Abstract Theorem Proving *

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Abstract

Informally, abstraction can be described as the process of mapping a representation of a problem into a new representation. The aim of the paper is to propose a theory of abstraction. The generality of the framework is tested by formalizing and analyzing some work done in the past [Sacerdoti, 1974, Hobbs, 1983, Plaisted, 1981]; its efficacy by giving a procedure which solves the “false proof” problem [Plaisted, 1981] by avoiding the use of inconsistent abstract spaces.

1 Introduction

Abstraction has been suggested as a very powerful technique for constraining search in automated reasoning. Informally, abstraction can be described as the process of mapping a representation of a problem (also called the “ground” representation) into a new representation (also called the “abstract” representation) which preserves certain desirable properties and is simpler to handle. The “desirable properties” amount to requiring that the abstract solution be of help in solving the problem in the original search space. The notion of “simplicity” depends on the application, it may mean decidability or lower complexity. As far as we know, no comprehensive theory of abstraction has been given. The only work in this direction [Plaisted, 1981] is concerned with one form of abstraction and is limited to the area of resolution theorem proving. This has caused the lack of a satisfactory characterization and general understanding of abstraction.

The aim of the work (partially) described in this paper is to provide a theory of abstraction and use it to: (1) classify the various forms of abstraction; (2) investigate their formal properties and the operations which can be defined on them; (3) analyze and classify past work; (4) define ways of building “useful abstractions” and (5) study how the proof in the abstract space can be used to aid the proof in the ground space. In this paper, for lack of space, only some issues are discussed and proofs are only outlined or not given (for a more complete treatment see [Giunchiglia and Walsh, 1990]).

In our formal framework (section 2), an abstraction is just a mapping between formal systems. As this is a very general concept, we concentrate on the classes of abstraction which preserve provability. This captures most of the relevant previous work in abstract theorem proving and planning (section 3). In section 4, we investigate the “false proof” problem [Plaisted, 1981]; when abstracting a problem, we may throw too much information away and leave an inconsistent abstract space. We prove that this problem cannot be avoided as it is always true for the class of abstractions we have come across in abstract theorem proving and planning. However, we are able to propose a (decidable) solution to this problem.

2 The formal framework

Definition 1 (Formal system) : A formal system $\Sigma$ is a triple $\langle \Lambda, \Delta, \Omega \rangle$, where $\Lambda$ is the Language, $\Omega$ is the set of axioms and $\Delta$ is the Deductive Machinery of $\Sigma$.

The language $\Lambda$ is composed of an alphabet, the set of (well formed) terms and the set of well formed formulae (wffs from now on). $\Omega$ is a subset of the wffs of $\Lambda$. The deductive machinery is a set of rules of inference for deriving theorems from axioms.

Definition 2 (Abstraction) : If $\Sigma_1 = \langle \Lambda_{\Sigma_1}, \Omega_{\Sigma_1}, \Delta_{\Sigma_1} \rangle$ and $\Sigma_2 = \langle \Lambda_{\Sigma_2}, \Omega_{\Sigma_2}, \Delta_{\Sigma_2} \rangle$ are two formal systems, an abstraction mapping $f$, written also $f : \Sigma_1 \rightarrow \Sigma_2$, is a triple of total functions $\langle f_\Lambda, f_\Omega, f_\Delta \rangle$ such that:

\[ f_\Lambda : \Lambda_{\Sigma_1} \rightarrow \Lambda_{\Sigma_2} \]
\[ f_\Omega : \Omega_{\Sigma_1} \rightarrow \Omega_{\Sigma_2} \]

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If not explicitly stated to the contrary we assume that $f_A$ and $f_B$ agree on the axioms; that is, for any wff $\omega$, if $\omega \in \Omega$, then $f_A(\omega) = f_B(\omega)$. When no confusion arises we drop the subfixes. Given a deduction tree $\Pi_{\Sigma_i}$ of $\forall \Sigma_i \in \Sigma_i$, we indicate by $f(\Pi_{\Sigma_i})$ a deduction trees $\Pi_{\Sigma_i}$ of $\forall \Sigma_i \in \Sigma_i$.

**Definition 3 (T*-abstractions):** An abstraction $f : \Sigma_1 \rightarrow \Sigma_2$ is said to be $^2$

TC-Abstraction iff, for any wff $\phi_{\Sigma_1}$, $\forall \Sigma_1 \in \Sigma_1$, $\phi_{\Sigma_1}$ iff $\Sigma_2 \in f(\phi_{\Sigma_1})$.

TD-Abstraction iff, for any wff $\phi_{\Sigma_1}$, if $\Sigma_1 \in f(\phi_{\Sigma_1})$ then $\Sigma_2 \in f(\phi_{\Sigma_1})$.

T*-Abstraction iff, for any wff $\phi_{\Sigma_1}$, if $\Sigma_1 \in f(\phi_{\Sigma_1})$ then $\Sigma_2 \in f(\phi_{\Sigma_1})$.

We write “T*-abstraction” to mean any of the above abstractions, TH$(\Sigma_1)$ to mean the set of wffs provable in $\Sigma_1$ and NTH$(\Sigma_1)$ to mean the set of wffs which, if added to the axioms of $\Sigma_1$, make it inconsistent. For example, a T*-abstraction can be represented by the following figure:

![Figure 1: T*-abstraction](image)

TC-abstractions map all the elements of TH$(\Sigma_1)$ into elements of TH$(\Sigma_2)$ and these are all and only the elements of TH$(\Sigma_2)$. Herbrand’s theorem can be formalized as a TC-abstraction. TC-abstractions are used, for instance, in decision theory, under the name of reduction methods, to prove the decidability of and build deciders for the validity problem for certain subclasses of the first order calculus. The trick is to find a class whose decidability is known and prove that there is a proof of a wff iff there is a proof of the “translated” wff in the new class.

In TD-abstractions a subset of the elements of TH$(\Sigma_1)$ is mapped into TH$(\Sigma_2)$ and these are all the elements of TH$(\Sigma_2)$. TD-abstractions are used, for instance, to implement derived inference rules and, as alternatives

To be precise, since we distinguish between wffs occurring as axioms and as anything else, we should consider occurrences of wffs and not wffs. Since, in this paper, for any wff $\omega$, if $\omega \in \Omega$, then $f_A(\omega) = f_B(\omega)$, to make things simpler, we consider $f_A$ and $f_B$ to range over wffs.

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\footnote{TC-abstractions map all the elements of TH$(\Sigma_1)$ into elements of TH$(\Sigma_2)$ and these are all and only the elements of TH$(\Sigma_2)$. Herbrand’s theorem can be formalized as a TC-abstraction. TC-abstractions are used, for instance, in decision theory, under the name of reduction methods, to prove the decidability of and build deciders for the validity problem for certain subclasses of the first order calculus. The trick is to find a class whose decidability is known and prove that there is a proof of a wff iff there is a proof of the “translated” wff in the new class.

T*-abstractions are classified on how provability is preserved between the ground space and the abstract space; they are thus useful when the deductive machinery is defined to generate theorems. On the other hand there are formal systems (ie. resolution) whose deductive machinery determines inconsistency. In these cases, abstractions must be classified on how inconsistent formal systems are mapped. This requires the definition of new classes of abstractions, called NT*-abstractions. Thus, for instance, NT*-abstractions are defined as follows:

\footnote{Note that we give the word “abstraction” a larger meaning than before (see definition 2).}

\footnote{NT*-abstractions and NT*-abstractions are defined analogously to TC-abstractions and TD-abstractions respectively, but preserving inconsistency instead of theoremhood (see definitions 3, 4).}

**Definition 4:** An abstraction $f : \Sigma_1 \rightarrow \Sigma_2$ is an NT*-Abstraction iff, for any wff $\phi_{\Sigma_1}$, if adding $\phi_{\Sigma_1}$ to the axioms of $\Sigma_1$ yields an inconsistent formal system, then adding $f(\phi_{\Sigma_1})$ to the axioms of $\Sigma_2$ yields an inconsistent formal system.

Various properties, equivalences, and relationships among T*, and NT*-abstractions can be proved [Giunchiglia and Walsh, 1990]. We can draw a similar figure for NT*-abstractions as for T*:

![Figure 2: NT*-abstractions (Falseful abstractions)](image)
In many cases, a refutation system, taking as input a goal formula $\alpha$ (usually automatically) negates it, adds the result to the axioms and tries to prove that the resulting theory is inconsistent. A TI-abstraction will therefore be useful if instead of adding $f(\alpha)$ to the abstract space, we add $\neg f(\alpha)$. Additionally there are TI-abstractions which can be used in resolution-based systems independently of whether the goal or its negation is abstracted.

**Definition 5 (Negation preserving)**: An abstraction $f: \Sigma_1 \mapsto \Sigma_2$ is 

negation preserving iff $f(\neg \alpha) = \neg f(\alpha)$.

**Theorem 1**: If $\Sigma_1 = \langle \Lambda_1, \Omega_1, \Delta_1 \rangle$ and $\Sigma_2 = \langle \Lambda_2, \Omega_2, \Delta_2 \rangle$ are two formal systems with (classical) negation, then a negation preserving abstraction $f: \Sigma_1 \mapsto \Sigma_2$ is a TI-abstraction iff $f'$: $\Sigma'_1 \mapsto \Sigma'_2$ is a NTI-abstraction, where $\Sigma'_1 = \langle \Lambda_1, \Omega_1, \Delta'_1 \rangle$, $\Sigma'_2 = \langle \Lambda_2, \Omega_2, \Delta'_2 \rangle$ and $\Delta'_1$ and $\Delta'_2$ are such that $TH(\Sigma'_1) = TH(\Sigma_1)$ and $TH(\Sigma'_2) = TH(\Sigma_2)$.

Examples are $f = f'$ with $\Sigma_1, \Sigma_2, \Sigma'_1$ and $\Sigma'_2$ being natural deduction, and $f \neq f'$ with $\Sigma_1, \Sigma_2$ being natural deduction and $\Sigma'_1, \Sigma'_2$ being resolution. As far as we know, all the abstractions proposed to work in resolution systems are negation preserving. However, there are useful abstractions which are not negation preserving (for instance when negation is not part of the language of $\Sigma_1$ or $\Sigma_2$ [Newell et al., 1963], or only partially preserved by the mapping).

3 Some examples of abstraction

The purpose of these examples, together with providing a rational reconstruction of the work described, is to convince the reader that the framework is very powerful and allows us to present an unified view of the work done in different areas and with different goals. For lack of space, only three examples are reported; more “historical” examples are reported in [Giunchiglia and Walsh, 1990].

**Example 1** [Planning]: Abstrips [Sacerdoti, 1974] was one of the first noticeable applications of abstraction. In Abstrips the preconditions to operators were abstracted according to their criticality. To formalize Strips-like planning we shall adopt a situation calculus in a natural deduction formal system. Let us consider the abstraction $f_{AB}$ where $\Sigma_1$ and $\Sigma_2$ are situation calculi with a first order language, $\Omega$ consists of frame, operator and theoretic axioms and $\Delta$ consists of natural deduction rules of inference. Operators are wffs of the form $\forall s, (\bigwedge_{1 \leq i \leq n} p_i(s) \rightarrow q(f(s)))$ where $p_i$ is a precondition, $s$ is a state of the world, $f$ is some action, and $q$ describes the new state of the world. Goals are wffs of the form $\exists s, r(s)$. $f_{AB}$ is applied to wffs and axioms as follows:

$f_{AB}(\alpha) = \alpha$ if $\alpha$ is an atomic formula.

$f_{AB}(\neg \alpha) = \neg f_{AB}(\alpha)$;

$f_{AB}(\phi \land \psi) = f_{AB}(\phi) \land f_{AB}(\psi)$, where “$\land$” is “AND” or “\lor”;

$f_{AB}(\forall x \alpha) = \exists x f_{AB}(\alpha)$, where “$\exists$” is “\exists” or “$\forall$”;

$f_{AB}(\alpha \rightarrow \beta) = f_{AB}(\alpha) \rightarrow f_{AB}(\beta)$, provided “$\rightarrow$” is not an operator;

$f_{AB}(\bigwedge_{1 \leq i \leq n} p_i(s) \rightarrow r) = \bigwedge_{i \in \text{crit}(\kappa)} p_i(s) \rightarrow f_{AB}(r)$, provided that “$\bigwedge_{1 \leq i \leq n} p_i(s) \rightarrow r$” is an operator, where $i \in \text{crit}(\kappa)$ if the criticality of $p_i$ is greater than $\kappa$.

**Theorem 2**: $f_{AB}$ is TI, namely, if $\vdash_{\Sigma_1} \varphi_{\Sigma_1}$, then $\vdash_{\Sigma_2} f_{AB}(\varphi_{\Sigma_1})$.

**Proof [Outline]**: By proving that a given deduction tree $\Pi_{\Sigma_1}$ of $\vdash_{\Sigma_1} \varphi_{\Sigma_1}$, we can build a deduction tree $\Pi_{\Sigma_2} = f_{AB}(\Pi_{\Sigma_1})$ of $\vdash_{\Sigma_2} f_{AB}(\varphi_{\Sigma_1})$. The proof proceeds by induction on the weight $\kappa$ of $\Pi_{\Sigma_1}$. For proofs of length 1, $f_{AB}$ is applied to the single wff; this generates a valid proof in $\Pi_{\Sigma_2}$. Assume that we have a deduction tree up to size $\kappa$. Any rule application that is not modus ponens involving an operator translates unmodified, in the sense that, for instance, an “\lor” on $\varphi$ in $\Pi_{\Sigma_1}$ becomes an “\lor” on $f_{AB}(\varphi)$ in $\Pi_{\Sigma_2}$. For an operator application, the following transformation is performed:

The abstract proof is valid since $f_{AB}(\ldots)$ is a valid deduction tree from the induction hypothesis, and the hypothesis of the (abstract) operator axiom is obtained from “$\bigwedge_{1 \leq i \leq n} p_i$” by a (possibly empty) sequence of applications of “and-elimination”.

Note that the abstract proof is longer than the one in the ground space. The purpose of abstracting is not to find these longer proofs; we hope that there are also going to be shorter proofs. These shorter proofs are those that don’t try to satisfy $p_j$ for $j \notin \text{crit}(\kappa)$. There is no guarantee that there will be a shorter proof than the one exhibited; we will always be able to devise an obtuse theory in which to prove the $p_i$ for $i \in \text{crit}(\kappa)$ we have to prove all the other $p_j$ for $j \notin \text{crit}(\kappa)$. ♦

*The weight of a deduction tree is the number of its formula occurrences.*
Example 2 [resolution theorem proving, logic programming]: The work by Plaisted is closest in spirit to ours. Plaisted proposes two classes of abstraction, ordinary abstractions and weak abstractions [Plaisted, 1981], which map a set of clauses onto a set of clauses and preserve inconsistency. His work is less general than ours as he restricts his attention to resolution systems and his classes of abstraction fail to capture all NTI-abstraction mappings that preserve inconsistency between resolution systems. In other words, Plaisted's abstractions are NTI, but not all NTI-abstractions are weak or ordinary. Moreover we claim that our definitions of abstraction are "more natural" in the sense that better reflect and capture the functionalities they are given for.

Ordinary abstractions are described as taking both $\Sigma_1$ and $\Sigma_2$ to be first order calculi with $\Lambda_\Sigma$ allowing clausal form, $\Delta_\Sigma$ being resolution and $\Omega_\Sigma$ being arbitrary. Any ordinary abstraction mapping $f$ maps a clause in $\Lambda_\Sigma$ onto a set of clauses in $\Lambda_\Sigma$ subject to the following conditions:

a) $f(\bot) = \{ \bot \}$;

b) if $\alpha_3$ is a resolvent of $\alpha_1$ and $\alpha_2$ in $\Sigma_1$, and $\beta_3 \in f(\alpha_3)$ then there exist $\beta_2 \in f(\alpha_2)$ and $\beta_1 \in f(\alpha_1)$ such that a resolvent of $\beta_1$ and $\beta_2$ subsumes $\beta_3$ in $\Sigma_2$;

c) if $\alpha_1$ subsumes $\alpha_2$ in $\Sigma_1$, then for every $\beta_2 \in f(\alpha_2)$ there exists $\beta_1 \in f(\alpha_1)$ such that $\beta_1$ subsumes $\beta_2$ in $\Sigma_2$.

Weak abstractions are identically defined to ordinary abstractions except condition b) is weakened to the property that if $\alpha_3$ is a resolvent of $\alpha_1$ and $\alpha_2$ in $\Sigma_1$, and $\beta_3 \in f(\alpha_3)$ then there exist $\beta_2 \in f(\alpha_2)$ and $\beta_1 \in f(\alpha_1)$ such that either $\beta_1$ subsumes $\beta_3$, or $\beta_2$ subsumes $\beta_3$, or a resolvent of $\beta_1$ and $\beta_2$ subsumes $\beta_3$ in $\Sigma_2$.

Theorem 3: Weak and ordinary abstractions are NTI.

Proof: The proof is a corollary to Theorem 2.5 on page 55 of [Plaisted, 1981].

Theorem 4: There exist NTI-abstractions between resolution systems that are not weak or ordinary abstractions.

Proof[Outline]: We can find NTI-abstractions that fail every one of the three conditions in the definition of weak and ordinary abstractions. Condition a) is failed by the NTI-abstraction $f$ such that, for any wff $\phi$ in $\Sigma_1$, $f(\phi) = \{ \phi \lor \bot \}$. The problem with condition b) is that we may also need to resolve with an axiom of the theory. Consider, for instance, the abstraction defined by $f(p \lor q) = \{ p \lor r \}$ and $f(\phi) = \{ \phi \}$ otherwise. If $\Sigma_1$ contains the axioms, $\neg q$, and $\neg r$ then $f$ is NTI. In particular, $p \lor q$ resolves with $\neg p$ in $\Sigma_1$ to give $q$. However, no clause in the abstraction of $p \lor q$, or $\neg p$ (or their resolvent) subsumes the clause $q$ found in the abstraction of $q$. For condition c), consider the abstraction defined by $f(p \lor q) = \{ r, p \lor q \}$ and $f(\phi) = \phi$ otherwise. Now $f$ is NTI. However, $f$ fails condition c) of the definition of weak and ordinary abstractions as $p$ subsumes $p \lor q$ but no clause in the abstraction of $p$ subsumes $r$ which is in $f(p \lor q)$.

The definition of weak and ordinary abstractions could be extended to overcome the first counter-example by replacing condition a) with the more general requirement that $\exists \psi \in f(\bot), \vdash_{\Sigma_1} \neg \psi$. However, this still leaves useful NTI-abstractions that fail conditions b) and c). For example, if $p_i \leftrightarrow p_i$ for many $i$ we might abstract many clauses of the form $p_i \lor q$ onto the one clause $\{ p \lor q \}$. One could argue that ordinary and weak abstractions have the advantage, over NTI-abstractions, that they always map into simpler theories, in the sense that there is always an abstract proof that is no longer than the shortest proof of the unabstracted theorem [Plaisted, 1981].

This does not seem a good point since, first of all, we intuitively expect NTI-abstractions (that are not NTC) to have this or similar properties and, second and more importantly, there are NTI-abstractions, which are not weak or ordinary, which build simpler theories (the last example is one possible case).

Example 3 [Common sense reasoning]: In [Hobbs, 1985], Hobbs presents a theory of granularity in which a complex theory is abstracted onto a simpler, more "coarse-grained" theory with a smaller domain; for example, we could map the real world of continuous time and positions onto a (micro)world of discrete time and positions. Hobbs' granularity theory can be formalized as a mapping (let us call it $f_{gran}$) that can be proved to be TI. Let us suppose that both $\Sigma_1$ and $\Sigma_2$ are calculi with a first order language, an arbitrary set of axioms and any complete deductive machinery for first order logic. $f_{gran}$ maps different objects in $\Sigma_1$ into (not necessarily different) objects in $\Sigma_1$ according to an indistinguishability relation, defined by the (second-order) axiom:

$$\forall x, y. x \sim y \leftrightarrow \forall p \in R. p(x) \leftrightarrow p(y)$$

where $R$ is the subset of the predicates of the theory determined to be relevant to the situation at hand. Thus $f_{gran}$ keeps the same logical structure of wffs and translates any constant into its equivlence class, namely $f_{gran}(p(a)) = p(\kappa(a))$ where $a$ is any constant symbol.

As in [Hobbs, 1985], we define indistinguishability for unary predicates; it can, however, be easily generalized to n-ary predicates.
Lemma 1 holds independently of the consistency of $\Sigma_1$. Once $f$ has been proved to be TI it may happen that $\Sigma_2$ is inconsistent. This is a major blow to the use of TI-abstractions to guide the proof in the ground space. When $\Sigma_2$ is inconsistent the structure of the proof in $\Sigma_2$ could still be used to shape the proof in $\Sigma_1$. However, any wff in $\Sigma_2$ is a theorem and thus $\Sigma_2$ does not filter out any of the wffs which are not theorems in $\Sigma_1$. In a way $\Sigma_2$ gives too little information. To make matters worse, in general it is not possible to decide in a finite amount of time whether a formal system is consistent.

When working with a fixed formal system (i.e. set theory + first order logic) a solution is to build abstractions which are proved a priori to have consistent $\Sigma_2$. In many cases, however, (i.e. planning, logic programming, knowledge-based systems), while the set of inference rules of $\Sigma_1$ is fixed, its axioms may vary and depend on the application. Tenenberg [Tenenberg, 1987] proposed a solution to the problem in the case of a form of predicate abstractions in a resolution-based system. However, the abstractions he proposes have many drawbacks: the first abstraction is TI but the construction of $\Omega_2$ is not decidable (even if recursively enumerable) and it may take an infinite amount of time to generate it; the other two types of abstraction are TD or similar to TD [Giunchiglia and Walsh, 1990] 10. This means that completeness is lost since there is at least one theorem in $\Sigma_1$ whose abstraction is not a theorem in $\Sigma_2$. We consider this the one property you do not want to lose.

The ideal solution would be to generalize the concept of abstraction mapping to be parameterized on the axioms of $\Sigma_1$ and then to find sufficient conditions which guarantee that a TI-abstraction maps $\Sigma_1$ into a consistent $\Sigma_2$, independently of the axioms of $\Sigma_1$ (as long as $\Sigma_1$ is consistent). This seems a reasonable request since there are abstractions which, fix $\Lambda_{\Sigma_1}$, $\Lambda_{\Sigma_2}$, $\Delta_{\Sigma_1}$, $\Delta_{\Sigma_2}$, $f_\Lambda$, $f_\Delta$ and $f_A$ are TI for any choice of the theoretical axioms (this is, for instance, true for the abstractions of the three examples) 11.

Let $\Lambda_{\Sigma_1}$ and $\Lambda_{\Sigma_2}$ be two languages, $\Delta_{\Sigma_1}$ and $\Delta_{\Sigma_2}$ two deductive machineries. Then, if $f_\Lambda : \Lambda_{\Sigma_1} \rightarrow \Lambda_{\Sigma_2}$, $g : \Lambda_{\Sigma_1} \rightarrow \Lambda_{\Sigma_2}$ and $f_\Delta : \Delta_{\Sigma_1} \rightarrow \Delta_{\Sigma_2}$ are three total functions, $F = (f_\Lambda, g, f_\Delta)$ is an abstraction from $\Sigma_1 = \langle \Lambda_{\Sigma_1}, \Lambda_{\Sigma_1}, \Delta_{\Sigma_1} \rangle$ to $\Sigma_2 = \langle \Lambda_{\Sigma_2}, \Lambda_{\Sigma_2}, \Delta_{\Sigma_2} \rangle$. Then for any $\Omega_{\Sigma_1} \subset \Lambda_{\Sigma_1}$, if by “$g \uparrow \Omega_{\Sigma_1}$” we indicate $g$ restricted to apply to $\Omega_{\Sigma_1}$, $F^{\Omega_{\Sigma_1}} = (f_\Lambda, g \uparrow \Omega_{\Sigma_1}, f_\Delta)$ is an abstraction from $\Sigma_1^{\Omega_{\Sigma_1}} = \langle \Lambda_{\Sigma_1}, \Omega_{\Sigma_1}, \Delta_{\Sigma_1} \rangle$ to $\Sigma_2^{\Omega_{\Sigma_1}} = \langle \Lambda_{\Sigma_2}, \Omega_{\Sigma_2}, \Delta_{\Sigma_2} \rangle$, with $\Omega_{\Sigma_2} = g(\Omega_{\Sigma_1})$.

8Predicate abstractions are abstractions where distinct predicate symbols in $\Sigma_1$ are mapped onto (possibly not distinct) predicate symbols in $\Sigma_2$ [Giunchiglia and Walsh, 1990].

10Note that it can be proved that, more generally, for any TD-abstraction, if $\Sigma_1$ is consistent, so is $\Sigma_2$.

11Of course theory independent TI-abstractions are in general less efficient than the ones geared towards one single theory as they do not exploit the structure of theorems axioms.
Theorem 7: Let $\Sigma_1$ and $\Sigma_2$ be two languages, $\Delta_1$, and $\Delta_2$ two deductive machineries, $f_1 : \Sigma_1 \rightarrow \Sigma_2$, $g : \Delta_1 \rightarrow \Delta_2$ and $f_\Delta : \Delta_1 \rightarrow \Delta_2$ three total functions. Then there exists $\Omega_{\Sigma_2} \subseteq \Sigma_1$, such that, if the abstraction $F^{\Sigma_2}$ : $(f_\Delta, g \uparrow \Omega_{\Sigma_2}, f_\Delta)$ is TI and NTI but not NTC, then $\Sigma_{\Sigma_2}^{\Omega_{\Sigma_2}}$ is consistent and $\Sigma_{\Sigma_2}^{\Omega_{\Sigma_2}}$ is inconsistent.

Proof[Outline]: By constructing $\Omega_{\Sigma_2}$. Because $F^{\Sigma_2}$, is NTI but not NTC, there exists a wff $\phi$ such that adding $F^{\Sigma_2}(\phi)$ as an axiom to $\Sigma_2$ makes an inconsistent formal system, but that adding $\phi$ as an axiom to $\Sigma_1$ doesn’t. $\square$

Theorem 7 can actually be proved in more powerful forms; however the hypotheses hold for most TI-abstractions. For instance negation preserving abstractions that are TI are also NTI and vice versa (theorem 1). Theorem 7 proves that we cannot find a TI-abstractions which maps a consistent $\Sigma_1$ into a consistent $\Sigma_2$ independently of the axioms of $\Sigma_1$. However, we can find (syntactic characterisations of) subsets of theories for which consistency is guaranteed. A different solution to the false proof problem is to vary the TI-abstraction until we can (decidably) show that $\Sigma_2$ is consistent. TI-abstractions applied to the same $\Sigma_1$ can be classified into a weak partial order, indicated by “$\subseteq$”.

Definition 6 ($\subseteq$): If $f_1 : \Sigma_1 \rightarrow \Sigma_2^1$ and $f_2 : \Sigma_1 \rightarrow \Sigma_2^2$ are two TI-abstractions then $f_1 \subseteq f_2$ iff for all wffs $\varphi_{\Sigma_1}$, if $\Gamma_{\Sigma_2} f_1(\varphi_{\Sigma_1})$ then $\Gamma_{\Sigma_2} f_2(\varphi_{\Sigma_1})$. We say that $f_1$ is weaker than $f_2$ or, dually, that $f_2$ is stronger than $f_1$.

If $f_1 \subseteq f_2$, then $f_2$ is stronger than $f_1$ in the sense that there are fewer wffs which are theorems in $\Sigma_2^1$ and not in $\Sigma_1$ than wffs which are theorems in $\Sigma_2^2$ and not in $\Sigma_1$. “$\subseteq$” is in general a weak partial order (respecting transitivity, antisymmetry and reflexivity) but not a total order. If, however, we have a set of totally ordered abstractions then the following result holds:

Theorem 8: If $f_1 : \Sigma_1 \rightarrow \Sigma_2^1$, ..., $f_n : \Sigma_1 \rightarrow \Sigma_n$ are TI-abstractions and $f_1 \subseteq \ldots \subseteq f_n$ ($f_1, \ldots, f_n$ are totally ordered), then if $\Sigma_{\Sigma_2}^i$ is consistent so is $\Sigma_{\Sigma_2}^j$ for any $1 \leq i \leq n$.

Theorem 8 suggests the following process:

- build sets of abstractions, $F_i = \{ f_i^1, \ldots, f_i^n \}$ where $f_i^1 \subseteq \ldots \subseteq f_i^n$ and $f_i^n(\Sigma_1)$ is decidable (e.g. it is propositional).
- find a set, $F_j$ in which the codomain of the strongest abstraction $f_j^n(\Sigma_1)$ is consistent. Note that, since $f_j^n(\Sigma_1)$ is decidable, its consistency or inconsistency can be proved in a finite amount of time.

starting with the strongest abstraction (that is with $l = n_j$), until $l > 1$ use the proof that the abstracted wff is a theorem in $f_i^n(\Sigma_1)$ to help construct a proof in $f_j^{l-1}(\Sigma_1)$. If, in any of the $f_j^l(\Sigma_1)$, the abstracted wff is not a theorem, then the wff cannot be a theorem in $\Sigma_1$ (since $f_j^n$ is a TI-abstraction).

Of course there is no guarantee that all the steps in unabstracting back to $\Sigma_1$ will go through or terminate. The overall performance depends on how the various abstractions in the total order are built and on how the process of unabstracting is performed. For instance, computing the consistency of $f_j^n(\Sigma_1)$ can be optimized by building a very simple, “minimal” $f_j^n(\Sigma_1)$. Further time can also be saved, when $f_j^n(\Sigma_1)$ is proved inconsistent by introducing (in an automated way) small variations in $f_j^l$ that are tuned to the source of the inconsistency.

5 Conclusions

In this paper we have proposed a theory of abstraction which extends the notions of abstraction previously used. We have focused on abstract theorem proving and have suggested that a certain class of provability preserving abstractions, TI- and NTI-abstractions (which are not TC and NTC) are the correct abstractions to use. TC- and NTC- abstractions are in general too strong, and the goal of having “simpler” abstract proofs does not seem achievable except in very special and limited forms (for instance, if $f : \Sigma_1 \rightarrow \Sigma_2$ is a TC-/ NTC- abstraction then if $\Sigma_1$ is undecidable then $\Sigma_2$ cannot be decidable). The dual class of provability preserving abstractions, TD- (and NTD-) abstractions (which are not TC- and NTC-) are of less use as they lose completeness; that is, there is at least one theorem whose abstraction is not a theorem. Unfortunately, TI- (and NTI-) abstractions are subject to the false proof problem; they can map a consistent theory into an inconsistent abstract theory. The last section has proposed a new (and decidable) solution to this problem.

References


