total score are selected (see also Brams and Kilgour 2014, p. 328).

approved candidates, and the goal is to maximize the sum of the voters’ utilities. Formally, the *PAV*-score of a set $W \subseteq C$ is defined as $\sum_{i \in N} r(|W \cap A_i|)$, where $r(p) = \sum_{j=1}^p \frac{1}{j}$, and *PAV* outputs a set $W \subseteq C$ of size k with the highest *PAV*-score. Though sometimes attributed to Forest Simmons, *PAV* was already proposed by the Danish polymath Thorvald N. Thiele in the 19th century (Thiele 1895).² *PAV* captures the idea of diminishing returns: an individual voter’s preferences should count less the more she is satisfied.

In fact, Thiele (1895) not only introduced *PAV*, but a whole family of *weighted PAV rules*: for every vector³ $\mathbf{w} = (w_1, w_2, \dots)$, where w_1, w_2, \dots are non-negative reals, the voting rule \mathbf{w} -*PAV* operates as follows. Given a ballot profile (A_1, \dots, A_n) and a target committee size k , \mathbf{w} -*PAV* returns a set W of size k with the highest \mathbf{w} -*PAV* score, defined by $\sum_{i \in N} r_{\mathbf{w}}(|W \cap A_i|)$, where $r_{\mathbf{w}}(p) = \sum_{j=1}^p w_j$. Usually, it is required that $w_1 = 1$ and $w_1 \geq w_2 \geq \dots$. The latter constraint is appropriate in the context of representative democracy: it is motivated by the intuition that once an agent already has one or more representatives in the committee, that agent should have less priority for further representation. In what follows, we will always impose the constraint $w_1 = 1$ (as we can always rescale the weight vector, this is equivalent to requiring that $w_1 > 0$; while the case $w_1 = 0$ may be of interest in some applications, we omit it in order to keep the length of the paper manageable⁴) and explicitly indicate which of our results require that $w_1 \geq w_2 \geq \dots$; in particular, for our characterization of *PAV* in Theorem 11 this is not the case.

Sequential Proportional Approval Voting (*SeqPAV*) *SeqPAV* converts *PAV* into a multi-round rule, by selecting a candidate in each round and then reweighing the approvals for the subsequent rounds. Specifically, *SeqPAV* starts by setting $W = \emptyset$. Then in round j , $j = 1, \dots, k$, it computes the *approval weight* of each candidate c in $C \setminus W$ as $\sum_{i:c \in A_i} \frac{1}{1+|W \cap A_i|}$, selects a candidate with the highest approval weight, and adds him to W . After k rounds, it outputs the set W . Several papers (including an earlier version of our work) refer to *SeqPAV* as “*Rewighted Approval Voting (RAV)*”; this rule was used briefly in Sweden during the early 1900s.

Thiele (1895) proposed *SeqPAV* as a tractable approximation to *PAV* (see Section 2.2 for a discussion of the computational complexity of these rules and the relationship between them). We note that there are several other examples of voting rules that were conceived as approximate versions of other rules, yet became viewed as legitimate voting rules in and of themselves; two representative examples are the Simplified Dodgson rule of Tideman (2006), which was designed as an approximate version of the Dodgson rule (see the discussion by Caragiannis et al. 2014), and the Greedy Monroe rule of Skowron et al. (2015a), which approximates the Monroe rule (Monroe 1995).

Just as for *PAV*, the definition of *SeqPAV* can be extended to score vectors other than $(1, \frac{1}{2}, \frac{1}{3}, \dots)$: every vector $\mathbf{w} = (w_1, w_2, \dots)$ defines a sequential voting rule \mathbf{w} -

² We are grateful to Svante Janson and Xavier Mora for pointing this out to us.

³ It is convenient to think of \mathbf{w} as an infinite vector; note that for an election with m candidates only the first m entries of \mathbf{w} matter. To analyze the complexity of \mathbf{w} -*PAV* rules, one would have to place additional requirements on \mathbf{w} ; however, we do not consider computational properties of such rules in this paper.

⁴ \mathbf{w} -*PAV* rules with $w_1 = 0$ have been considered by Fishburn and Pekec (2004) and Skowron et al. (2016). We note that such rules do not satisfy justified representation (as defined in Section 3).

SeqPAV, which proceeds as *SeqPAV*, except that it computes the approval weight of a candidate c in round j as $\sum_{i:c \in A_i} w_{|W \cap A_i|+1}$, where W is the winning set after the first $j - 1$ rounds. Again, we impose the constraint $w_1 = 1$ (note that if $w_1 = 0$, then *w-SeqPAV* can pick an arbitrary candidate at the first step, which is obviously undesirable).

A particularly interesting rule in this class is $(1, 0, \dots)$ -*SeqPAV*: this rule, which we will refer to as *Greedy Approval Voting (GreedyAV)*, is the sequential version of what Thiele (1895) called the “weak method.” *GreedyAV* can be seen as a variant of the SweetSpotGreedy (SSG) algorithm of Lu and Boutilier (2011), and admits a very simple description: we pick candidates one by one, trying to ‘cover’ as many currently ‘uncovered’ voters as possible. In more detail, a winning committee under this rule can be computed by the following algorithm. We start by setting $C' = C$, $A' = A$, and $W = \emptyset$. As long as $|W| < k$ and A' is non-empty, we pick a candidate $c \in C'$ that has the highest approval score with respect to A' , and set $W := W \cup \{c\}$, $C' := C' \setminus \{c\}$. Also, we remove from A' all ballots A_i such that $c \in A_i$. If at some point we have $|W| < k$ and A' is empty, we add an arbitrary set of $k - |W|$ candidates from C' to W and return W ; if this does not happen, we terminate after having picked k candidates.

We will also consider a variant of *GreedyAV*, where, at each step, after selecting a candidate c , instead of removing all ballots in $N_c = \{i \in N \mid c \in A_i\}$ from A' , we remove a subset of N_c of size $\min\left\{\left\lceil \frac{n}{k} \right\rceil, |N_c|\right\}$. This rule can be seen as an adaptation of the classic STV rule to approval ballots, and we will refer to it as *HareAV*.⁵

Minimax Approval Voting (MAV) *MAV* returns a committee W that minimizes the maximum *Hamming distance* between W and the voters’ ballots; this rule was proposed by Brams et al. (2007). Formally, for any $Q, T \subseteq C$, let $d(Q, T) = |Q \setminus T| + |T \setminus Q|$. Define the *MAV*-score of a set $W \subseteq C$ as $\max(d(W, A_1), \dots, d(W, A_n))$. *MAV* outputs a size- k set with the lowest *MAV*-score.

Chamberlin–Courant and Monroe Approval Voting (CCAV and MonroeAV)

The Chamberlin–Courant rule (Chamberlin and Courant 1983) is usually defined for the setting where each voter provides a full ranking of the candidates. Each voter $i \in N$ is associated with a scoring vector $\mathbf{u}^i = (u_1^i, \dots, u_{|C|}^i)$ whose entries are non-negative reals; we think of u_j^i as voter i ’s satisfaction from being represented by candidate c_j . A voter’s satisfaction from a committee W is defined as $\max_{c_j \in W} u_j^i$, and the rule returns a committee of size k that maximizes the sum of voters’ satisfactions. For the case of approval ballots, it is natural to define the scoring vectors by setting $u_j^i = 1$ if $c_j \in A_i$ and $u_j^i = 0$ otherwise; that is, a voter is satisfied by a committee if this committee contains one of her approved candidates. Thus, the resulting rule is equivalent to $(1, 0, \dots)$ -*PAV* (and therefore we will not discuss it separately).

The Monroe rule (Monroe 1995) is a modification of the Chamberlin–Courant rule where each committee member represents roughly the same number of voters. Just as under the Chamberlin–Courant rule, we have a scoring vector $\mathbf{u}^i =$

⁵ For readability, we use the Hare quota $\left\lceil \frac{n}{k} \right\rceil$; this choice of quota motivates our name for this rule. However, all our proofs go through if we use the Droop quota $\left\lfloor \frac{n}{k+1} \right\rfloor + 1$ instead. For a discussion of differences between these two quotas, see the article of Tideman (1995).

$(u_1^i, \dots, u_{|C|}^i)$ for each voter $i \in N$. Given a committee W of size k , we say that a mapping $\pi : N \rightarrow W$ is *valid* if it satisfies $|\pi^{-1}(c)| \in \left\{ \left\lfloor \frac{n}{k} \right\rfloor, \left\lceil \frac{n}{k} \right\rceil \right\}$ for each $c \in W$. The Monroe score of a valid mapping π is given by $\sum_{i \in N} u_{\pi(i)}^i$, and the Monroe score of a committee W is the maximum Monroe score of a valid mapping from N to W . The Monroe rule returns a size- k committee with the maximum Monroe score. For approval ballots, we define the scoring vectors in the same manner as for the Chamberlin–Courant Approval Voting rule; we call the resulting rule the Monroe Approval Voting rule (*MonroeAV*).

We note that for $k = 1$, *AV*, *PAV*, *SeqPAV*, *GreedyAV*, *HareAV* and *MonroeAV* produce the same output if there is a unique candidate with the highest approval score. However, such a candidate need not be a winner under *SAV* or *MAV*.

2.2 Computational Complexity

The rules listed above differ from a computational perspective. For some of these rules, namely, *AV*, *SAV*, *SeqPAV*, *GreedyAV* and *HareAV*, a winning committee can be computed in polynomial time; this is also true for *w-SeqPAV* as long as the entries of the weight vector are rational numbers that can be efficiently computed given the number of candidates. In contrast, *PAV*, *MAV*, and *MonroeAV* are computationally hard (Aziz et al. 2015; Skowron et al. 2016; LeGrand et al. 2007; Procaccia et al. 2008); for *w-PAV*, the hardness result holds for most weight vectors, including $(1, 0, \dots)$ (i.e. it holds for *CAV*). However, both *PAV* and *MAV* admit efficient approximation algorithms (i.e. algorithms that output committees which are approximately optimal with respect to the optimization criteria of these rules) and have been analyzed from the perspective of parameterized complexity. Specifically, *w-PAV* admits an efficient $(1 - \frac{1}{e})$ -approximation algorithm as long as the weight vector \mathbf{w} is efficiently computable and non-increasing; in fact, such an algorithm is provided by *w-SeqPAV* (Skowron et al. 2016). For *MAV*, LeGrand et al. (2007) proposed a simple 3-approximation algorithm; Caragiannis et al. (2010) improve the approximation ratio to 2 and Byrka and Sornat (2014) develop a polynomial-time approximation scheme. Misra et al. (2015) show that *MAV* is fixed-parameter tractable for a number of natural parameters; Elkind and Lackner (2015) obtain fixed-parameter tractability results for *PAV* when voters' preferences are, in some sense, single-dimensional. There is also a number of tractability results for *CAV*, and, to a lesser extent, for *MonroeAV*; we refer the reader to the work of Skowron et al. (2016) and references therein.

3 Justified Representation

We will now define one of the main concepts of this paper.

Definition 1 (Justified representation (JR)) Given a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ over a candidate set C and a target committee size k , we say that a set of candidates W of size $|W| = k$ provides *justified representation* for (\mathbf{A}, k) if there does not exist

a set of voters $N^* \subseteq N$ with $|N^*| \geq \frac{n}{k}$ such that $\bigcap_{i \in N^*} A_i \neq \emptyset$ and $A_i \cap W = \emptyset$ for all $i \in N^*$. We say that an approval-based voting rule *satisfies justified representation (JR)* if for every profile $\mathbf{A} = (A_1, \dots, A_n)$ and every target committee size k it outputs a winning set that provides justified representation for (\mathbf{A}, k) .

The logic behind this definition is that if k candidates are to be selected, then, intuitively, each group of $\frac{n}{k}$ voters “deserves” a representative. Therefore, a set of $\frac{n}{k}$ voters that have at least one candidate in common should not be completely unrepresented. We refer the reader to Section 6 for a discussion of alternative definitions.

3.1 Existence and Computational Properties

We start our analysis of justified representation by observing that, for every ballot profile \mathbf{A} and every value of k , there is a committee that provides justified representation for (\mathbf{A}, k) , and, moreover, such a committee can be computed efficiently. In fact, both *GreedyAV* and *HareAV* output a committee that provides *JR*.

Theorem 1 *GreedyAV and HareAV satisfy JR.*

Proof We present a proof that applies to both *GreedyAV* and *HareAV*. Suppose for the sake of contradiction that for some ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ and some $k > 0$, *GreedyAV* (respectively, *HareAV*) outputs a committee that does not provide justified representation for (\mathbf{A}, k) . Then there exists a set $N^* \subseteq N$ with $|N^*| \geq \frac{n}{k}$ such that $\bigcap_{i \in N^*} A_i \neq \emptyset$ and, when *GreedyAV* (respectively, *HareAV*) terminates, every ballot A_i with $i \in N^*$ is still in \mathbf{A}' . Consider some candidate $c \in \bigcap_{i \in N^*} A_i$. At every point in the execution of our algorithm, c 's approval score is at least $|N^*| \geq \frac{n}{k}$. As c was not elected, at every stage the algorithm selected a candidate whose approval score was at least as high as that of c . Thus, at the end of each stage the algorithm removed from \mathbf{A}' at least $\lceil \frac{n}{k} \rceil$ ballots containing the candidate added to W at that stage, so altogether the algorithm has removed at least $k \cdot \frac{n}{k}$ ballots from \mathbf{A}' . This contradicts the assumption that \mathbf{A}' contains at least $\frac{n}{k}$ ballots when the algorithm terminates. \square

Theorem 1 shows that it is easy to find a committee that provides justified representation for a given ballot profile. It is also not too hard to check whether a given committee W provides *JR*. Indeed, while it may seem that we need to consider every subset of voters of size $\frac{n}{k}$, in fact it is sufficient to consider the candidates one by one, and, for each candidate c , compute $s(c) = |\{i \in N \mid c \in A_i, A_i \cap W = \emptyset\}|$; the set W fails to provide justified representation for (\mathbf{A}, k) if and only if there exists a candidate c with $s(c) \geq \frac{n}{k}$. We obtain the following theorem.

Theorem 2 *There exists a polynomial-time algorithm that, given a ballot profile \mathbf{A} over a candidate set C , and a committee W , $|W| = k$, decides whether W provides justified representation for (\mathbf{A}, k) .*

3.2 Justified Representation and Unanimity

A desirable property of single-winner approval-based voting rules is *unanimity*: a voting rule is unanimous if, given a ballot profile (A_1, \dots, A_n) with $\bigcap_{i \in N} A_i \neq \emptyset$, it outputs a candidate in $\bigcap_{i \in N} A_i$. This property is somewhat similar in spirit to *JR*, so the reader may expect that for $k = 1$ it is equivalent to *JR*. However, it turns out that the *JR* axiom is strictly weaker than unanimity for $k = 1$: while unanimity implies *JR*, the converse is not true, as illustrated by the following example.

Example 1 Let $N = \{1, \dots, n\}$, $C = \{a, b_1, \dots, b_n\}$, $A_i = \{a, b_i\}$ for $i \in N$. Consider a voting rule that for $k = 1$ outputs b_1 on this profile and coincides with *GreedyAV* in all other cases. Clearly, this rule is not unanimous; however, it satisfies *JR*, as it is impossible to find a group of $\frac{n}{k} = n$ unrepresented voters for (A_1, \dots, A_n) .

It is not immediately clear how to define unanimity for multi-winner voting rules; however, any reasonable definition would be equivalent to the standard definition of unanimity when $k = 1$, and therefore would be different from justified representation.

We remark that a rule can be unanimous for $k = 1$ and provide *JR* for all values of k : this is the case, for instance, for *GreedyAV*.

4 Justified Representation under Approval-Based Rules

We have argued that justified representation is a reasonable condition: there always exists a committee that provides it, and, moreover, such a committee can be computed efficiently. It is therefore natural to ask whether prominent voting rules satisfy *JR*. In this section, we will answer this question for *AV*, *SAV*, *MAV*, *PAV*, *SeqPAV*, and *MonroeAV*. We will also identify conditions on \mathbf{w} that are sufficient/necessary for *w-PAV* and *w-SeqPAV* to satisfy *JR*.

In what follows, for each rule we will try to identify the range of values of k for which this rule satisfies *JR*. Trivially, all rules that we consider satisfy *JR* for $k = 1$. It turns out that *AV* fails *JR* for $k > 2$, and for $k = 2$ the answer depends on the tie-breaking rule.

Theorem 3 *For $k = 2$, AV satisfies JR if ties are broken in favor of sets that provide JR. For $k \geq 3$, AV fails JR.*

Proof Suppose first that $k = 2$. Fix a ballot profile \mathbf{A} . If every candidate is approved by fewer than $\frac{n}{2}$ voters in \mathbf{A} , *JR* is trivially satisfied. If some candidate is approved by more than $\frac{n}{2}$ voters in \mathbf{A} , then *AV* selects some such candidate, in which case no group of $\lceil \frac{n}{2} \rceil$ voters is unrepresented, so *JR* is satisfied in this case as well. It remains to consider the case where $n = 2n'$, some candidates are approved by n' voters, and no candidate is approved by more than n' voters. Then *AV* necessarily picks at least one candidate approved by n' voters; denote this candidate by c . In this situation *JR* can only be violated if the n' voters who do not approve c all approve the same candidate (say, c'), and this candidate is not elected. But the approval score of c' is n' , and, by our assumption, the approval score of every candidate is at most n' , so

this is a contradiction with our tie-breaking rule. This argument also illustrates why the assumption on the tie-breaking rule is necessary: it can be the case that n' voters approve c and c'' , and the remaining n' voters approve c' , in which case the approval score of $\{c, c''\}$ is the same as that of $\{c, c'\}$.

For $k \geq 3$, we let $C = \{c_0, c_1, \dots, c_k\}$, $n = k$, and consider the profile where the first voter approves c_0 , whereas each of the remaining voters approves all of c_1, \dots, c_k . JR requires c_0 to be selected, but AV selects $\{c_1, \dots, c_k\}$. \square

On the other hand, SAV and MAV fail JR even for $k = 2$.

Theorem 4 *SAV and MAV do not satisfy JR for $k \geq 2$.*

Proof We first consider SAV . Fix $k \geq 2$, let $X = \{x_1, \dots, x_k, x_{k+1}\}$, $Y = \{y_1, \dots, y_k\}$, $C = X \cup Y$, and consider the profile (A_1, \dots, A_k) , where $A_1 = X$, $A_2 = \{y_1, y_2\}$, $A_i = \{y_i\}$ for $i = 3, \dots, k$. JR requires each voter to be represented, but SAV will choose Y : the SAV -score of Y is $k - 1$, whereas the SAV -score of every committee W with $W \cap X \neq \emptyset$ is at most $k - 2 + \frac{1}{2} + \frac{1}{k+1} < k - 1$. Therefore, the first voter will remain unrepresented.

For MAV , we use the following construction. Fix $k \geq 2$, let $X = \{x_1, \dots, x_k\}$, $Y = \{y_1, \dots, y_k\}$, $C = X \cup Y \cup \{z\}$, and consider the profile (A_1, \dots, A_{2k}) , where $A_i = \{x_i, y_i\}$ for $i = 1, \dots, k$, $A_i = \{z\}$ for $i = k + 1, \dots, 2k$. Every committee of size k that provides JR for this profile contains z . However, MAV fails to select z . Indeed, the MAV -score of X is $k + 1$: we have $d(X, A_i) = k$ for $i \leq k$ and $d(X, A_i) = k + 1$ for $i > k$. Now, consider some committee W with $|W| = k$, $z \in W$. We have $A_i \cap W = \emptyset$ for some $i \leq k$, so $d(W, A_i) = k + 2$. Thus, MAV prefers X to any committee that includes z . \square

The constructions used in the proof of Theorem 4 show that MAV and SAV may behave very differently: SAV appears to favor voters who approve very few candidates, whereas MAV appears to favor voters who approve many candidates.

Interestingly, we can show that MAV satisfies JR if we assume that each voter approves exactly k candidates and ties are broken in favor of sets that provide JR .

Theorem 5 *If the target committee size is k , $|A_i| = k$ for all $i \in N$, and ties are broken in favor of sets that provide JR , then MAV satisfies JR .*

Proof Consider a profile $\mathbf{A} = (A_1, \dots, A_n)$ with $|A_i| = k$ for all $i \in N$.

Observe that if there exists a set of candidates W with $|W| = k$ such that $W \cap A_i \neq \emptyset$ for all $i \in N$, then MAV will necessarily select some such set. Indeed, for any such set W we have $d(W, A_i) \leq 2k - 1$ for each $i \in N$, whereas if $W' \cap A_i = \emptyset$ for some set W' with $|W'| = k$ and some $i \in N$, then $d(W', A_i) = 2k$. Further, by definition, every set W such that $|W| = k$ and $W \cap A_i \neq \emptyset$ for all $i \in N$ provides justified representation for (\mathbf{A}, k) .

On the other hand, if there is no k -element set of candidates that intersects each A_i , $i \in N$, then the MAV -score of every set of size k is $2k$, and therefore MAV can pick an arbitrary size- k subset. Since we assumed that the tie-breaking rule favors sets that provide JR , our claim follows. \square

While Theorem 5 provides an example of a setting where *MAV* satisfies *JR*, this result is not entirely satisfactory: first, we had to place a strong restriction on voters' preferences, and, second, we used a tie-breaking rule that was tailored to *JR*.

We will now show that *PAV* satisfies *JR*, for all ballot profiles and irrespective of the tie-breaking rule.

Theorem 6 *PAV satisfies JR.*

Proof Fix a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ and a $k > 0$ and let $s = \lceil \frac{n}{k} \rceil$. Let W be the output of *PAV* on (\mathbf{A}, k) . Suppose for the sake of contradiction that there exists a set $N^* \subseteq N$, $|N^*| \geq s$, such that $\bigcap_{i \in N^*} A_i \neq \emptyset$, but $W \cap (\bigcup_{i \in N^*} A_i) = \emptyset$. Let c be some candidate approved by all voters in N^* .

For each candidate $w \in W$, define its *marginal contribution* as the difference between the *PAV*-score of W and that of $W \setminus \{w\}$. Let $m(W)$ denote the sum of marginal contributions of all candidates in W . Observe that if c were to be added to the winning set, this would increase the *PAV*-score by at least s . Therefore, it suffices to argue that the marginal contribution of some candidate in W is less than s : this would mean that swapping this candidate with c increases the *PAV*-score, a contradiction. To this end, we will prove that $m(W) \leq s(k-1)$; as $|W| = k$, our claim would then follow by the pigeonhole principle.

Consider the set $N \setminus N^*$; we have $n \leq sk$, so $|N \setminus N^*| \leq n - s \leq s(k-1)$. Pick a voter $i \in N \setminus N^*$, and let $j = |A_i \cap W|$. If $j > 0$, this voter contributes exactly $\frac{1}{j}$ to the marginal contribution of each candidate in $A_i \cap W$, and hence her contribution to $m(W)$ is exactly 1. If $j = 0$, this voter does not contribute to $m(W)$ at all. Therefore, we have $m(W) \leq |N \setminus N^*| \leq s(k-1)$, which is what we wanted to prove. \square

The reader may observe that the proof of Theorem 6 applies to all voting rules of the form *w-PAV* where the weight vector satisfies $w_1 = 1$ and $w_j \leq \frac{1}{j}$ for all $j > 1$. In Section 5 we will see that this condition on \mathbf{w} is also necessary for *w-PAV* to satisfy *JR*.

Next, we consider *SeqPAV*. As this voting rule can be viewed as a tractable approximation of *PAV* (recall that *PAV* is NP-hard to compute), one could expect that *SeqPAV* satisfies *JR* as well. However, this turns out not to be the case, at least if k is sufficiently large.

Theorem 7 *SeqPAV satisfies JR for $k = 2$, but fails it for $k \geq 10$.*

Proof For $k = 2$, we can use essentially the same argument as for *AV*; however, we do not need to assume anything about the tie-breaking rule. This is because if there are three candidates, c , c' , and c'' , such that c and c'' are approved by the same $\frac{n}{2}$ voters, whereas c' is approved by the remaining $\frac{n}{2}$ voters, and *SeqPAV* selects c in the first round, then in the second round *SeqPAV* favors c' over c'' .

Now, suppose that $k = 10$. Consider a profile over a candidate set $C = \{c_1, \dots, c_{11}\}$ with 1199 voters who submit the following ballots:

81×{ c_1, c_2 },	81×{ c_1, c_3 },	80×{ c_2 },	80×{ c_3 },
81×{ c_4, c_5 },	81×{ c_4, c_6 },	80×{ c_5 },	80×{ c_6 },
49×{ c_7, c_8 },	49×{ c_7, c_9 },	49×{ c_7, c_{10} },	
96×{ c_8 },	96×{ c_9 },	96×{ c_{10} },	120×{ c_{11} }.

Candidates c_1 and c_4 are each approved by 162 voters, the most of any candidate, and these blocks of 162 voters do not overlap, so *SeqPAV* selects c_1 and c_4 first. This reduces the *SeqPAV* scores of c_2, c_3, c_5 and c_6 from $80+81 = 161$ to $80+40.5 = 120.5$, so c_7 , whose *SeqPAV* score is 147, is selected next. Now, the *SeqPAV* scores of c_8, c_9 and c_{10} become $96 + 24.5 = 120.5$. The selection of any of $c_2, c_3, c_5, c_6, c_8, c_9$ or c_{10} does not affect the *SeqPAV* score of the others, so all seven of these candidates will be selected before c_{11} , who has 120 approvals. Thus, after the selection of 10 candidates, there are $120 > \frac{1199}{10} = \frac{n}{k}$ unrepresented voters who jointly approve c_{11} .

To extend this construction to $k > 10$, we create $k - 10$ additional candidates and $120(k - 10)$ additional voters such that for each new candidate, there are 120 new voters who approve that candidate only. Note that we still have $120 > \frac{n}{k}$. *SeqPAV* will proceed to select c_1, \dots, c_{10} , followed by $k - 10$ additional candidates, and c_{11} or one of the new candidates will remain unselected. \square

While *SeqPAV* itself does not satisfy *JR*, one could hope that this can be fixed by tweaking the weights, i.e. that *w-SeqPAV* satisfies *JR* for a suitable weight vector \mathbf{w} . However, it turns out that $(1, 0, \dots)$ is essentially the only weight vector for which this is the case: Theorem 7 extends to *w-SeqPAV* for every weight vector \mathbf{w} with $w_1 = 1, w_2 > 0$.

Theorem 8 *For every vector $\mathbf{w} = (w_1, w_2, \dots)$ with $w_1 = 1, w_2 > 0$, there exists a value of $k_0 > 0$ such that *w-SeqPAV* does not satisfy *JR* for $k > k_0$.*

Proof Pick a positive integer $s \geq 8$ such that $w_2 \geq \frac{1}{s}$. Let $C = C_1 \cup C_2 \cup \{x, y\}$, where

$$C_1 = \{c_{i,j} \mid i = 1, \dots, 2s+3, j = 1, \dots, 2s+1\} \quad \text{and} \quad C_2 = \{c_i \mid i = 1, \dots, 2s+3\}.$$

For each $i = 1, \dots, 2s+3$ and each $j = 1, \dots, 2s+1$ we construct $2s^3 - s$ voters who approve $c_{i,j}$ only and s^2 voters who approve $c_{i,j}$ and c_i only. Finally, we construct $2s^3 - 1$ voters who approve x only and $s^2 - 7s - 5$ voters who approve y only (note that the number of voters who approve y is positive by our choice of s).

Set $k_0 = (2s+2)(2s+3) = |C_1 \cup C_2|$. Note that the number of voters n is given by

$$\begin{aligned} & (2s+3)(2s+1)(2s^3 + s^2 - s) + (2s^3 - 1) + (s^2 - 7s - 5) \\ &= (2s+2)(2s+3)(2s^3 - 1) = (2s^3 - 1)k_0, \end{aligned}$$

and hence $\frac{n}{k_0} = 2s^3 - 1$.

Under *w-SeqPAV* initially the score of each candidate in C_2 is $s^2(2s+1) = 2s^3 + s^2$, the score of each candidate in C_1 is $2s^3 + s^2 - s$, the score of x is $2s^3 - 1$, and the score of y is $s^2 - 7s - 5$, so in the first $2s+3$ rounds the candidates from C_2 get elected. After that, the score of every candidate in C_1 becomes $2s^3 - s + w_2 s^2 \geq 2s^3 - s + s = 2s^3$, while the scores of x and y remains unchanged. Therefore, in the next $(2s+3)(2s+1)$ rounds the candidates from C_1 get elected. At this point, k candidates are elected, and x is not elected, even though the $2s^3 - 1 = \frac{n}{k_0}$ voters who approve him do not approve of any of the candidates in the winning set.

To extend this argument to larger values of k , we proceed as in the proof of Theorem 7: for $k > k_0$, we add $k - k_0$ new candidates, and for each new candidate we

construct $2s^3 - 1$ new voters who approve that candidate only. Let the resulting number of voters be n' ; we have $\frac{n'}{k} = 2s^3 - 1$, so \mathbf{w} -*SeqPAV* will first select the candidates in C_2 , followed by the candidates in C_1 , and then it will choose $k - k_0$ winners among the new candidates and x . As a result, either x or one of the new candidates will remain unselected. \square

Remark 1 Theorem 8 partially subsumes Theorem 7: it implies that *SeqPAV* fails *JR*, but the proof only shows that this is the case for $k \geq 18 \cdot 19 = 342$, while Theorem 7 states that *SeqPAV* fails *JR* for $k \geq 10$ already. We chose to include the proof of Theorem 7 because we feel that it is useful to know what happens for relatively small values of k . Note, however, that Theorem 7 leaves open the question of whether *SeqPAV* satisfies *JR* for $k = 3, \dots, 9$. Very recently, Sánchez-Fernández et al. (2016, 2017) have answered this question by showing that *SeqPAV* satisfies *JR* for $k \leq 5$ and fails it for $k \geq 6$.

If we allow the entries of the weight vector to depend on the number of voters n , we can obtain another class of rules that provide justified representation: the argument used to show that *GreedyAV* satisfies *JR* extends to \mathbf{w} -*SeqPAV* where the weight vector \mathbf{w} satisfies $w_1 = 1$, $w_j \leq \frac{1}{n}$ for $j > 1$. In particular, the rule $(1, \frac{1}{n}, \frac{1}{n^2}, \dots)$ -*SeqPAV* is somewhat more appealing than *GreedyAV*: for instance, if $\bigcap_{i \in N} A_i = \{c\}$ and $k > 1$, *GreedyAV* will pick c , and then behave arbitrarily, whereas $(1, \frac{1}{n}, \frac{1}{n^2}, \dots)$ -*SeqPAV* will also pick c , but then it will continue to look for candidates approved by as many voters as possible.

We conclude this section by showing that *MonroeAV* satisfies *JR*.

Theorem 9 *MonroeAV* satisfies *JR*.

Proof Fix a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ and a $k > 0$. Let W be an output of *MonroeAV* on (\mathbf{A}, k) . If $A_i \cap W \neq \emptyset$ for all $i \in N$, then W provides justified representation for (\mathbf{A}, k) . Thus, assume that this is not the case, i.e. there exists some voter i with $A_i \cap W = \emptyset$. Consider a valid mapping $\pi : N \rightarrow W$ whose Monroe score equals the Monroe score of W , let $c = \pi(i)$, and set $s = |\pi^{-1}(c)|$; note that $s \in \left\{ \left\lfloor \frac{n}{k} \right\rfloor, \left\lceil \frac{n}{k} \right\rceil \right\}$.

Suppose for the sake of contradiction that W does not provide justified representation for (\mathbf{A}, k) . Then by our choice of s there exists a set $N^* \subseteq N$, $|N^*| = s$, such that $\bigcap_{i \in N^*} A_i \neq \emptyset$, but $W \cap (\bigcup_{i \in N^*} A_i) = \emptyset$. Let c' be some candidate approved by all voters in N^* , and set $W' = (W \setminus \{c\}) \cup \{c'\}$. To obtain a contradiction, we will argue that W' has a higher Monroe score than W .

To this end, we will modify π by first swapping the voters in N^* with voters in $\pi^{-1}(c)$ and then assigning the voters in N^* to c' . Formally, let $\sigma : \pi^{-1}(c) \setminus N^* \rightarrow N^* \setminus \pi^{-1}(c)$ be a bijection between $\pi^{-1}(c) \setminus N^*$ and $N^* \setminus \pi^{-1}(c)$. We construct a mapping $\hat{\pi} : N \rightarrow W'$ by setting

$$\hat{\pi}(i) = \begin{cases} \pi(\sigma(i)) & \text{for } i \in \pi^{-1}(c) \setminus N^*, \\ c' & \text{for } i \in N^*, \\ \pi(i) & \text{for } i \notin \pi^{-1}(c) \cup N^*. \end{cases}$$

Note that $\hat{\pi}$ is a valid mapping: we have $|\hat{\pi}^{-1}(c')| = s$ and $|\hat{\pi}^{-1}(c'')| = |\pi^{-1}(c'')|$ for each $c'' \in W' \setminus \{c'\}$. Now, let us consider the impact of this modification on the

Monroe score. The s voters in N^* contributed nothing to the Monroe score of π , and they contribute s to the Monroe score of $\hat{\pi}$. By our choice of c , the voters in $\pi^{-1}(c)$ contributed at most $s - 1$ to the Monroe score of π , and their contribution to the Monroe score of $\hat{\pi}$ is non-negative. For all other voters their contribution to the Monroe score of π is equal to their contribution to the Monroe score of π' . Thus, the total Monroe score of $\hat{\pi}$ is higher than that of π . Since the Monroe score of W is equal to the Monroe score of π , and, by definition, the Monroe score of W' is at least the Monroe score of $\hat{\pi}$, we obtain a contradiction. \square

5 Extended Justified Representation

We have identified four (families of) voting rules that satisfy *JR* for arbitrary ballot profiles: **w-PAV** with $w_1 = 1$, $w_j \leq \frac{1}{j}$ for $j > 1$ (this class includes *PAV*), **w-SeqPAV** with $w_1 = 1$, $w_j \leq \frac{1}{n}$ for $j > 1$ (this class includes *GreedyAV*), *HareAV* and *MonroeAV*. The obvious advantage of *GreedyAV* and *HareAV* is that their output can be computed efficiently, whereas computing the outputs of *PAV* or *MonroeAV* is NP-hard. However, *GreedyAV* puts considerable emphasis on representing *every* voter, at the expense of ensuring that large sets of voters with shared preferences are allocated an adequate number of representatives. This approach may be problematic in a variety of applications, such as selecting a representative assembly, or choosing movies to be shown on an airplane, or foods to be provided at a banquet (see the discussion by Skowron et al. 2016). In particular, it may be desirable to have several assembly members that represent a widely held political position, both to reflect the popularity of this position, and to highlight specific aspects of it, as articulated by different candidates. Consider, for instance, the following example.

Example 2 Let $k = 3$, $C = \{a, b, c, d\}$, and $n = 100$. One voter approves c , one voter approves d , and 98 voters approve a and b . *GreedyAV* would include both c and d in the winning set, whereas in many settings it would be more reasonable to choose both a and b (and one of c and d); indeed, this is exactly what *HareAV* would do.

This issue is not addressed by the *JR* axiom, as this axiom does not care if a given voter is represented by one or more candidates. Thus, if we want to capture the intuition that large cohesive groups of voters should be allocated several representatives, we need a stronger condition. Recall that *JR* says that each group of $\frac{n}{k}$ voters that all approve the same candidate “deserves” at least one representative. It seems reasonable to scale this idea and say that, for every $\ell > 0$, each group of $\ell \cdot \frac{n}{k}$ voters that all approve the same ℓ candidates “deserves” at least ℓ representatives. This approach can be formalized as follows.

Definition 2 (Extended justified representation (EJR)) Given a ballot profile (A_1, \dots, A_n) over a candidate set C , a target committee size k , $k \leq |C|$, and an integer ℓ , $1 \leq \ell \leq k$, we say that a set of candidates W , $|W| = k$, *provides ℓ -justified representation for (\mathbf{A}, k)* if there does not exist a set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \cdot \frac{n}{k}$ such that $|\bigcap_{i \in N^*} A_i| \geq \ell$, but $|A_i \cap W| < \ell$ for each $i \in N^*$; we say that W *provides extended justified representation (EJR) for (\mathbf{A}, k)* if it provides ℓ -JR for (\mathbf{A}, k) for all

ℓ , $1 \leq \ell \leq k$. We say that an approval-based voting rule *satisfies ℓ -justified representation (ℓ -JR)* if for every profile $\mathbf{A} = (A_1, \dots, A_n)$ and every target committee size k it outputs a committee that provides ℓ -JR for (\mathbf{A}, k) . Finally, we say that a rule *satisfies extended justified representation (EJR)* if it satisfies ℓ -JR for all ℓ , $1 \leq \ell \leq k$.

Observe that *EJR* implies *JR*, because the latter coincides with 1-JR.

The definition of *EJR* interprets “a group N^* deserves at least ℓ representatives” as “at least one voter in N^* gets ℓ representatives.” Of course, other interpretations are also possible: for instance, we can require that each voter in N^* is represented by ℓ candidates in the winning committee or, alternatively, that the winning committee contains at least ℓ candidates each of which is approved by some member of N^* . However, the former requirement is too strong: it differs from *JR* for $\ell = 1$ and in Section 6 we show that there are ballot profiles for which committees with this property do not exist even for $\ell = 1$. The latter approach, which was very recently proposed by Sánchez-Fernández et al. (2016, 2017) (see the discussion in Section 7), is not unreasonable; in particular, it coincides with *JR* for $\ell = 1$. However, it is strictly less demanding than the approach we take: clearly, every rule that satisfies *EJR* also satisfies this condition. As it turns out (Theorem 10) that every ballot profile admits a committee that provides *EJR*, the *EJR* axiom offers more guidance in choosing a good winning committee than its weaker cousin, while still leaving us with a non-empty set of candidate committees to choose from. Finally, the *EJR* axiom in its present form is very similar to a core stability condition for a natural NTU game associated with the input profile (see Section 5.2); it is not clear if the axiom of Sánchez-Fernández et al. (2016, 2017) admits a similar interpretation.

5.1 Extended Justified Representation under Approval-Based Rules

It is natural to ask which of the voting rules that satisfy *JR* also satisfy *EJR*. Example 2 immediately shows that for *GreedyAV* the answer is negative. Consequently, no *w-SeqPAV* rule such that the entries of \mathbf{w} do not depend on n satisfies *EJR*: if $w_2 = 0$, this rule behaves like *GreedyAV* on the ballot profile from Example 2 and if $w_2 > 0$, our claim follows from Theorem 8. Moreover, Example 2 also implies that *w-SeqPAV* rules with $w_j \leq \frac{1}{n}$ for $j > 1$ also fail *EJR*.

The next example shows that *MAV* fails *EJR* even if each voter approves exactly k candidates (recall that under this assumption *MAV* satisfies *JR*).

Example 3 Let $k = 4$, $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where $|C_1| = |C_2| = |C_3| = |C_4| = 4$ and the sets C_1, C_2, C_3, C_4 are pairwise disjoint. Let $\mathbf{A} = (A_1, \dots, A_8)$, where $A_i = C_i$ for $i = 1, 2, 3$, and $A_i = C_4$ for $i = 4, 5, 6, 7, 8$. *MAV* will select exactly one candidate from each of the sets C_1, C_2, C_3 and C_4 , but *EJR* dictates that at least two candidates from C_4 are chosen.

Further, *MonroeAV* fails *EJR* as well.

Example 4 Let $k = 4$, $C = \{c_1, c_2, c_3, c_4, a, b\}$, $N = \{1, \dots, 8\}$, $A_i = \{c_i\}$ for $i = 1, \dots, 4$, $A_i = \{c_{i-4}, a, b\}$ for $i = 5, \dots, 8$. *MonroeAV* outputs $\{c_1, c_2, c_3, c_4\}$ on this

profile, as this is the unique set of candidates with the maximum Monroe score. Thus, every voter is represented by a single candidate, though the voters in $N^* = \{5, 6, 7, 8\}$ “deserve” two candidates.

Example 4 illustrates the conflict between the *EJR* axiom and the requirement to represent all voters whenever possible. We discuss this issue in more detail in Section 7.

For *HareAV*, it is not hard to construct an example where this rule fails *EJR* for some way of breaking intermediate ties.

Example 5 Let $N = \{1, \dots, 8\}$, $C = \{a, b, c, d, e, f\}$, $A_1 = A_2 = \{a\}$, $A_3 = A_4 = \{a, b, c\}$, $A_5 = A_6 = \{d, b, c\}$, $A_7 = \{d, e\}$, $A_8 = \{d, f\}$. Suppose that $k = 4$. Note that all voters in $N^* = \{3, 4, 5, 6\}$ approve b and c , and $|N^*| = 2 \cdot \frac{n}{k}$. Under *HareAV*, at the first step candidates a, b, c and d are tied, so we can select a and remove voters 3 and 4. Next, we have to select d ; we can then remove voters 5 and 6. In the remaining two steps, we add e and f to the committee. The resulting committee violates *EJR*, as each voter in $N^* = \{3, 4, 5, 6\}$ is only represented by a single candidate.

We note that in Example 5 we can remove voters 1 and 2 after selecting a , which enables us to select b or c in the second step and thereby obtain a committee that provides *EJR*. In fact, we were unable to construct an example where *HareAV* fails *EJR* for all ways of breaking intermediate ties; we now conjecture that it is always possible to break intermediate ties in *HareAV* so as to satisfy *EJR*. However, it is not clear if a tie-breaking rule with this property can be formulated in a succinct manner. Thus, *HareAV* does not seem particularly useful if we want to find a committee that provides *EJR*: even if our conjecture is true, we may have to explore all ways of breaking intermediate ties.

In contrast, we will now show that *PAV* satisfies *EJR* irrespective of the tie-breaking rule.

Theorem 10 *PAV satisfies EJR.*

Proof Suppose that *PAV* violates *EJR* for some value of k , and consider a ballot profile A_1, \dots, A_n , a value of $\ell > 0$ and a set of voters N^* , $|N^*| = s \geq \ell \cdot \frac{n}{k}$, that witness this. Let W , $|W| = k$, be the winning set. We know that at least one of the ℓ candidates approved by all voters in N^* is not elected; let c be some such candidate. Each voter in N^* has at most $\ell - 1$ representatives in W , so the marginal contribution of c (if it were to be added to W) would be at least $s \cdot \frac{1}{\ell} \geq \frac{n}{k}$. On the other hand, the argument in the proof of Theorem 6 can be modified to show that the sum of marginal contributions of candidates in W is at most n .

Now, consider some candidate $w \in W$ with the smallest marginal contribution; clearly, his marginal contribution is at most $\frac{n}{k}$. If it is strictly less than $\frac{n}{k}$, we are done, as we can improve the total *PAV*-score by swapping w and c , a contradiction. Therefore suppose it is exactly $\frac{n}{k}$, and therefore the marginal contribution of each candidate in W is exactly $\frac{n}{k}$. Since *PAV* satisfies *JR*, we know that $A_i \cap W \neq \emptyset$ for some $i \in N^*$. Pick some candidate $w' \in W \cap A_i$, and set $W' = (W \setminus \{w'\}) \cup \{c\}$. Observe that after w' is removed, adding c increases the total *PAV*-score by at least $(s - 1) \cdot \frac{1}{\ell} + \frac{1}{\ell - 1} > \frac{n}{k}$. Indeed, i approves at most $\ell - 2$ candidates in $W \setminus \{w'\}$ and therefore adding c to $W \setminus \{w'\}$ contributes at least $\frac{1}{\ell - 1}$ to her satisfaction. Thus, the *PAV*-score of W' is higher than that of W , a contradiction again. \square

Interestingly, Theorem 10 does not extend to weight vectors other than $(1, \frac{1}{2}, \frac{1}{3}, \dots)$: our next theorem shows that *PAV* is essentially the unique *w-PAV* rule that satisfies *EJR*.

Theorem 11 *For every weight vector \mathbf{w} with $w_1 = 1$, $\mathbf{w} \neq (1, \frac{1}{2}, \frac{1}{3}, \dots)$, the rule *w-PAV* does not satisfy *EJR*.*

Theorem 11 follows immediately from Lemmas 1 and 2, which are stated below.

Lemma 1 *Consider a weight vector \mathbf{w} with $w_1 = 1$. If $w_j > \frac{1}{j}$ for some $j > 1$, then *w-PAV* fails *JR*.*

Proof Suppose that $w_j = \frac{1}{j} + \varepsilon$ for some $j > 1$ and $\varepsilon > 0$. Pick $k > \left\lceil \frac{1}{\varepsilon j} \right\rceil + 1$ so that j divides k ; let $t = \frac{k}{j}$. Let $C = C_0 \cup C_1 \cup \dots \cup C_t$, where $C_0 = \{c\}$, $|C_1| = \dots = |C_t| = j$, and the sets C_0, C_1, \dots, C_t are pairwise disjoint. Note that $|C| = tj + 1 = k + 1$. Also, construct $t + 1$ pairwise disjoint groups of voters N_0, N_1, \dots, N_t so that $|N_0| = k$, $|N_1| = \dots = |N_t| = j(k-1)$, and for each $i = 0, 1, \dots, t$ the voters in N_i approve the candidates in C_i only. Observe that the total number of voters is given by $n = k + tj(k-1) = k^2$.

We have $|N_0| = k = \frac{n}{k}$, so every committee that provides justified representation for this profile must elect c . However, we claim that *w-PAV* elects all candidates in $C \setminus \{c\}$ instead. Indeed, if we replace an arbitrary candidate in $C \setminus \{c\}$ with c , then under *w-PAV* the total score of our committee changes by

$$k - j(k-1) \cdot \left(\frac{1}{j} + \varepsilon \right) = 1 - j(k-1)\varepsilon < 1 - j\varepsilon \left\lceil \frac{1}{\varepsilon j} \right\rceil \leq 0,$$

i.e. $C \setminus \{c\}$ has a strictly higher score than any committee that includes c . \square

Lemma 2 *Consider a weight vector \mathbf{w} with $w_1 = 1$. If $w_j < \frac{1}{j}$ for some $j > 1$, then *w-PAV* fails *j-JR*.*

Proof Suppose that $w_j = \frac{1}{j} - \varepsilon$ for some $j > 1$ and $\varepsilon > 0$. Pick $k > j + \left\lceil \frac{1}{\varepsilon} \right\rceil$. Let $C = C_0 \cup C_1$, where $|C_0| = j$, $C_1 = \{c_1, \dots, c_{k-j+1}\}$ and $C_0 \cap C_1 = \emptyset$. Note that $|C| = k + 1$. Also, construct $k - j + 2$ pairwise disjoint groups of voters $N_0, N_1, \dots, N_{k-j+1}$ so that $|N_0| = j(k-j+1)$, $|N_1| = \dots = |N_{k-j+1}| = k-j$, the voters in N_0 approve the candidates in C_0 only, and for each $i = 1, \dots, k-j+1$ the voters in N_i approve c_i only. Note that the number of voters is given by $n = j(k-j+1) + (k-j+1)(k-j) = k(k-j+1)$.

We have $\frac{n}{k} = k-j+1$ and $|N_0| = j \cdot \frac{n}{k}$, so every committee that provides *EJR* must select all candidates in C_0 . However, we claim that *w-PAV* elects all candidates from C_1 and $j-1$ candidates from C_0 instead. Indeed, let c be some candidate in C_0 , let c' be some candidate in C_1 , and let $W = C \setminus \{c\}$, $W' = C \setminus \{c'\}$. The difference between the total score of W and that of W' is

$$j(k-j+1) \left(\frac{1}{j} - \varepsilon \right) - (k-j) < 1 - j \cdot \frac{1}{\varepsilon} \cdot \varepsilon < 1 - j < 0,$$

i.e. *w-PAV* assigns a higher score to W . As this argument does not depend on the choice of c in C_0 and c' in C_1 , the proof is complete. \square

5.2 JR, EJR and Core Stability

One can view (extended) justified representation as a stability condition, by associating committees that provide JR/EJR with outcomes of a certain NTU game that are resistant to certain types of deviations.

Specifically, given a pair (\mathbf{A}, k) , where $\mathbf{A} = (A_1, \dots, A_n)$, we define an NTU game $\mathcal{G}(\mathbf{A}, k)$ with the set of players N as follows. We assume that each coalition of size x , $\ell \frac{n}{k} \leq x < (\ell+1) \frac{n}{k}$, where $\ell \in \{1, \dots, k\}$, can “purchase” ℓ alternatives. Moreover, each player evaluates a committee of size ℓ , $\ell \in \{1, \dots, k\}$, using the PAV utility function, i.e. i derives a utility of $1 + \frac{1}{2} + \dots + \frac{1}{j}$ from a committee that contains exactly j of her approved alternatives (the argument goes through for \mathbf{w} -PAV utilities, as long as $w_1 \geq \dots \geq w_k > 0$). Thus, for each coalition S with $\ell \frac{n}{k} \leq |S| < (\ell+1) \frac{n}{k}$ a payoff vector $\mathbf{x} \in \mathbb{R}^n$ is considered to be feasible for S if and only if there exists a committee $W \subseteq C$ with $|W| \leq \ell$ such that $x_i = u_i(W)$ for each $i \in S$, where $u_i(W) = 1 + \dots + \frac{1}{|A_i \cap W|}$. We denote the set of all payoff vectors that are feasible for a coalition $S \subseteq N$ by $V(S)$.

We say that a coalition $S \subseteq N$ has a *profitable deviation* from a payoff vector $\mathbf{x} \in V(N)$ if there exists a payoff vector $\mathbf{y} \in V(S)$ such that $y_i > x_i$ for all $i \in S$. A payoff vector \mathbf{x} is *stable* if it is feasible for N and no coalition $S \subseteq N$ has a profitable deviation from it; the set of all stable payoff vectors is the *core* of $\mathcal{G}(\mathbf{A}, k)$.

The following theorem describes the relationship between JR, EJR, and profitable deviations in $\mathcal{G}(\mathbf{A}, k)$.

Theorem 12 *A committee W , $|W| = k$, provides justified representation for (\mathbf{A}, k) if and only if no coalition of size $\lceil \frac{n}{k} \rceil$ or less has a profitable deviation from the payoff vector \mathbf{x} associated with W . Moreover, W provides extended justified representation for (\mathbf{A}, k) if and only if for every integer $\ell \geq 0$ no coalition N^* with $\ell \cdot \frac{n}{k} \leq |N^*| < (\ell+1) \cdot \frac{n}{k}$, $|\cap_{i \in N^*} A_i| \geq \ell$ has a profitable deviation from \mathbf{x} .*

Proof Suppose that W fails to provide justified representation for (\mathbf{A}, k) , i.e. there exists a set of voters N^* , $|N^*| = \lceil \frac{n}{k} \rceil$, who all approve some candidate $c \notin W$, but none of them approves any of the candidates in W . Then we have $x_i = 0$ for each $i \in N^*$, and players in N^* can successfully deviate: the payoff vector \mathbf{y} that is associated with the committee $\{c\}$ is feasible for N^* and satisfies $y_i = 1$ for each $i \in N^*$.

Conversely, suppose that W provides justified representation for (\mathbf{A}, k) , and consider a coalition N^* . If $|N^*| < \lceil \frac{n}{k} \rceil$, then for every $\mathbf{y} \in V(N^*)$ we have $y_i = 0$ for all $i \in N^*$, so N^* cannot profitably deviate. On the other hand, if $|N^*| = \lceil \frac{n}{k} \rceil$, then every payoff vector $\mathbf{y} \in V(N^*)$ is associated with a committee of size 1. Hence, for every $\mathbf{y} \in V(N^*)$ we have $y_i \leq 1$ for all $i \in N^*$, and if $y_i = 1$ for all $i \in N^*$, then $\cap_{i \in N^*} A_i \neq \emptyset$, and therefore, since W provides JR, we have $x_i \geq 1$ for some $i \in N^*$. Thus, N^* has no profitable deviation.

For EJR the argument is similar. If W fails to provide extended justified representation for (\mathbf{A}, k) , there exists an $\ell > 0$ and a set of voters N^* , $|N^*| \geq \ell \cdot \frac{n}{k}$, such that $|\cap_{i \in N^*} A_i| \geq \ell$, but $|A_i \cap W| < \ell$ for each $i \in N^*$. Then we have $x_i < 1 + \dots + \frac{1}{\ell}$ for each $i \in N^*$, and players in N^* can successfully deviate: if S is a committee that consists of some ℓ candidates in $\cap_{i \in N^*} A_i$, then the payoff vector \mathbf{y} that is associated with S is feasible for N^* and satisfies $y_i = 1 + \dots + \frac{1}{\ell}$ for each $i \in N^*$.

Conversely, suppose that W provides extended justified representation for (\mathbf{A}, k) , and consider some $\ell \geq 0$ and some coalition N^* with $\ell \cdot \frac{n}{k} \leq |N^*| < (\ell + 1) \frac{n}{k}$. We have argued above that if $\ell = 0$, then N^* cannot profitably deviate. Thus, assume $\ell > 0$. Every payoff vector $\mathbf{y} \in V(N^*)$ is associated with a committee of size ℓ . Hence, for every $\mathbf{y} \in V(N^*)$ we have $y_i \leq 1 + \dots + \frac{1}{\ell}$ for all $i \in S$, and if $y_i = 1 + \dots + \frac{1}{\ell}$ for all $i \in N^*$, then $|\cap_{i \in N^*} A_i| \geq \ell$. Since W provides *EJR*, we have $x_i \geq 1 + \dots + \frac{1}{\ell}$ for some $i \in N^*$. Again, this implies that N^* has no profitable deviation. \square

The second part of Theorem 12 considers deviations by cohesive coalitions. The reader may wonder if it can be strengthened to *arbitrary* coalitional deviations, i.e. whether a committee provides *EJR* if and only if the associated payoff vector is in the core of $\mathcal{G}(\mathbf{A}, k)$. The following example shows that this is not the case.

Example 6 Let $k = 10$, $C = \{x_1, x_2, \dots, x_{10}, y, z\}$, $N = \{1, 2, \dots, 20\}$, and

$$\begin{aligned} A_1 &= A_2 = A_3 = \{x_1, y\}, \\ A_4 &= A_5 = A_6 = \{x_1, z\}, \\ A_7 &= \dots = A_{20} = \{x_2, \dots, x_{10}\}. \end{aligned}$$

Then *PAV* outputs the committee $W = \{x_1, x_2, \dots, x_{10}\}$ for (\mathbf{A}, k) ; in particular, W provides *EJR* for (\mathbf{A}, k) . However, the associated payoff vector \mathbf{x} is not in the core, as the players in $\{1, 2, 3, 4, 5, 6\}$, a coalition of size $3 \frac{n}{k}$, can successfully deviate: the payoff vector associated with $\{x_1, y, z\}$ is feasible for $\{1, 2, 3, 4, 5, 6\}$ and provides a higher payoff than \mathbf{x} to each of the first six players. We remark that the core of $\mathcal{G}(\mathbf{A}, k)$ is not empty: in particular, it contains the payoff vector associated with $\{x_1, \dots, x_8, y, z\}$.

It remains an open question whether the core of $\mathcal{G}(\mathbf{A}, k)$ is non-empty for every pair (\mathbf{A}, k) . Further, while it would be desirable to have a voting rule that outputs a committee whose associated payoff vector is in the core whenever the core is not empty, we are not aware of any such rule: every voting rule that fails *EJR* also fails this more demanding criterion, and Example 6 illustrates that *PAV* fails this criterion as well.

5.3 Computational Issues

In Section 3 we have argued that it is easy to find a committee that provides *JR* for a given ballot profile, and to check whether a specific committee provides *JR*. In contrast, for *EJR* these questions appear to be computationally difficult. Specifically, we were unable to design an efficient algorithm for computing a committee that provides *EJR*; while *PAV* is guaranteed to find such a committee, computing its output is NP-hard. We remark, however, that when ℓ is bounded by a constant, we can efficiently compute a committee that provides ℓ -*JR*.

Theorem 13 *A committee that provides ℓ -JR can be computed in time polynomial in n and $|C|^\ell$.*

Proof Consider the following greedy algorithm, which we will refer to as ℓ -GreedyAV. We start by setting $C' = C$, $\mathbf{A}' = \mathbf{A}$, and $W = \emptyset$. As long as $|W| \leq k - \ell$, we check if there exists a set of candidates $\{c_1, \dots, c_\ell\} \subseteq C'$ that is unanimously approved by at least $\ell \frac{n}{k}$ voters in \mathbf{A}' (this can be done in time $n \cdot |C|^{\ell+1}$). If such a set exists, we set $W := W \cup \{c_1, \dots, c_\ell\}$ and we remove from \mathbf{A}' all ballots A_i such that $|A_i \cap W| \geq \ell$ (note that this includes all ballots A_i with $\{c_1, \dots, c_\ell\} \subseteq A_i$). If at some point we have $|W| \leq k - \ell$ and no set $\{c_1, \dots, c_\ell\}$ satisfies our criterion or $|W| > k - \ell$, we add an arbitrary subset of $k - |W|$ candidates from C' to W and return W ; if this does not happen, we terminate after having picked k candidates.

Suppose for the sake of contradiction that for some profile $\mathbf{A} = (A_1, \dots, A_n)$ and some $k > 0$, ℓ -GreedyAV outputs a committee that does not provide ℓ -JR for (\mathbf{A}, k) . Then there exists a set $N^* \subseteq N$ with $|N^*| \geq \ell \frac{n}{k}$ such that $|\bigcap_{i \in N^*} A_i| \geq \ell$ and, when ℓ -GreedyAV terminates, every ballot A_i such that $i \in N^*$ is still in \mathbf{A}' . Consider some subset of candidates $\{c_1, \dots, c_\ell\} \subseteq \bigcap_{i \in N^*} A_i$. At every point in the execution of ℓ -GreedyAV this subset is unanimously approved by at least $|N^*| \geq \ell \frac{n}{k}$ ballots in \mathbf{A}' . As at least one of $\{c_1, \dots, c_\ell\}$ was not elected, at every stage the algorithm selected a set of ℓ candidates that was approved by at least $\ell \frac{n}{k}$ ballots (until more than $k - \ell$ candidates were selected). Since at the end of each stage the algorithm removed from \mathbf{A}' all ballots containing the candidates that had been added to W at that stage, it follows that altogether the algorithm has removed at least $\lfloor \frac{k}{\ell} \rfloor \cdot \ell \frac{n}{k} > (\frac{k}{\ell} - 1) \cdot \ell \frac{n}{k} = n - \ell \frac{n}{k}$ ballots from \mathbf{A}' . This is a contradiction, since we assumed that, when the algorithm terminates, the $\ell \frac{n}{k}$ ballots $(A_i)_{i \in N^*}$ are still in \mathbf{A}' . \square

For the problem of checking whether a given committee provides *EJR* for a given input, we can establish a formal hardness result.

Theorem 14 *Given a ballot profile \mathbf{A} , a target committee size k , and a committee W , $|W| = k$, it is coNP-complete to check whether W provides *EJR* for (\mathbf{A}, k) .*

Proof It is easy to see that this problem is in coNP: to show that W does not provide *EJR* for (\mathbf{A}, k) , it suffices to guess an integer $\ell > 0$ and a set of voters N^* of size at least $\ell \cdot \frac{n}{k}$ such that $|\bigcap_{i \in N^*} A_i| \geq \ell$, but $|A_i \cap W| < \ell$ for all $i \in N^*$.

To prove coNP-completeness, we reduce the classic BALANCED BICLIQUE problem (Garey and Johnson 1979, [GT24]) to the complement of our problem. An instance of BALANCED BICLIQUE is given by a bipartite graph (L, R, E) with parts L and R and edge set E , and an integer ℓ ; it is a “yes”-instance if we can pick subsets of vertices $L' \subseteq L$ and $R' \subseteq R$ so that $|L'| = |R'| = \ell$ and $(u, v) \in E$ for each $u \in L', v \in R'$; otherwise, it is a “no”-instance.

Given an instance $\langle (L, R, E), \ell \rangle$ of BALANCED BICLIQUE with $R = \{v_1, \dots, v_s\}$, we create an instance of our problem as follows. Assume without loss of generality that $s \geq 3$, $\ell \geq 3$. We construct 4 pairwise disjoint sets of candidates C_0, C_1, C'_1, C_2 , so that $C_0 = L$, $|C_1| = |C'_1| = \ell - 1$, $|C_2| = s\ell + \ell - 3s$, and set $C = C_0 \cup C_1 \cup C'_1 \cup C_2$. We then construct 3 sets of voters N_0, N_1, N_2 , so that $N_0 = \{1, \dots, s\}$, $|N_1| = \ell(s - 1)$, $|N_2| = s\ell + \ell - 3s$ (note that $|N_2| > 0$ as we assume that $\ell \geq 3$). For each $i \in N_0$ we set $A_i = \{u_j \mid (u_j, v_i) \in E\} \cup C_1$, and for each $i \in N_1$ we set $A_i = C_0 \cup C'_1$. The candidates in C_2 are matched to voters in N_2 : each voter in N_2 approves exactly one candidate in C_2 , and each candidate in C_2 is approved by exactly one voter in N_2 . Denote the

resulting list of ballots by \mathbf{A} . Finally, we set $k = 2\ell - 2$, and let $W = C_1 \cup C'_1$. Note that the number of voters n is given by $s + \ell(s - 1) + s\ell + \ell - 3s = 2s(\ell - 1)$, so $\frac{n}{k} = s$.

Suppose first that we started with a “yes”-instance of BALANCED BICLIQUE, and let (L', R') be the respective ℓ -by- ℓ biclique. Let $C^* = L'$, $N^* = N_1 \cup \{i \in N_0 \mid v_i \in R'\}$. Then $|N^*| = \ell s$, all voters in N^* approve all candidates in C^* , $|C^*| = \ell$, but each voter in N^* is only represented by $\ell - 1$ candidates in W . Hence, W fails to provide ℓ -justified representation for (\mathbf{A}, k) .

Conversely, suppose that W fails to provide *EJR* for (\mathbf{A}, k) . That is, there exists a value $j > 0$, a set N^* of js voters and a set C^* of j candidates so that all voters in N^* approve of all candidates in C^* , but for each voter in N^* at most j of her approved candidates are in W . Note that, since $s > 1$, we have $N^* \cap N_2 = \emptyset$. Further, each voter in $N \setminus N_2$ is represented by $\ell - 1$ candidates in W , so $j \geq \ell$. As $N^* = js \geq \ell s \geq s$, it follows that $|N^* \cap N_0| \geq \ell$, $|N^* \cap N_1| > 0$. Since N^* contains voters from both N_0 and N_1 , it follows that $C^* \subseteq C_0$. Thus, there are at least ℓ voters in $N^* \cap N_0$ who approve the same $j \geq \ell$ candidates in C_0 ; any set of ℓ such voters and ℓ such candidates corresponds to an ℓ -by- ℓ biclique in the input graph. \square

6 Variants of Justified Representation

The definition of *JR* requires that if there is a group of $\lceil \frac{n}{k} \rceil$ voters who jointly approve some candidate, then the elected committee has to contain at least one candidate approved by *some* member of this group. This condition may appear to be too weak; it may seem more natural to require that *every* group member approves some candidate in the committee, or—stronger yet—that the committee contains at least one candidate approved by *all* group members. This intuition is captured by the following definitions.

Definition 3 Given a ballot profile $\mathbf{A} = (A_1, \dots, A_n)$ and a target committee size k , we say that a committee W of size k provides

- *semi-strong justified representation for (\mathbf{A}, k)* if for each group $N^* \subseteq N$ with $|N^*| \geq \frac{n}{k}$ and $\bigcap_{i \in N^*} A_i \neq \emptyset$ it holds that $W \cap A_i \neq \emptyset$ for all $i \in N^*$.
- *strong justified representation for (\mathbf{A}, k)* if for each group $N^* \subseteq N$ with $|N^*| \geq \frac{n}{k}$ and $\bigcap_{i \in N^*} A_i \neq \emptyset$ it holds that $W \cap (\bigcap_{i \in N^*} A_i) \neq \emptyset$.

By definition, a committee providing strong justified representation also provides semi-strong justified representation, and a committee providing semi-strong justified representation also provides (standard) justified representation.

However, it turns out that satisfying these stronger requirements is not always feasible: there are ballot profiles for which no committee provides semi-strong justified representation.

Example 7 Let $k = 3$ and consider the following profile with $C = \{a, b, c, d\}$ and $n = 9$.

$$\begin{array}{llll} A_1 = A_2 = \{a\} & A_3 = \{a, b\} & A_4 = \{b\} & A_5 = \{b, c\} \\ A_6 = \{c\} & A_7 = \{c, d\} & A_8 = A_9 = \{d\} & \end{array}$$

For each candidate $x \in C$, there are $\frac{n}{k} = 3$ voters such that $\cap_i A_i = \{x\}$, and at least one of those voters has $A_i = \{x\}$. Thus, a committee that satisfies semi-strong justified representation would have to contain all four candidates, which is impossible.

While Example 7 shows that no approval-based voting rule can always find a committee that provides strong or semi-strong justified representation, it may be interesting to identify voting rules that output such committees whenever they exist.

Finally, we remark that strong justified representation does not imply *EJR*.

Example 8 Let $C = \{a, b, c, d, e\}$, $n = 4$, $k = 4$, and consider the following ballot profile.

$$A_1 : \{a, b\} \quad A_2 : \{a, b\} \quad A_3 : \{c\} \quad A_4 : \{d, e\}$$

EJR requires that we choose both a and b , but $\{a, c, d, e\}$ provides strong justified representation.

7 Related Work

It is instructive to compare *JR* and *EJR* to alternative approaches towards fair representation, such as *representativeness* (Duddy 2014) and *proportional justified representation* (Sánchez-Fernández et al. 2016, 2017).

Duddy (2014) proposes the notion of *representativeness*, which applies to probabilistic voting rules. The property Duddy proposes is incomparable with *JR*: in situations he considers ($k = 2$, n voters approve x , $n + 1$ voters approve y and z), *JR* requires that one of y and z should be selected, whereas Duddy requires x to be selected with positive probability. Both are reasonable requirements, but they address different concerns. Duddy's axiom say nothing about situations where voters are split equally (say, n voters approve $\{x, y\}$, n voters approve $\{z, t\}$), whereas *JR* requires that each voter is represented. Another obvious difference is that he allows for randomized rules.

Very recently (after the conference version of our paper was published), Sánchez-Fernández et al. (2016, 2017) came up with the notion of *proportional justified representation (PJR)*, which can be seen as an alternative to *EJR*. A committee is said to provide *PJR* for a ballot profile (A_1, \dots, A_n) over a candidate set C and a target committee size k if, for every positive integer ℓ , $\ell \leq k$, there does not exist a set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \cdot \frac{n}{k}$ such that $|\cap_{i \in N^*} A_i| \geq \ell$, but $|(\cup_{i \in N^*} A_i) \cap W| < \ell$. In contrast to *EJR*, the *PJR* condition does not require one of the voters in N^* to have ℓ representatives. Rather, a committee provides *PJR* as long as it contains ℓ candidates that are approved by (possibly different) voters in N^* , for every group N^* satisfying the size and cohesiveness constraints. An attractive feature of *PJR* is that it is compatible with the idea of *perfect representation*: a committee W provides perfect representation for a group of n voters and a target committee size k if $n = ks$ for some positive integer s and the voters can be split into k pairwise disjoint groups N_1, \dots, N_k of size s each in such a way that there is a one-to-one mapping $\mu : W \rightarrow \{N_1, \dots, N_k\}$ such that for each candidate $c \in W$ all voters in $\mu(c)$ approve c . Sánchez-Fernández et al. (2016,

2017) prove that every committee that provides perfect representation also provides *PJR*; in contrast, *EJR* may rule out all committees that provide perfect representation, as illustrated by Example 4. It is easily seen that *PJR* is a weaker requirement than *EJR*, and a stronger one than *JR*. Interestingly, Sánchez-Fernández et al. (2016, 2017) show that many results that we have established for *EJR* also hold for *PJR*: in particular, *w-SeqPAV* violates *PJR* for every weight vector \mathbf{w} with $w_1 = 1$, and *w-PAV* with $w_1 = 1$ satisfies *PJR* if and only if $\mathbf{w} = (1, \frac{1}{2}, \frac{1}{3}, \dots)$.

8 Conclusions

We have formulated a desirable property of approval-based committee selection rules, which we called justified representation (*JR*). While *JR* is fairly easy to satisfy, it turns out that many well-known approval-based rules fail it. A prominent exception is the *PAV* rule, which also satisfies a stronger version of this property, namely extended justified representation (*EJR*). Indeed, *EJR* characterizes *PAV* within the class of *w-PAV* rules, and we are not aware of any other natural voting rule that satisfies *EJR* irrespective of the tie-breaking rule (of course, we can construct voting rules that differ from *PAV*, yet satisfy *EJR*, by modifying the output of *PAV* on profiles on which *EJR* places no constraints on the output). Table 1 summarizes the representation properties and computational complexity of approval-based voting rules.

Perhaps the most pressing open question suggested by our work is whether there is an efficient algorithm for finding a committee that provides *EJR* for a given profile. In particular, we would like to understand whether we can break ties in the execution of *HareAV* to always produce such a committee, and whether some tie-breaking rule with this property is polynomial-time computable. Also, it would be interesting to see if *EJR*, in combination with other natural axioms, can be used to axiomatize *PAV*. Concerning (semi-)strong justified representation, an interesting computational problem is whether there are efficient algorithm for checking the existence of committees satisfying these requirements.

Rule	<i>JR</i>	<i>EJR</i>	Complexity
<i>AV</i>	–	–	in P
<i>SAV</i>	–	–	in P
<i>MAV</i>	–	–	NP-hard
<i>SeqPAV</i>	–	–	in P
<i>GreedyAV</i>	✓	–	in P
<i>HareAV</i>	✓	–*	in P
<i>MonroeAV</i>	✓	–	NP-hard
<i>PAV</i>	✓	✓	NP-hard

Table 1 Satisfaction of *JR* and *EJR* and computational complexity of approval-based voting rules; the superscript ‘*’ indicates that the rule fails the respective axiom for some way of breaking intermediate ties.

JR and *EJR* can also be used to formulate new approval-based rules. We mention two types of rules that seem particularly attractive:

The *utilitarian (E)JR rule* returns a committee that, among all committees that satisfy *(E)JR*, has the highest *AV* score.

The *egalitarian (E)JR rule* returns a committee that, among all committees that satisfy *(E)JR*, maximizes the number of representatives of the voter who has the least number of representatives in the winning committee.

The computational complexity of winner determination for these rules is an interesting problem. Since *PAV* is NP-hard to compute, our study also provides additional motivation for the use of approximation and parameterized algorithms to compute *PAV* outcomes. Finally, analyzing the compatibility of *(E)JR* with other important properties, such as, e.g., strategyproofness for dichotomous preferences, is another avenue of future research.

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