Male optimal and unique stable marriages with partially ordered preferences

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Abstract. The stable marriage problem has a wide variety of practical applications, including matching resident doctors to hospitals, and students to schools. In the classical stable marriage problem, both men and women express a strict order over the members of the other sex. Here we consider a more realistic case, where both men and women can express their preferences via partial orders, i.e., by allowing ties and incomparability. This may be useful, for example, when preferences are elicited via compact preference representations like soft constraint or CP-nets that produce partial orders, as well as when preferences are obtained via multi-criteria reasoning. We study male optimality and uniqueness of stable marriages in this setting. Male optimality gives priority to one gender over the other, while uniqueness means that the solution is optimal, since it is as good as possible for all the participating agents. Uniqueness of solution is also a barrier against manipulation. We give an algorithm to find stable marriages that are male optimal. Moreover, we give sufficient conditions on the preferences (that are also necessary in some special case), that occur often in real-life scenarios, which guarantee the uniqueness of a stable marriage.

1 Introduction

The stable marriage problem (SM) [9] is a well-known collaboration problem. Given n men and n women, where each expresses a strict ordering over the members of the opposite sex, the problem is to match the men to the women so that there are no two people of opposite sex who would both rather be matched with each other than their current partners. In [6] Gale and Shapley proved that it is always possible to find a matching that makes all marriages stable, and provided a quadratic time algorithm which can be used to find one of two extreme stable marriages, the so-called *male optimal* or *female optimal* solutions. The Gale-Shapley algorithm has been used in many real-life scenarios, such as in matching hospitals to resident doctors, medical students to hospitals [10], sailors to ships, primary school students to secondary schools, as well as in market trading.

In the classical stable marriage problem, both men and women express a strict order over the members of the other sex. We consider a potentially more realistic case, where men and women express their preferences via *partial orders*, i.e., given a pair of men (resp., women), the women (resp., the men) can strictly order the elements of the pair, they may say that these elements are in a tie, or that they are incomparable. This is useful in practical applications when a person may not wish (or be able) to choose between alternatives, thus allowing ties in the preference list (or more generally, allowing each preference list to be a partial order) [11]. For example, in the context of centralized matching scheme, some participating hospitals with many applicants have found the task of producing a strictly ordered preference list difficult, and have expressed a desire to use ties [12]. Ties also naturally occur when assigning students to schools, since many students are indistinguishable from the point of view of a given school. Another situation where partial orders are useful is when preferences are elicited with a compact preference representation formalism like soft constraints [1] or CP-nets [2] that give partial orders. Another context where partial orders naturally occur is when preferences are obtained via multi-criteria reasoning.

We study male optimality and uniqueness of solution in this more general context. *Male optimality* can be a useful property since it allows us to give priority to one gender over the other. For example, in matching residents to hospitals in the US, priority is given to the residents. We present an algorithm, based on an extended version of the Gale-Shapely algorithm, to find a male optimal solution in stable marriage problems with partially ordered preferences (SMPs). This algorithm is sound but not complete: it may fail to find a male optimal solution even when one exists. We conjecture, however, that the incompleteness is rare. We also give a sufficient condition on the preference profile that guarantees to find a male optimal solution, and we show how to find it.

Uniqueness is another interesting concept. For instance, it guarantees that the solution is optimal since it is as good as possible for all the participating agents. Uniqueness is also a barrier against manipulation. This is important as *all* stable marriage procedures can be manipulated. In [5] sufficient conditions on the preferences are given, that guarantee uniqueness of stable marriages when only strictly ordered preferences are allowed. Such conditions identify classes of preferences that are broad and of particular interest in many real-life scenarios [4]. In particular, a class of preference orderings that satisfy one of these conditions requires that all the agents of the same sex have identical preferences over the mates of the opposite sex, i.e., there is a common ordering over the mates. Another class of preference orderings that satisfy one of these conditions of uniqueness requires that each agent has a different most preferred mate, i.e., there is a subjective ranking over the mates. We show that it is possible to generalize these sufficient conditions for uniqueness to SMs with partially ordered preferences, by considering in some cases uniqueness up to indifference and incomparability.

All the proofs have been omitted due to lack of space. They can be found in [8]. A brief overview of some of the theoretical results shown in this paper is contained in [7].

2 Background

2.1 Stable marriage problems

Definition 1 (profile). *Given n men and n women, a profile is a sequence of $2n$ strict total orders (i.e., transitive and complete binary relations), n over the men and n over the women.*

Given a profile, the stable marriage problem (SM) [6] is the problem of finding a matching between men and women so that there are no two people of opposite sex who would both rather be married to each other than their current partners. If there are no such people, the marriage is said to be stable.

Definition 2 (feasible partner). *Given an SM P, a feasible partner for a man m (resp., a woman w) is a woman w (resp., a man m) such that there is a stable marriage for P where m and w are married.*

The set of the stable marriages for an SM forms a lattice w.r.t. the men's (or women's) preferences. This is a graph where vertices correspond bijectively to the stable marriages and a marriage is above another if every man (resp., every woman) is married with a woman (resp., man) is at least as happy with the first marriage as with the second. The top of this lattice is the stable marriage, called male optimal (resp., female optimal), where men (resp., women) are mostly satisfied. Conversely, the bottom is the stable marriage where men's (resp., women's) preferences are least satisfied [9].

Definition 3 (male (resp., female) optimal marriage). *Given an SM P, a marriage is male (resp., female) optimal iff every man (resp., woman) is paired with his (resp., her) highest ranked feasible partner in P.*

2.2 Gale-Shapley algorithm

The *Gale-Shapley* (GS) *algorithm* [6] is a well-known algorithm to solve the SM problem. At the start of the algorithm, each person is free and becomes engaged during the execution of the algorithm. Once a woman is engaged, she never becomes free again (although to whom she is engaged may change), but men can alternate between being free and being engaged. The following step is iterated until all men are engaged: choose a free man m, and let m propose to the most preferred woman w on his preference list, such that w has not already rejected m. If w is free, then w and m become engaged. If w is engaged to man m', then she rejects the man (m or m') that she least prefers, and becomes, or remains, engaged to the other man. The rejected man becomes, or remains, free. When all men are engaged, the engaged pairs are a male optimal stable marriage.

This algorithm needs a number of steps that is quadratic in n (that is, the number of men), and it guarantees that, if the number of men and women coincide, and all participants express a strict order over all the members of the other group, everyone gets married, and the returned marriage is stable. Since the input includes the profiles, the algorithm is linear in the size of the input.

Example 1. Assume $n = 3$. Let $W = \{w_1, w_2, w_3\}$ and $M = \{m_1, m_2, m_3\}$ be respectively the set of women and men. The following sequence of strict total orders defines a profile: $\{m_1 : w_1 > w_2 > w_3$ (i.e., man m_1 prefers woman w_1 to w_2 to w_3); $m_2 : w_2 > w_1 > w_3$; $m_3 : w_3 > w_2 > w_1\}$ $\{w_1 : m_1 > m_2 > m_3$; $w_2 : m_3 > m_1 > m_2$; $w_3 : m_2 > m_1 > m_3\}$. For this profile, the Gale-Shapley algorithm returns the male optimal solution $\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$. On the other hand, the female optimal solution is $\{(w_1, m_1), (w_2, m_3), (w_3, m_2)\}$. □

The *Extended Gale-Shapely algorithm* [9] is the GS algorithm [6] where, whenever the proposal of a man m to a woman w is accepted, in w's preference list all men less desirable than m are deleted, and w is deleted from the preference lists of all such men. This means that, every time that a woman receives a proposal from a man, she accepts since only most preferred men can propose to her.

3 Stable marriage problems with partial orders

We assume now that men and women express their preferences via partial orders. The notions given in Section 2 can be generalized as follows.

Definition 4 (partially ordered profile). *Given n men and n women, a profile is a sequence of $2n$ partial orders (i.e., reflexive, antisymmetric and transitive binary relations), n over the men and n over the women.*

Definition 5 (SMP). *A stable marriage problem with partial orders (SMP) is just a SM where men's preferences and women's preference are partially ordered.*

Definition 6 (linearization of an SMP). *A linearization of an SMP is an SM that is obtained by giving a strict ordering to all the pairs that are not strictly ordered such that the resulting ordering is transitive.*

Definition 7 (weakly stable marriage in SMP). *A marriage in an SMP is weakly stable if there is no pair (x, y) such that each one strictly prefers the other to his/her current partner.*

Definition 8 (feasible partner in SMP). *Given an SMP P, a feasible partner for a man m (resp., woman w) is a woman w (resp., man m) such that there is a weakly stable marriage for P where m and w are married.*

A weakly stable marriage is male optimal if there is no man that can get a strictly better partner in some other weakly stable marriage.

Definition 9 (male optimal weakly stable marriage). *Given an SMP P, a weakly stable marriage of P is male optimal iff there is no man that prefers to be married with another feasible partner of P.*

In SMs there is always exactly one male optimal stable marriage. In SMPs, however, we can have zero or more male optimal weakly stable marriages. Moreover, given an SMP P, all the stable marriages of the linearizations of P are weakly stable marriages. However, not all these marriages are male optimal.

Example 2. In a setting with 2 men and 2 women, consider the profile P: $\{m_1 : w_1 \bowtie w_2$ ($\bowtie$ means incomparable); $m_2 : w_2 > w_1;\}$ $\{w_1 : m_1 \bowtie m_2;$ $w_2 : m_1 \bowtie m_2;\}$. Then consider the following linearization of P, say Q: $\{m_1 : w_2 > w_1;$ $m_2 : w_2 > w_1;\}$ $\{w_1 : m_2 > m_1;$ $w_2 : m_1 > m_2;\}$. If we apply the extended GS algorithm to Q, we obtain the weakly stable marriage μ_1 where m_1 marries w_2 and m_2 marries w_1. However, w_1 is not the most preferred woman for m_2 amongst all weakly stable marriages. In fact, if we consider the linearization Q', obtained from Q, by changing m_1's preferences as follows: $m_1 : w_1 > w_2$, and if we apply the extended GS algorithm, we obtain the weakly stable marriage μ_2, where m_1 is married with w_1 and m_2 is married with w_2, i.e., m_2 is married with a woman that m_2 prefers more than w_1. Notice that μ_2 is male optimal, while μ_1 is not. Also, μ_1 and μ_2 are the only weakly stable marriages for this example. □

4 Finding male optimal weakly stable marriages

We now present an algorithm, called *MaleWeaklyStable* 1, that takes as input an SMP P and, either returns a male optimal weakly stable marriage for P, or the string 'I don't know'. This algorithm is sound but not complete: if the algorithm returns a marriage, then it is weakly stable and male optimal; however, it may fail to return a male optimal marriage even if there is one.

We assume that the women express strict total orders over the men. If they don't, we simply pick any linearization.

The algorithm exploits the extended GS algorithm [9], and at every step orders the free men by increasing number of their current top choices (i.e., the alternatives that are undominated). List L contains the current ordered sequence of free men.

More precisely, our algorithm works as follows. It takes in input an SMP P, and it computes the list L of free men. At the beginning all the men are unmarried, and thus L contains them all. Then, we continue to check the following cases on the man m which is the first element of L, until they do not occur any longer:

- If the set of top choices of m contains exactly one unmarried woman, say w, m proposes to w and, since we are using the extended GS algorithm, the proposal is accepted. Then, all men that are strictly worse than m in w's preferences are removed from w's preference list, and w is removed from the preference lists of these men. Then, m is removed from L and L is ordered again, since the top choices of some men may now be smaller.
- If m has a single top choice, say w, that is already married, m proposes to w, w accepts the proposal, and she breaks the engagement with her current partner, say m'. Then, m is removed from L, m' becomes free and is put back in L, and L is ordered again.

When we exit from this cycle, we check if L is empty or not:

- if L is empty, the algorithm returns the current marriage. Notice that the current marriage, say (m_i, w_i), for $i = 1, \ldots, n$, is weakly stable, since it is the solution of a linearization of P where, for every m_i with ties or incomparability in current set of top choices, we have set w_i strictly better than all the other women in the top choice. Also, the returned marriage is male optimal since we have applied the extended GS algorithm.
- If L is not empty, it means that the next free man in L has several current top choices and more than one is unmarried.
  - If there is a way to assign to the men currently in L different unmarried women from their current top choices then these men make these proposals, that are certainly accepted by the women, since every woman receives a proposal from a different man. Therefore, we add to the current marriage these new pairs and we return the resulting marriage. Such a marriage is weakly stable and male optimal by construction.
  - If it is not possible to make the above assignment, the algorithm removes unfeasible women from the current top choices of the men until it is possible to make the assignment or until all unfeasible women have been removed. More precisely, if there is a set S of men in L with the same Top T and the cardinality of T is smaller than the cardinality of S, we check if there is a man m^* such that, for every $w \in T$, m^* is worse than m_i for every m_i in $S - m^*$. If this is the case, for every woman w in T, we remove w from the preferences of m^* in p and we apply again the algorithm MaleWeaklyStable to the profile obtained so far.
    This could make now possible to make the assignment. If so, the algorithm adds to the current marriage these new pairs and returns the resulting marriage; otherwise, it performs the same reasoning for another pair of men that have some woman in common in their current top choices until all such pairs of men have been considered and no marriage has been returned. At this point the algorithm stops returning the string 'I don't know'.

Example 3. Consider the profile $\{m_1 : w_1 \bowtie w_2 > w_3;\ m_2 : w_1 \bowtie w_2 > w_3;\ m_3 : w_1 \bowtie w_2 > w_3;\}$ $\{w_1 : m_1 > m_2 > m_3;\ w_2 : m_1 > m_2 > m_3;\ w_3 : m_1 > m_2 > m_3;\}$. The algorithm first computes the ordered list $L = [m_1,\ m_2, m_3]$. The elements of L are men with more than one top choice and all these top choices are unmarried, but there is no way to assign them with different women from their top choices, since they are three men and the union of their top choices contains only two women. However, in every linearization, m_3 will not be matched with w_1 or w_2, due to w_1 and w_2's preferences. In fact, m_1 and m_2 will choose between $\{w_1, w_2\}$, while m_3 will always propose to his next best choice, i.e., w_3. Hence, the considered profile is one of the profiles where only two of the three men with multiple top choices are feasible with w_1 and w_2, i.e. m_1 and m_2, and there is a way to assign to these men different unmarried women in their top choices. In such a case there are two male optimal weakly stable solutions, i.e., $\{(m_1, w_1)(m_2, w_2)(m_3, w_3)\}$ and $\{(m_1, w_2)(m_2, w_1)(m_3, w_3)\}$. Our algorithm returns the first one. □

Algorithm 1: *MaleWeaklyStable*

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Input: p: a profile;
Output: μ: a weakly stable marriage or the string 'I don't know';
μ ← ∅;
L ← list of the men of p;
L ← ComputeOrderedList(L);
while Top((first(L)) contains exactly one unmarried woman) or (first(L) has a
single top choice already married) do
    m ← first(L);
    if Top(m) contains exactly one unmarried woman then
        w ← UnmarriedTop(m);
        Add the pair (m, w) to μ;
        foreach strict successor m* of m on w's preferences do
            delete m* from w's preferences and w from m's preferences ;
        L ← L \ {m};
        L ← ComputeOrderedList(L);
    if m has a single top choice already married then
        w ← Top(m);
        m' ← μ(w);
        Remove the pair (m', w) from μ;
        Add the pair (m, w) to μ;
        foreach strict successor m* of m on w's preferences do
            delete m* from w's preferences and w from m's preferences;
        L ← L ∪ {m'} \ {m};
        L ← ComputeOrderedList(L);
if (L = ∅) or (AllDiffUnmarried(L)=true) then
    Add to μ AllDiffUnmarriedMatching(L);
    return μ
else
    if there is a set S of men in L with the same Top T and |T| < |S| then
        if there is a man m* ∈ S s.t., ∀w ∈ T, m* ≺_w m_i, ∀m_i ∈ S − m* then
            ∀w ∈ T, remove w from the preferences of m* in p;
            MaleWeaklyStable(p);
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Example 4. Consider the profile obtained from the profile shown in Example 3 by changing the preferences of w_1 as follows: $m_1 > m_3 > m_2$. We now show that there is no male optimal solution. It is easy to see that in any weakly stable marriage m_1 is married with w_1 or w_2. In the weakly stable marriage where m_1 is married with w_1, m_2 must be married with w_2 and m_3 must be married with w_3, while in the weakly stable marriage where m_1 is married with w_2, m_2 must be married with w_3 and m_3 must be married with w_1. Therefore, in any weakly stable marriage, exactly one of these conditions holds: either m_2 prefers to be married with w_2, or m_3 prefers to be married with w_2. Therefore, there is no male optimal solution. Our algorithm works as follows. Since AllDiffUnmarried(L)=false and since we cannot remove any unfeasible woman from the top choices of m_1, m_2, and m_3, the algorithm returns the string 'I don't know'. □

The *MaleWeaklyStable* algorithm has a time complexity which is $O(n^{\frac{5}{2}})$. In fact, the first part has the same complexity of the extended GS algorithm, which is $O(n^2)$. The second part requires performing an all-different check between the current set of free men and the union of their top choices. Since there are at most n free men and n top choices for each man, we can build a bipartite graph where nodes are men and women, and each arc connects a man with one of his unmarried top choices. Performing the all-different check means finding a subset of the arcs which forms a matching in this graph and involves all men. This can be done in $O(m\sqrt{n})$ where m is the number of edges, which is $O(n^2)$.

The *MaleWeaklyStable Algorithm* is sound, but not complete, i.e., if it returns a marriage, then such a marriage is male optimal and weakly stable, but if it returns the string 'I don't know', we don't know if there is a weakly stable marriage that is male optimal. A case where our algorithm returns the string 'I don't know' is when L is *not empty* and there is a free man with more than one top choice and *all his top choices are already married.* We conjecture that in this case there is a male optimal weakly stable marriage a few times, since it seems there are some very specific circumstances for our algorithm to mot return a male optimal weakly stable marriage (i.e., it has to pass through all the conditions we test) when one exists.

As we noticed above, there are SMPs with no male optimal weakly stable marriages. We now want to identify a class of SMPs where it is always possible to find a linearization which has a male optimal stable marriage.

Definition 10 (male-alldifference property). *An SMP P satisfies the male-alldifference property iff men's preferences satisfy the following conditions:*

- *all the men with a single top choice have top choices that are different;*
- *it is possible to assign to all men with multiple top choices an alternative in their top choices that is different from the one of all the other men of P.*

Theorem 1. *If an SMP is male-alldifferent, then there is a weakly stable marriage that is male optimal and we can find it in polynomial time.*

The *MaleWeaklyStable Algorithm* exploits this same sufficient condition, plus some other sufficient condition. Notice that if an SMP satisfies the male-alldifference property, then, not only is there at least one weakly stable marriage that is male optimal, but there is an unique stable marriage up to ties and incomparability.

5 On the uniqueness of weakly stable marriage in SMPs

For strict total orders, [5] gives sufficient conditions on preference for the uniqueness of the stable marriage. We now extend these results to partial orders. Notice that, if there is an unique stable marriage, then it is clearly male optimal. A class of preference profiles in [5] giving an unique stable marriage, when the preferences are strict total orders, is defined as follows. The set of the men and the set of the women are ordered sets, the preferences require that no man or woman prefers the mate of the opposite sex with the same rank order below his/her own order. Given such a preference ordering, by a recursive argument starting at the highest ranked mates, any other stable marriage would be blocked by the identity marriage, i.e., the marriage in which we match mates of the same rank.

Theorem 2. [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$. If the profile satisfies the following conditions:*

$$\forall w_i \in W : \; m_i >_{w_i} m_j, \; \forall j > i \tag{1}$$

$$\forall m_i \in M : \; w_i >_{m_i} w_j, \; \forall j > i \tag{2}$$

then there is a unique stable marriage $\mu^*(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, \frac{N}{2}\}$.

Notice that the condition above is also necessary when the economies are small, i.e., $N = 4$ and $N = 6$.

There are two particular classes of preference profiles that generate a unique stable marriage, and that are commonly assumed in economic applications [5]. The first assumes that all the women have identical preferences over the men, and that all the men have identical preferences over the women. In such a case there is a common (objective) ranking over the other sex.

Definition 11 (vertical heterogeneity). [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$. A profile satisfies the vertical heterogeneity property iff it satisfies the following conditions:*

- $\forall w_i \in W : m_k >_{w_i} m_j, \;\; \forall k < j$
- $\forall m_i \in M : w_k >_{m_i} w_j, \;\; \forall k < j$

Example 5. An example of a profile that satisfies vertical heterogeneity for $N = 6$ is the following. $\{m_1 : w_1 > w_2 > w_3;\; m_2 : w_1 > w_2 > w_3;\; m_3 : w_1 > w_2 > w_3;\}$ $\{w_1 : m_2 > m_3 > m_1;\; w_2 : m_2 > m_3 > m_1;\; w_3 : m_2 > m_3 > m_1.\}$ □

Corollary 1. [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ and a profile P. If P satisfies the vertical heterogeneity property, then there is a unique stable marriage $\mu^*(w_i) = m_i$.*

When agents have different preferences over the other sex, but each agent has a different most preferred mate and in addition is the most preferred by the mate, then the preference profile satisfies horizontal heterogeneity. In this situation there is a subjective ranking over the other sex.

Definition 12 (horizontal heterogeneity). [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$. A profile satisfies the horizontal heterogeneity property iff it satisfies the following conditions:*

- $\forall w_i \in W : m_i >_{w_i} m_j, \;\; \forall j$

– $\forall m_i \in M : w_i >_{m_i} w_j, \ \forall j$

Example 6. The following profile over 3 men and 3 women satisfies horizontal heterogeneity. $\{m_1 : w_1 > \ldots; \ m_2 : w_2 > \ldots; \ m_3 : w_3 > \ldots\} \ \{w_1 : m_1 > \ldots; \ w_2 : m_2 > \ldots; \ w_3 : m_3 > \ldots.\}$ □

Corollary 2. [5] *Consider two ordered sets $M = (m_i)$ and $W = (w_i)$ and a profile P. If P satisfies the horizontal heterogeneity property, then there is a unique stable marriage $\mu^*(w_i) = m_i$.*

We now check if the results given above for strictly ordered preferences can be generalized to the case of partially ordered preferences. Theorem 2 holds also when the men's preferences and/or women's preferences are partially ordered.

Theorem 3. *In SMPs, if there is an ordering of men and women such that the preference profile satisfies the conditions described in Theorem 2, then there is a unique weakly stable marriage $\mu(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, n\}$.*

Notice that the condition above is also necessary when the economies are small. For example, this holds when $N = 6$ (that is, three men and three women).

We now check if the *vertical heterogeneity* result (Corollary 1) holds also when the preferences are partially ordered. We recall that vertical heterogeneity assumes that all the agents of the same sex have the same strict preference ordering over the mates of the opposite sex. It is possible to see that, even if there is only one incomparable element in the ordering given by the men (or the women), then vertical heterogeneity does not hold and there may be more than one weakly stable marriage, as shown in the following example.

Example 7. Consider the following profile: $\{m_1 : w_1 > w_2 \bowtie w_3; \ m_2 : w_1 > w_2 \bowtie w_3; \ m_3 : w_1 > w_2 \bowtie w_3;\} \ \{w_1 : m_1 > m_2 > m_3; \ w_2 : m_1 > m_2 > m_3; \ w_3 : m_1 > m_2 > m_3\}$. In this profile all the agents of the same sex have the same preference ordering over the mates of the opposite sex, however, there are two weakly stable marriages, i.e., $\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ and $\mu_2 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$. Notice however that these two weakly stable marriages differ only for incomparable or tied partners. □

It is possible to show that if all the agents of the same sex have the same preference ordering over the mates of the opposite sex and there is at least one incomparable or tied pair, then there is a unique weakly stable marriage up to ties and incomparability.

Let us consider now Corollary 2 regarding the *horizontal heterogeneity* property. From Theorem 3, it follows immediately that Corollary 2 holds also when partially ordered preferences are allowed.

Corollary 3. *In SMPs, if there is an ordering of men and women such that the preference profile satisfies horizontal heterogeneity, there is a unique weakly stable marriage $\mu(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, n\}$.*

For partially ordered preferences, we can also guarantee uniqueness of weakly stable marriages by relaxing the horizontal heterogeneity property as follows.

Theorem 4. *In an SMP, let us denote with m_k is the first man with more than one top choice, if he exists. If there is an ordering of men and women in increasing number of their top choices such that the preference profile satisfies the following conditions:*

- $\forall m_i \in M$ *with* $m_i < m_k$, $w_i >_{m_i} w_j$, $\forall j$;
- $\forall m_i \in M$ *with* $m_i \geq m_k$,
  - $(w_i >_{m_i}$ *(or* $\bowtie_{m_i}$*)* $w_j)$, $\forall j < i$, *and*
  - $(w_i >_{m_i} w_j)$, $\forall j > i$;
- $\forall w_i \in W$, *with* $w_i < w_k$, $m_i >_{w_i} m_j$, $\forall j$;
- $\forall w_i \in W$, *with* $w_i \geq w_k$,
  - $(m_i >_{w_i}$ *(or* $\bowtie_{w_i}$*)* $m_j)$, $\forall j < i$, *and*
  - $(m_i >_{w_i} m_j)$, $\forall j > i$,

there is a unique weakly stable marriage $\mu(w_i) = m_i$, $\forall i \in \{1, 2, \ldots, n\}$.

In words, the conditions above require that every man m_i (resp., woman w_i) with a single alternative, i.e., w_i (resp., m_i) has as unique top choice w_i (resp., m_i), and every m_i (resp., w_i) with more than one top choice has exactly one alternative that must be chosen in every weakly stable marriage, that is, w_i (resp., m_i).

6 Related work

In this paper, as in [11, 12], we permit non-strictly ordered preferences (i.e., preferences may contain ties and incomparable pairs) and we focus on weakly stable marriages. However, while in [11, 12], an algorithm is given that finds a weakly stable marriage by solving a specific linearization obtained by breaking arbitrarily the ties, we present an algorithm that looks for weakly stable marriages that are male optimal, i.e., we look for those linearizations that favor one gender over the other one. Moreover, since there is no guarantee that a male optimal weakly stable marriage exists, we give a sufficient condition on the preference profile that guarantees to find a weakly stable marriage that is male optimal, and we show how to find such a marriage. Other work focuses on providing sufficient conditions when a certain property is not assured for all marriages. For example, in [3] a sufficient condition is given for the existence of a stable roommate marriage when we have preferences with ties.

7 Conclusions

We have given an algorithm to find male optimal weakly stable solutions when the men's preferences are partially ordered. The algorithm is sound but not complete. We conjecture, however, that incompleteness is rare since very specific circumstances are required for our algorithm not to return a male optimal weakly stable marriage when one exists. We have then provided a sufficient condition, which is polynomial to check, for the existence of male optimal weakly stable marriages. We have also analyzed the issue of uniqueness of weakly stable marriages, providing sufficient conditions, which are likely to occur in real life problems, that are also necessary in special cases.

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