Interchangeable variables and values

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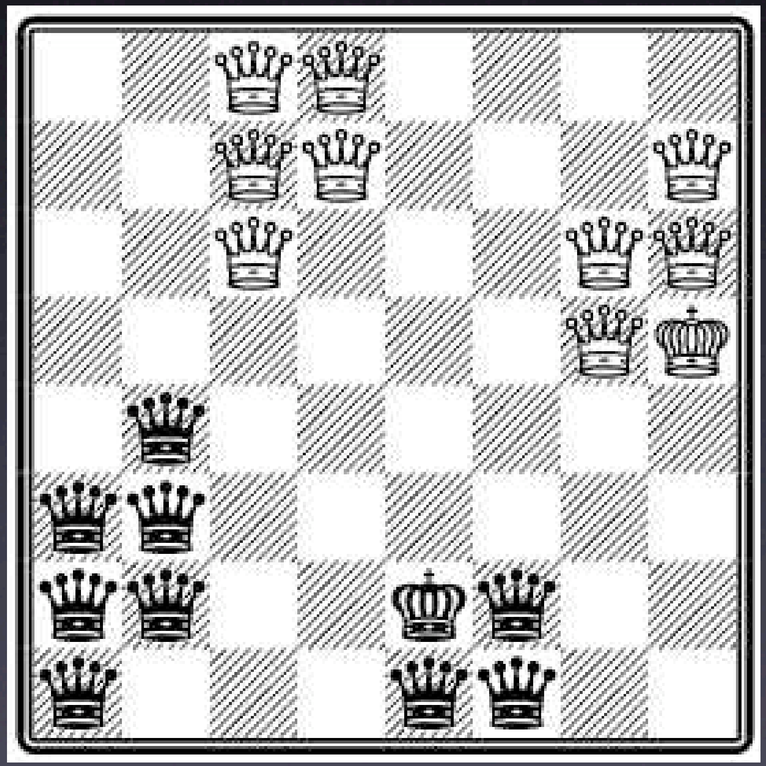
Symmetry

- Symmetry is bijection, σ on assignments that preserves solutions/constraints
  - In armies of queens problem, swap colours!
  - X[1,3]=white queen, X[6,2] =black queen ..
  - X[1,3]=black queen, X[6,2] =white queen ..



Types of symmetry

- Which part of the assignment does the bijection act upon?
  - Variable symmetry
  - Value symmetry
  - Variable/value symmetry

Symmetry breaking

- General method for variable symmetries
  [Crawford, Ginsberg, Luks and Roy KR96]
  - Look for lexicographically least assignment
    - $(Z[1],Z[2],\ldots) \leq_{lex} (Z[\sigma(1)],Z[\sigma(2)],\ldots)$
    - reversal symmetry:
      $(X[1],X[2],\ldots,X[n-1],X[n]) \leq_{lex}$
      $(X[n],X[n-1],\ldots,X[2],X[1])$

Adding constraints

- Same method works with symmetries in general [Walsh CP06]
  - Including those that act both on variables and values simultaneously
  - Look for assignment that is lex smaller than all its symmetries
  - So, we're done? Symmetry solved problem?

Adding constraints

- No! Too many constraints in general
  - For instance, *m* interchangeable values gives *m!* symmetry breaking constraints
  - Look for special cases where we can do better

Interchangeable variables and values

- Often variables and values partition interchangeable sets
  - Pigeonhole problem
  - P[i]=j iff pigeon i in hole j
  - Pigeons (variables) interchangeable
  - Holes (values) interchangeable

Interchangeable variables and values

- Often variables and values partition interchangeable sets
  - Timetabling
  - Class[i]=j iff class i occurs at time j
  - Classes taken by same students interchangeable (partition into sets)
  - All times interchangeable

Signature of an assignment

- Can break all symmetry due to interchangeable variables and values
  - By ordering "signatures" of an assignment
- Signature is an abstract view of an assignment
  - Each equivalence class of symmetric assignments has unique signature

Signature of an assignment

- Suppose variables partition into *a* interchangeable classes
  - X[1] to X[p(1)-1],
  - X[p(1)] to X[p(2)-1],
  - X[p(2)] to X[p(3)-1],
  - ..
  - X[p(a-1)] to X[p(a)-1]

Signature of an assignment

- Suppose values partition into *b* interchangeable classes
  - 1 to q(1)-1,
  - q(1) to q(2)-1,
  - q(2) to q(3)-1,
  - ..
  - q(b-1) to q(b)-1

Signature of an assignment

- Signature of the value k
  - Sig[k] = (O[1],..O[a])
  - Where O(i) = |{ j | X[j]=k}|
- Example
  - X[1],X[2] and X[3],X[4] are 2 partitions of vars
  - 1,2 and 3,4 are 2 partitions of values
  - X[1]=3, X[2]=3, X[3]=1, X[4]=4

Signature of an assignment

- Signature of the value k
  - Sig[k] = (O[1],..O[a])
  - Where $O[i] = |\{ j \mid X[j]=k \;\&\; p(i) \leq j < p(i+1)\}|$
- Signature invariant of permutation of variables within an equivalence class

Ordering signatures

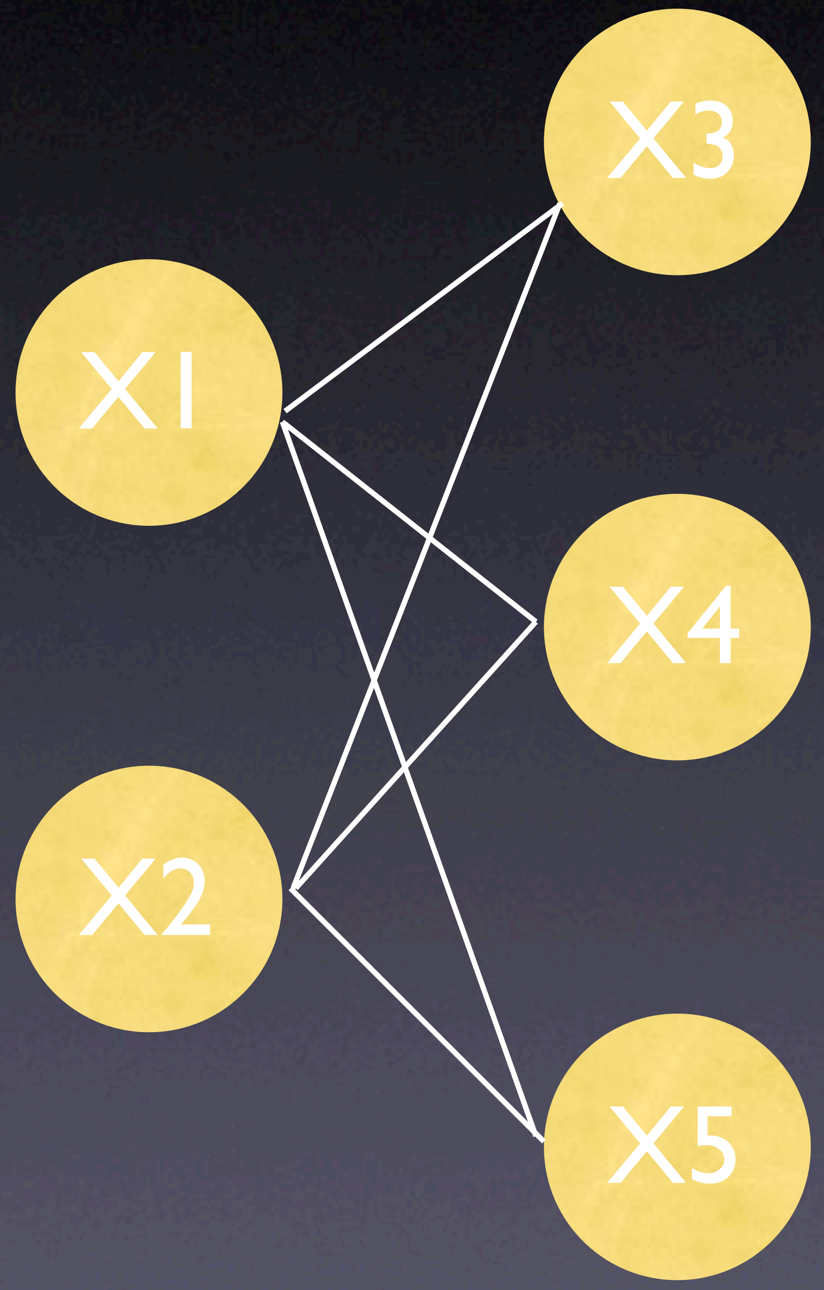
- To break all symmetry
  - Order variables within each partition
    - X[1] ≤ .. ≤ X[p(1)-1]
    - X[2] ≤ .. ≤ X[p(2)-1]
    - ..
    - X[p(a-1)] ≤ .. ≤ X[p(a)-1]

Ordering signatures

- To break all symmetry
  - Order signatures of values within each partition
    - $Sig[1] \geq_{lex} .. \geq_{lex} Sig[q(1)-1]$
    - $Sig[q(1)] \geq_{lex} .. \geq_{lex} Sig[q(2)-1]$
    - ..
    - $Sig[q(b-1)] \geq_{lex} .. \geq_{lex} Sig[q(b)-1]$

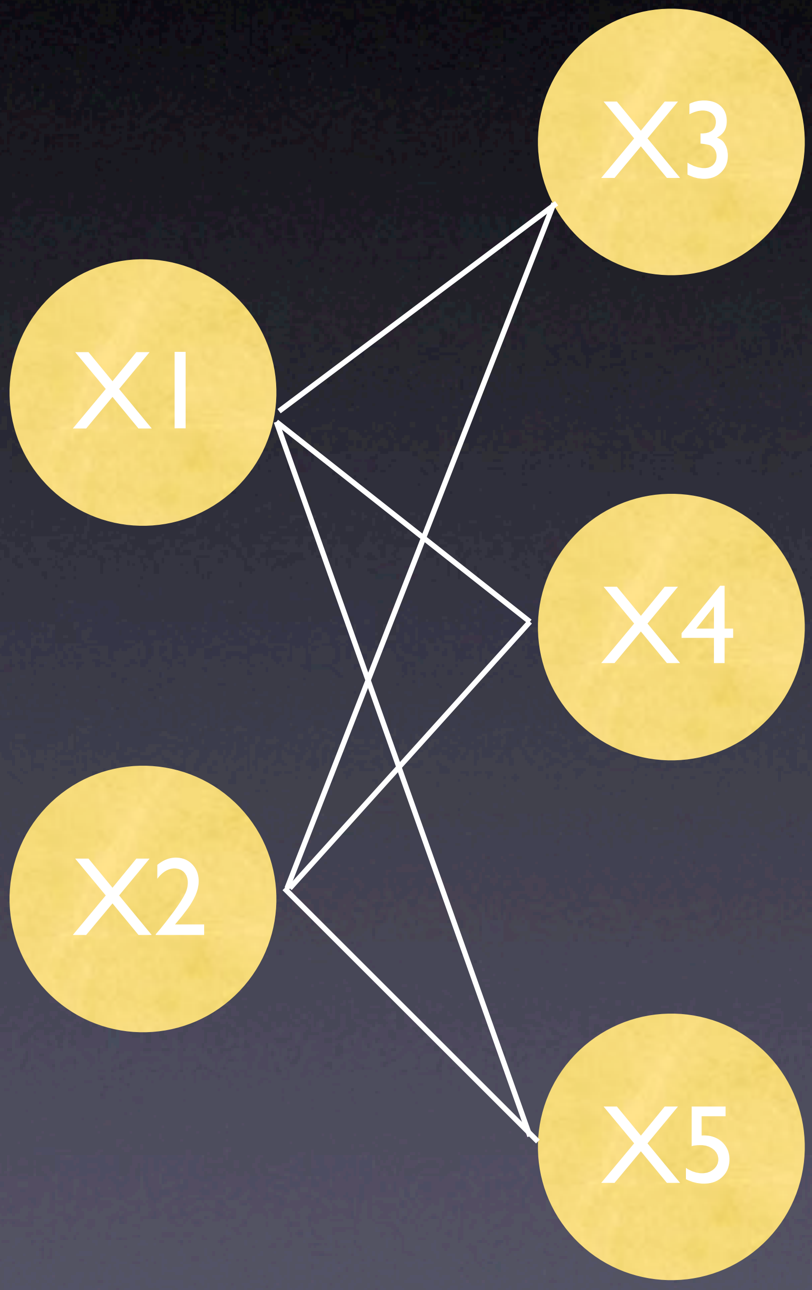
An example

- Two nodes are interchangeable
  - If they have the same neighbours
  - Variables partition into interchangeable classes
- All colours are interchangeable
  - Values are fully interchangeable



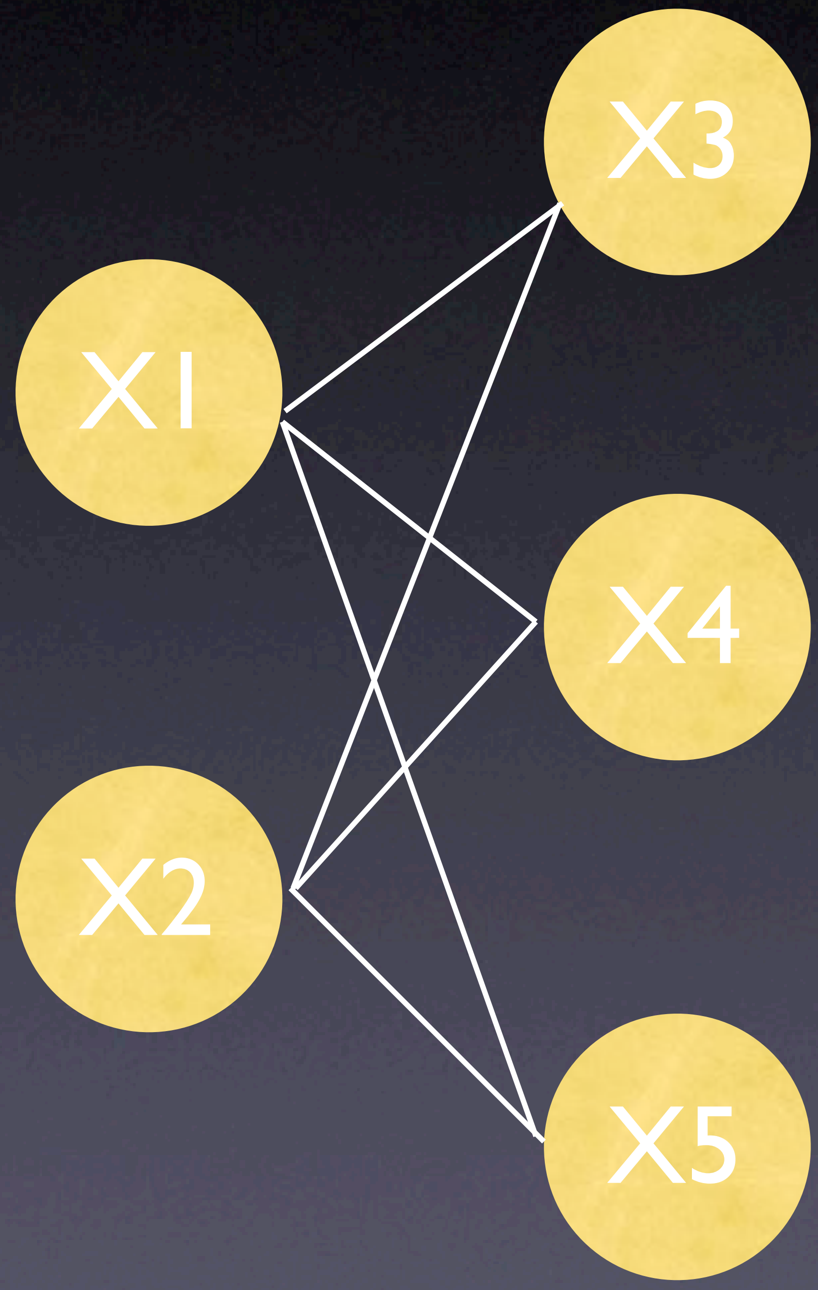
An example

- Variables partition into two equivalence classes
  - X1, X2
  - X3, X4, X5
- Values partition into one equivalence class
  - 1, 2, 3
  - (Or red, green, blue if you prefer)



An example

- 30 proper colourings
    - But only 3 if we break symmetry
    - X1,..X5 is 1,1,2,2,2, or 1,1,2,2,3 or 1,2,3,3,3
- Some of the colourings eliminated by ordering signatures
    - 2,1,3,3,3 and 1,1,3,3,3 and 1,1,2,3,3



Ordering signatures

- Equivalent to the exponential number of LEX LEADER constraints
  - But only requires a polynomial number of constraints
  - Channel into a count on occurrences of values within each equivalence class
    - $O[i,k] = |\{ j \mid X[j]=k \ \& \ p(i) \leq j < p(i+1)\}|$
  - Lex order signatures/counts

Some special cases

- Interchangeable values but not variables
  - i.e. $a=n$
  - Ordering signatures = value precedence
  - $(X1=k, X2=k, \ldots) \geq_{lex} (X1=k+1, X2=k+1, \ldots)$
  - Consider k=3, and (X1,..,X6)=(1,1,2,1,3,4)

Some special cases

- All variables and values are interchangeable
  - i.e. $a=b=1$
  - Ordering signatures = "decreasing sequence"
    - $X1 \leq X2 \leq .. \leq Xn$
    - $|\{ i \mid Xi=k\}| \geq |\{ i \mid Xi=k+1\}|$ for all k
    - Consider 1,1,1,2,2,3,4 and 1,1,1,2,2,3,5 or 1,1,2,2,2,3,4

Dynamic methods

- "Efficient" dynamic methods exist for interchangeable variables and values
  - Dominance detection between two assignments in $O(nd+d^{5/2})$ time
  - Eliminate all symmetry at each node in $O(nd^{7/2}+n^2d^2)$ time
  - Not seen this implemented!

Conclusions

- Symmetry occurs in many problems
  - We must deal with it or face a combinatorial explosion!
- We have a generic method (for small numbers of symmetries)
  - In special cases, we can break all symmetries
  - One such case is when vars and vals partition into interchangeable sets

Questions?